Appendix

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A Proof

In Theorem 3.1, Corollary 3.3 and Remark 3.4, we deal with an arm with index $k \in [K]$. To simplify notations, we drop the subscript k in $s_{t,k}$, $w_{t,k}$, $x_{t,k}$, $\widehat{\theta}_{\text{ALEE},k}$ and θ_k^* throughout the proof, and use s_t, w_t, x_t, $\widehat{\theta}_{\text{ALEE}}$ and θ^*, respectively.

A.1 Proof of Theorem 3.1

Condition (17) serves as an important role in proving (18). Therefore, we start our proof by verifying the condition (17). Since function f is a positive decreasing function, we first have

$$\max_{1\le t\le n} f^2(\frac{s_t}{s_0})\frac{x_t^2}{s_0} \le f^2(1)\frac{1}{s_0}. \tag{32}$$

Furthermore, since function f'/f is increasing, we have

$$\begin{aligned}\max_{1\le t\le n}\left(1-\frac{f(s_t/s_0)}{f(s_{t-1}/s_0)}\right) &= \max_{1\le t\le n}\frac{f(s_{t-1}/s_0)-f(s_t/s_0)}{f(s_{t-1}/s_0)}\\ \overset{(i)}{\le} \frac{1}{s_0}\max_{1\le t\le n}\frac{-f'(s_{t-1}/s_0)}{f(s_{t-1}/s_0)} &= \frac{1}{s_0}\frac{-f'(1)}{f(1)},\end{aligned} \tag{33}$$

where inequality (i) follows from mean value theorem and the monotonicity of the function f'/f. Thus, by assuming $1/s_0 = o_p(1)$ and $s_0/s_n = o_p(1)$, condition (17) follows directly from equation (32) and equation (33).

By the construction of ALEE estimator, we have

$$\left\{\sum_{t=1}^n w_t x_t\right\}\cdot(\widehat{\theta}_{\text{ALEE}}-\theta^*) = \sum_{t=1}^n w_t\epsilon_t. \tag{34}$$

Note that

$$\sum_{t=1}^n w_t x_t = \sqrt{s_0}\sum_{t=1}^n f(s_t/s_0)\frac{x_t^2}{s_0} = \sqrt{s_0}\int_1^{s_n/s_0} f(x)dx\cdot\frac{\sum_{t\le n} f(s_t/s_0)x_t^2/s_0}{\int_1^{s_n/s_0} f(x)dx}. \tag{35}$$

By the mean value theorem, we have that for $t\in[n]$, $\xi_t\in[s_{t-1},s_t]$

$$\int_{s_{t-1}/s_0}^{s_t/s_0} f(x)dx = f(\xi_t/s_0)\frac{x_t^2}{s_0}.$$

Therefore, we have

$$\frac{\sum_{t\le n} f(s_t/s_0)x_t^2/s_0}{\int_1^{s_n/s_0} f(x)dx} = 1 + \underbrace{\frac{\sum_{t\le n}(\frac{f(s_t/s_0)}{f(\xi_t/s_0)} - 1)f(\xi_t/s_0)x_t^2/s_0}{\sum_{t\le n} f(\xi_t/s_0)x_t^2/s_0}}_{\triangleq R}.$$

Observe that

$$\begin{aligned}
|R| &\le \frac{\sum_{t\le n}|\frac{f(s_t/s_0)}{f(\xi_t/s_0)} - 1|f(\xi_t/s_0)x_t^2/s_0}{\sum_{t\le n} f(\xi_t/s_0)x_t^2/s_0} \\
&\le \frac{\sum_{t\le n}|\frac{f(s_t/s_0)}{f(s_{t-1}/s_0)} - 1|f(\xi_t/s_0)x_t^2/s_0}{\sum_{t\le n} f(\xi_t/s_0)x_t^2/s_0} \\
&\le \max_{1\le t\le n}\left(1 - \frac{f(s_t/s_0)}{f(s_{t-1}/s_0)}\right) \overset{(ii)}{=} o_p(1).
\end{aligned}$$

Equality (ii) follows from condition (17). Consequently, applying Slutsky's theorem yields

$$\frac{\sum_{i=1}^n w_t x_t}{\sqrt{s_0}\int_1^{s_n/s_0} f(x)dx} \xrightarrow{p} 1.$$

Similarly, we can derive

$$\sum_{t=1}^n w_t^2 = \sum_{t=1}^n f^2(s_t/s_0)\frac{x_t^2}{s_0} = (1 + o_p(1))\int_1^{s_n/s_0} f^2(x)dx = 1 + o_p(1). \tag{36}$$

Knowing $\max_{1\le t\le n} w_t^2 = \max_{1\le t\le n} f^2(s_t/s_0)x_t^2/s_0 = o_p(1)$, which is a consequence of equation (17), martingale central limit theorem together with an application of Slutsky's theorem yields

$$(\widehat{\theta}_{\text{ALEE}} - \theta^*)\cdot\int_1^{s_n/s_0}\frac{\sqrt{s_0}}{\widehat{\sigma}}f(x)dx \xrightarrow{d} \mathcal{N}(0,1).$$

Lastly, we recall that

$$\frac{\widehat{\theta}_{\text{ALEE}} - \theta^*}{\widehat{\sigma}\sqrt{\sum_{t\le n} w_t^2}}\cdot\left(\sum_{t=1}^n w_t x_t\right) = \frac{1}{\widehat{\sigma}\sqrt{\sum_{t\le n} w_t^2}}\sum_{t=1}^n w_t\epsilon_t.$$

Therefore, equation (19) follows from martingale central limit theorem and Slutsky's theorem.

Remark A.1. *Equation* (18) *sheds light on the asymptotic variance of the ALEE estimator, thereby aiding in the selection of a suitable function f to improve the efficiency of ALEE estimator. On the other hand, equation* (19) *offers a practical approach to obtaining an asymptotically precise confidence interval.*

Remark A.2. *Condition* (17) *is a general requirement that governs equation* (18)*, and is not specific to bandit problems. However, the difficulty in verifying* (17) *can vary depending on the problem at hand.*

A.2 Proof of Remark 3.4

Corollary 3.3 follows directly from Theorem 1 in [1]. In this section, we provide a proof of Remark 3.4. By considering $\lambda_0 = 1$ in Corollary 3.3, we have with probability at least $1-\delta$

$$\left|\sum_{t=1}^n w_t x_t\right|\cdot|\widehat{\theta}_{\text{ALEE}} - \theta^*| \le \sigma_g\sqrt{(1 + \sum_{t=1}^n w_t^2)\cdot\log\left(\frac{1 + \sum_{t=1}^n w_t^2}{\delta^2}\right)}. \tag{37}$$

By the construction of the weights in Corollary 3.2, we have

$$\sum_{t=1}^n w_t^2 = \sum_{t=1}^n f^2(\frac{s_t}{s_0})\frac{x_t^2}{s_0} \le \int_1^\infty f^2(x)dx = 1. \tag{38}$$

Therefore, to complete the proof, it suffices to characterize a lower bound for $\sum_{1\le t\le n} w_t x_t$. By definition, we have

$$
\begin{aligned}
\sum_{t=1}^{n} w_t x_t &= \sum_{t=1}^{n} f(s_t/s_0)\frac{x_t^2}{\sqrt{s_0}} \\
&\overset{(i)}{=} \sum_{t=1}^{n} \frac{x_t^2}{(s_t\log(e^2 s_t/s_0))^{1/2}\log\log(e^2 s_t/s_0)} \\
&\ge \frac{1}{(2+\log(s_n/s_0))^{1/2}\log(2+\log(s_n/s_0))}\sum_{t=1}^{n}\frac{x_t^2}{\sqrt{s_t}} \\
&\overset{(ii)}{\ge} \frac{1}{(2+\log(s_n/s_0))^{1/2}\log(2+\log(s_n/s_0))}\cdot 2(\sqrt{s_n}-\sqrt{s_0})\sqrt{\frac{s_0}{1+s_0}} \\
&\overset{(iii)}{\ge} \frac{1}{(2+\log(s_n/s_0))^{1/2}\log(2+\log(s_n/s_0))}\cdot \sqrt{2}(\sqrt{s_n}-\sqrt{s_0}).
\end{aligned}
\tag{39}
$$

In equation (i), we plug in the expression of function f and hence $\sqrt{s_0}$ cancels out. Since x_t is either 0 or 1, inequality (ii) follows from the integration of the function $h(x) = 1/\sqrt{x}$. Inequality (iii) follows from $s_0 > 1$. Putting things together, we have

$$
\begin{aligned}
|\widehat{\theta}_{\text{ALEE}} - \theta^*| &\le \sigma_g \frac{\sqrt{2\log(2/\delta^2)}}{\sum_{1\le t\le n} w_t x_t} \\
&\le \sigma_g\sqrt{\log(2/\delta^2)}\,\frac{\sqrt{2+\log(s_n/s_0)}\,\log\{2+\log(s_n/s_0)\}}{\sqrt{s_n}-\sqrt{s_0}}.
\end{aligned}
\tag{40}
$$

This completes our proof of Remark 3.4.

A.3 Proof of Corollary 3.5

Note that it suffices to verify the following condition (41)

$$
\max_{1\le t\le n} f^2\Big(\frac{s_t}{s_0}\Big)\frac{y_{t-1}^2}{s_0} + \max_{1\le t\le n}\Big(1-\frac{f(s_t/s_0)}{f(s_{t-1}/s_0)}\Big) + \int_{s_n/s_0}^{\infty} f^2(x)dx = o_p(1) \tag{41}
$$

for $\theta^* \in [-1,1]$ in order to complete the proof of Corollary 3.5. The other part of the proof can be adapted from the proof of Theorem 3.1. To simplify notations, we let

$$
T_1 \triangleq \max_{1\le t\le n} f^2\Big(\frac{s_t}{s_0}\Big)\frac{y_{t-1}^2}{s_0},\quad T_2 \triangleq \max_{1\le t\le n}\Big(1-\frac{f(s_t/s_0)}{f(s_{t-1}/s_0)}\Big),\quad \text{and}\quad T_3 \triangleq \int_{s_n/s_0}^{\infty} f^2(x)dx.
$$

Therefore, proving equation (41) is equivalent to showing that T_1, T_2, and T_3 converge to zero in probability. We will now demonstrate the convergence of each of these three terms to zero in probability.

T_1 with θ^* = 1: To prove $T_1 = o_p(1)$, we make use of a result in [19, Equation 3.23], which is

$$
\mathbb{P}\left(\liminf_{n\to\infty} n^{-2}(\log\log n)\sum_{t=1}^{n} y_{t-1}^2 = \sigma^2/4\right) = 1. \tag{42}
$$

Observe that

$$
\begin{aligned}
T_1 = \max_{1\le t\le n} f^2(\frac{s_t}{s_0})\frac{y_{t-1}^2}{s_0} &= \max_{1\le t\le n}\frac{y_{t-1}^2}{s_t\log(e^2 s_t/s_0)\{\log\log(e^2 s_t/s_0)\}^{1+\beta}} \\
&\le \max_{1\le t\le n}\frac{y_{t-1}^2}{s_t\log(e^2)\{\log\log(e^2)\}^{1+\beta}} \\
&= \frac{1}{2(\log 2)^{1+\beta}}\max_{1\le t\le n}\frac{y_{t-1}^2}{s_t} \\
&\le \frac{1}{2(\log 2)^{1+\beta}}\max\{\max_{1\le t\le \lfloor n^{2/3}\rfloor}\frac{y_{t-1}^2}{s_0}, \max_{\lfloor n^{2/3}\rfloor+1\le t\le n}\frac{y_{t-1}^2}{s_{\lfloor n^{2/3}\rfloor}-s_0}\}
\end{aligned}
\tag{43}
$$

In equation (43), we split the sequence into two parts and set different lower bounds for s_t. The major benefit of this step is to help us derive a better choice of s_0. Now we bound $\max_{1\le t\le\lfloor n^{2/3}\rfloor} y_{t-1}^2$ and

$\max_{\lfloor n^{2/3}\rfloor+1\le t\le n} y_{t-1}^2$. Note that

$$\begin{aligned}
&\mathbb{P}\left(\max_{1\le t\le \lfloor n^{2/3}\rfloor} y_{t-1}^2 \ge \epsilon\right)\\
=&\mathbb{P}\left(\max\{\max_{1\le t\le \lfloor n^{2/3}\rfloor} y_{t-1}, \max_{1\le t\le \lfloor n^{2/3}\rfloor} -y_{t-1}\} \ge \sqrt{\epsilon}\right)\\
\le&\frac{\mathbb{E}\left[\max\{\max_{1\le t\le \lfloor n^{2/3}\rfloor} y_{t-1}, \max_{1\le t\le \lfloor n^{2/3}\rfloor} -y_{t-1}\}\right]}{\sqrt{\epsilon}}\\
\overset{(i)}{\le}&\sqrt{\frac{2n^{2/3}\sigma_g^2\log(2n^{2/3})}{\epsilon}},
\end{aligned} \tag{44}$$

where inequality is derived from [33, Exercise 2.12] and the fact that y_i is sub-Gaussian with sub-Gaussian parameter $\sigma_g^2 n^{2/3}$ for $i \le \lfloor n^{2/3}\rfloor$. Therefore, we conclude that

$$\max_{1\le t\le \lfloor n^{2/3}\rfloor} y_{t-1}^2 = O_p(n^{2/3}\log n).$$

Consequently, we have

$$\max_{1\le t\le \lfloor n^{2/3}\rfloor} \frac{y_{t-1}^2}{s_0} = \frac{n^{2/3}\log n}{n/\log\log n}\cdot O_p(1) = o_p(1). \tag{45}$$

By applying the same trick to $\max_{\lfloor n^{2/3}\rfloor+1\le t\le n} y_{t-1}^2$, we can derive

$$\max_{\lfloor n^{2/3}\rfloor+1\le t\le n} y_{t-1}^2 = O_p(n\log n).$$

Hence we have

$$\max_{\lfloor n^{2/3}\rfloor+1\le t\le n} \frac{y_{t-1}^2}{s_{\lfloor n^{2/3}\rfloor}-s_0} = \frac{O_p(n\log n)}{n^{4/3}/\log\log n^{2/3}}\cdot\frac{n^{4/3}/\log\log n^{2/3}}{s_{\lfloor n^{2/3}\rfloor}-s_0} \overset{(ii)}{=} o_p(1)\cdot O_p(1) = o_p(1). \tag{46}$$

Equality (ii) makes use of equation (42). Combining equation (44) with equations (45) and (46), we conclude that $T_1 = o_p(1)$.

T_1 with $\theta^* = -1$: When $\theta^* = -1$, the proof is essentially the same as the case when $\theta^* = -1$. The only difference lies in the order of $\sum_{1\le i\le n} y_{i-1}^2$. However, by pairing ϵ_{2t-1} with ϵ_{2t} for $t\in\mathbb{N}^+$, we can arrive at the same result. Specifically, for $t\in\mathbb{N}^+$, we let $\epsilon_t' = \epsilon_{2t}-\epsilon_{2t-1}$ and define

$$y_t' = \sum_{k=1}^{t}\epsilon_k'$$

where $y_0' \triangleq 0$ and $\{\epsilon_t'\}_{t\ge1}$ are random variables with mean zero, variance $2\sigma^2$ and sub-Gaussian parameter $2\sigma_g^2$. Therefore, applying equation (44) yields

$$\liminf_{n\to\infty} n^{-2}(\log\log n)\sum_{t=1}^{n}(y_{t-1}')^2 = \sigma^2. \tag{47}$$

Setting $n_0 = \lfloor(\lfloor n^{2/3}\rfloor-1)/2\rfloor$, we have

$$s_{\lfloor n^{2/3}\rfloor} - s_0 = \sum_{t=1}^{\lfloor n^{2/3}\rfloor-1} y_t^2 \ge \sum_{t=1}^{n_0}(y_t')^2 = \sum_{t=1}^{n_0+1}(y_{t-1}')^2. \tag{48}$$

According to equation (47) and equation (48), we have

$$\begin{aligned}
\max_{\lfloor n^{2/3}\rfloor+1\le t\le n}\frac{y_{t-1}^2}{s_{\lfloor n^{2/3}\rfloor}-s_0} &\le \frac{\max_{\lfloor n^{2/3}\rfloor+1\le t\le n} y_{t-1}^2}{\sum_{1\le t\le n_0+1}(y_{t-1}')^2}\\
&= \frac{\max_{\lfloor n^{2/3}\rfloor+1\le t\le n} y_{t-1}^2}{(n_0+1)^2/\log\log(n_0+1)}\cdot\frac{(n_0+1)^2/\log\log(n_0+1)}{\sum_{1\le t\le n_0+1}(y_{t-1}')^2}\\
&= o_p(1)\cdot O_p(1) = o_p(1),
\end{aligned} \tag{49}$$

which completes the proof of $T_1 = o_p(1)$ for the case when $\theta^* = -1$.

T_1 with $\theta^* \in (-1,1)$: Given $\theta^* \in (-1,1)$, we observe that y_t is a sub-Gaussian random variable with sub-Gaussian parameter $\frac{\sigma_g^2}{1-(\theta^*)^2}$ for any $t \in \mathbb{N}^+$. Therefore, following equation (43), we have

$$T_1 \leq \frac{1}{2(\log 2)^{1+\beta}} \max_{1\leq t\leq n} \frac{y_{t-1}^2}{s_0} \tag{50}$$

where in the above inequality we use s_0 as a lower bound for s_t. By applying [33, Exercise 2.12], we have

$$\max_{1\leq t\leq n} y_{t-1}^2 = O_p(\log n), \tag{51}$$

leading to the conclusion that $T_1 = o_p(1)$.

T_2 with $\theta^* \in [-1,1]$: Similar to equation (33), we have

$$\begin{aligned} T_2 = \max_{1\leq t\leq n} \left(1 - \frac{f(s_t/s_0)}{f(s_{t-1}/s_0)}\right) &= \max_{1\leq t\leq n} \frac{f(s_{t-1}/s_0) - f(s_t/s_0)}{f(s_{t-1}/s_0)} \\ &\leq \max_{1\leq t\leq n} \frac{-f'(s_{t-1}/s_0)}{f(s_{t-1}/s_0)} \frac{y_{t-1}^2}{s_0}. \end{aligned}$$

Define $g(x) = -f'(x)/f(x)$ and we can compute that

$$\int g(x)dx = -\int \frac{f'(x)}{f(x)}dx = -\int \frac{1}{f}df = -\log f + C,$$

where C is some constant. Doing some calculation yields

$$\begin{aligned} g(x) &= \frac{d}{dx} - \log f = \frac{d}{dx}\left\{\frac{1}{2}(\log(x) + \log\log(e^2x)) + (1+\beta)\log\log\log(e^2x))\right\} \\ &= \frac{1}{2x}\left\{1 + \frac{1}{\log(e^2x)} + \frac{1+\beta}{\log(e^2x)} \cdot \frac{1}{\log\log(e^2x)}\right\}. \end{aligned}$$

Therefore, we have

$$\begin{aligned} T_2 &\leq \max_{1\leq t\leq n} \frac{-f'(s_{t-1}/s_0)}{f(s_{t-1}/s_0)} \frac{y_{t-1}^2}{s_0} = \max_{1\leq t\leq n} g(s_{t-1}/s_0) \frac{y_{t-1}^2}{s_0} \\ &\leq \frac{1}{2}\left(\frac{3}{2} + \frac{1+\beta}{2\log 2}\right) \max_{1\leq t\leq n} \frac{y_{t-1}^2}{s_{t-1}}. \end{aligned}$$

We note that demonstrating $\max_{1\leq t\leq n} y_{t-1}^2/s_{t-1} = o_p(1)$ follows the same approach as the proof of $\max_{1\leq t\leq n} y_{t-1}^2/s_t = o_p(1)$. Hence, we omit it. To conclude, we show that $T_2 = o_p(1)$ for $\theta^* \in [-1,1]$.

T_3 with $\theta^* \in [-1,1]$: To prove $T_3 = o_p(1)$, it suffices to verify that

$$\frac{s_0}{\sum_{1\leq t\leq n} y_t^2} = o_p(1). \tag{52}$$

For convenience, in equation (52) we use y_t instead of y_{t-1}. Note that when $\theta^* = 1$ or $\theta^* = -1$, we have provided almost sure lower bounds for $\sum_{1\leq t\leq n} y_t^2$ in the proof of $T_1 = o_p(1)$. Therefore, equation (52) follows from these lower bounds. To prove equation (52) when $\theta^* \in (-1,1)$, we begin by rewriting $\sum_{1\leq t\leq n} y_t^2$ in quadratic form. Without confusion and loss of generality, we replace θ^* by θ, consider $\mathrm{var}(\epsilon_t) = 1$, and set $\boldsymbol{\varepsilon}_n = (\epsilon_1, \epsilon_2, \ldots, \epsilon_n)^\top$. For $t \in [n]$, we have

$$y_t = \sum_{k=1}^{t} \theta^{t-k}\epsilon_k = \boldsymbol{a}_t^\top \boldsymbol{\varepsilon}_n,$$

where $\boldsymbol{a}_t \in \mathbb{R}^n$ and $a_{t,j} = \theta^{t-j}$ for $j \leq t$ and $a_{t,j} = 0$ for $j > t$. Therefore, $\sum_{1\leq t\leq n} y_t^2$ can be written as

$$\sum_{1\leq t\leq n} y_t^2 = \sum_{1\leq t\leq n} \boldsymbol{\varepsilon}_n^\top \boldsymbol{a}_t \boldsymbol{a}_t^\top \boldsymbol{\varepsilon}_n = \boldsymbol{\varepsilon}_n^\top \mathbf{A} \boldsymbol{\varepsilon}_n, \tag{53}$$

where $\mathbf{A} = \sum_{1\leq t\leq n} \boldsymbol{a}_t \boldsymbol{a}_t^\top$. Applying Hanson-Wright inequality (e.g. see [32]), we have

$$\mathbb{P}\left(|\boldsymbol{\varepsilon}_n^\top \mathbf{A}\boldsymbol{\varepsilon}_n - \mathbb{E}\boldsymbol{\varepsilon}_n^\top \mathbf{A}\boldsymbol{\varepsilon}_n| > t\right) \leq 2\exp\left[-c\min\left(\frac{t^2}{K^4\|\mathbf{A}\|_{\mathrm{F}}^2}, \frac{t}{K^2\|\mathbf{A}\|_{\mathrm{F}}}\right)\right], \tag{54}$$

where c and K are some universal constants. Observe that

$$\begin{aligned}\mathbb{E}\boldsymbol{\varepsilon}_n^\top \mathbf{A}\boldsymbol{\varepsilon}_n &= \mathrm{trace}(A) = \mathrm{trace}(\sum_{1\le t\le n} \boldsymbol{a}_t\boldsymbol{a}_t^\top) = \mathrm{trace}(\sum_{1\le t\le n} \boldsymbol{a}_t^\top\boldsymbol{a}_t)\\ &= \sum_{1\le t\le n}(1+\theta^2+\cdots+\theta^{2(t-1)})\\ &= \sum_{1\le t\le n}\frac{1-\theta^{2t}}{1-\theta^2}\\ &= \frac{n}{1-\theta^2}-\frac{\theta^2(1-\theta^{2n})}{(1-\theta^2)^2}.\end{aligned}$$

Furthermore, we have

$$\begin{aligned}\|\mathbf{A}\|_{\mathrm{F}}^2 &= \mathrm{trace}(\mathbf{A}^\top\mathbf{A}) = \mathrm{trace}(\sum_{1\le i\le n}\boldsymbol{a}_i\boldsymbol{a}_i^\top\cdot\sum_{1\le j\le n}\boldsymbol{a}_j\boldsymbol{a}_j^\top)\\ &= \sum_{1\le i\le n}\sum_{1\le j\le n}(\boldsymbol{a}_i^\top\boldsymbol{a}_j)^2\\ &= \sum_{1\le i\le n}\|\boldsymbol{a}_i\|_2^4+2\sum_{1\le i<j\le n}\|\boldsymbol{a}_i\|_2^4\cdot\theta^{2(j-i)}.\end{aligned}\tag{55}$$

Subsequently, we have

$$\sum_{1\le i\le n}\|\boldsymbol{a}_i\|_2^4\le\|\mathbf{A}\|_{\mathrm{F}}^2\le(1+\frac{2}{1-\theta^2})\sum_{1\le i\le n}\|\boldsymbol{a}_i\|_2^4,\tag{56}$$

where

$$\sum_{1\le i\le n}\|\boldsymbol{a}_i\|_2^4=\frac{n}{(1-\theta^2)^2}-\frac{2\theta^2(1-\theta^{2n})}{(1-\theta^2)^3}+\frac{\theta^4(1-\theta^{4n})}{(1-\theta^2)^2(1-\theta^4)}.$$

Assuming $\delta\le 2e^{-c}$ and $t=\frac{1}{c}K^2\|\mathbf{A}\|_{\mathrm{F}}\log(\frac{2}{\delta})$, we have with probability at least $1-\delta$,

$$\boldsymbol{\varepsilon}_n^\top\mathbf{A}\boldsymbol{\varepsilon}_n\ge\mathbb{E}\boldsymbol{\varepsilon}_n^\top\mathbf{A}\boldsymbol{\varepsilon}_n-\frac{1}{c}K^2\|\mathbf{A}\|_{\mathrm{F}}\log(\frac{2}{\delta}).\tag{57}$$

We note that the term on the right hand side of equation (57) has order n. For any $\epsilon>0$, consider the following probability

$$\begin{aligned}\limsup_{n\to\infty}\mathbb{P}\left(\frac{s_0}{\boldsymbol{\varepsilon}_n^\top\mathbf{A}\boldsymbol{\varepsilon}_n}>\epsilon\right)&\le\limsup_{n\to\infty}\mathbb{P}\left(\frac{s_0}{\boldsymbol{\varepsilon}_n^\top\mathbf{A}\boldsymbol{\varepsilon}_n}>\epsilon,\boldsymbol{\varepsilon}_n^\top\mathbf{A}\boldsymbol{\varepsilon}_n\ge\mathbb{E}\boldsymbol{\varepsilon}_n^\top\mathbf{A}\boldsymbol{\varepsilon}_n-\frac{1}{c}K^2\|\mathbf{A}\|_{\mathrm{F}}\log(\frac{2}{\delta})\right)\\ &\quad+\limsup_{n\to\infty}\mathbb{P}\left(\boldsymbol{\varepsilon}_n^\top\mathbf{A}\boldsymbol{\varepsilon}_n<\mathbb{E}\boldsymbol{\varepsilon}_n^\top\mathbf{A}\boldsymbol{\varepsilon}_n-\frac{1}{c}K^2\|\mathbf{A}\|_{\mathrm{F}}\log(\frac{2}{\delta})\right)\\ &\le\limsup_{n\to\infty}\mathbb{P}\left(\frac{s_0}{\mathbb{E}\boldsymbol{\varepsilon}_n^\top\mathbf{A}\boldsymbol{\varepsilon}_n-\frac{1}{c}K^2\|\mathbf{A}\|_{\mathrm{F}}\log(\frac{2}{\delta})}>\epsilon\right)+\delta.\end{aligned}\tag{58}$$

By fixing δ and comparing the order of s_0 with the order of $\boldsymbol{\varepsilon}_n^\top\mathbf{A}\boldsymbol{\varepsilon}_n-\frac{1}{c}K^2\|\mathbf{A}\|_{\mathrm{F}}\log(\frac{2}{\delta})$, we have

$$\limsup_{n\to\infty}\mathbb{P}\left(\frac{s_0}{\mathbb{E}\boldsymbol{\varepsilon}_n^\top\mathbf{A}\boldsymbol{\varepsilon}_n-\frac{1}{c}K^2\|\mathbf{A}\|_{\mathrm{F}}\log(\frac{2}{\delta})}>\epsilon\right)=0.$$

Since δ can be arbitrarily small, we conclude that

$$\frac{s_0}{\boldsymbol{\varepsilon}_n^\top\mathbf{A}\boldsymbol{\varepsilon}_n}=o_p(1),\tag{59}$$

which completes the proof of $T_3=o_p(1)$.

A.4 Proof of Theorem 3.6

Note that for any $t\ge1$, we have

$$\|\mathbf{V}_t\|_{\mathrm{op}}\le1\quad\text{and}\quad\mathbf{V}_t=\mathbf{V}_{t-1}-\mathbf{V}_{t-1}\boldsymbol{z}_t\boldsymbol{z}_t^\top\mathbf{V}_{t-1}/(1+\boldsymbol{z}_t^\top\mathbf{V}_{t-1}\boldsymbol{z}_t).\tag{60}$$

The second part of equation (60) follows from the Sherman–Morrison formula. Let $\boldsymbol{u}_t=\mathbf{V}_t\boldsymbol{z}_t$ and we adopt the notation $\mathbf{V}_0=\mathbf{I}_d$. By multiplying $\boldsymbol{z}_t$ on the right hand side of $\mathbf{V}_t$, we have

$$\begin{aligned}\mathbf{V}_t\boldsymbol{z}_t&=\mathbf{V}_{t-1}\boldsymbol{z}_t-\mathbf{V}_{t-1}\boldsymbol{z}_t\boldsymbol{z}_t^\top\mathbf{V}_{t-1}\boldsymbol{z}_t/(1+\boldsymbol{z}_t^\top\mathbf{V}_{t-1}\boldsymbol{z}_t)\\ &=\mathbf{V}_{t-1}\boldsymbol{z}_t\left(1-\frac{\boldsymbol{z}_t^\top\mathbf{V}_{t-1}\boldsymbol{z}_t}{1+\boldsymbol{z}_t^\top\mathbf{V}_{t-1}\boldsymbol{z}_t}\right)=\frac{\mathbf{V}_{t-1}\boldsymbol{z}_t}{1+\boldsymbol{z}_t^\top\mathbf{V}_{t-1}\boldsymbol{z}_t}.\end{aligned}\tag{61}$$

Therefore, following the definition of $\boldsymbol{u}_t$, we have $(1+\boldsymbol{z}_t^\top \mathbf{V}_{t-1}\boldsymbol{z}_t)\boldsymbol{u}_t = \mathbf{V}_{t-1}\boldsymbol{z}_t$. Consequently,

$$\sum_{t=1}^{n}(1+\boldsymbol{z}_t^\top \mathbf{V}_{t-1}\boldsymbol{z}_t)\boldsymbol{u}_t\boldsymbol{u}_t^\top = \sum_{t=1}^{n}\mathbf{V}_{t-1}(\mathbf{V}_t^{-1}-\mathbf{V}_{t-1}^{-1})\mathbf{V}_t = \mathbf{I}_d - \mathbf{V}_n. \tag{62}$$

By recognizing $\boldsymbol{w}_t = \sqrt{1+\boldsymbol{z}_t^\top \mathbf{V}_{t-1}\boldsymbol{z}_t}\cdot \boldsymbol{u}_t$, we come to

$$\sum_{t=1}^{n}\boldsymbol{w}_t\boldsymbol{w}_t^\top = \sum_{t=1}^{n}\mathbf{V}_{t-1}(\mathbf{V}_t^{-1}-\mathbf{V}_{t-1}^{-1})\mathbf{V}_t = \mathbf{I}_d - \mathbf{V}_n.$$

What remains now is to verify conditions in (6). Notably, assumption $\|\mathbf{V}_n\|_{\text{op}} = o_p(1)$ implies

$$\sum_{t=1}^{n}\boldsymbol{w}_t\boldsymbol{w}_t^\top \xrightarrow{p} \mathbf{I}_d. \tag{63}$$

Since $\|\boldsymbol{\Sigma}_0^{-1}\|_{\text{op}} = o_p(1)$, $\|\mathbf{V}_t\|_{\text{op}} \le 1$ and $\|\boldsymbol{x}_t\|_2 \le 1$, we can show

$$\max_{1\le t\le n}\boldsymbol{z}_t^\top \mathbf{V}_t\boldsymbol{z}_t = \max_{1\le t\le n}\boldsymbol{x}_t^\top \boldsymbol{\Sigma}_{t-1}^{-\frac{1}{2}}\mathbf{V}_t\boldsymbol{\Sigma}_{t-1}^{-\frac{1}{2}}\boldsymbol{x}_t = o_p(1). \tag{64}$$

Besides, equation (61) together with equation (64) implies

$$\max_{1\le t\le n}\boldsymbol{z}_t^\top \mathbf{V}_{t-1}\boldsymbol{z}_t = \max_{1\le t\le n}\frac{\boldsymbol{z}_t^\top \mathbf{V}_t\boldsymbol{z}_t}{1-\boldsymbol{z}_t^\top \mathbf{V}_t\boldsymbol{z}_t} = o_p(1). \tag{65}$$

Thus, it follows that

$$\begin{aligned}\max_{1\le t\le n}\|\boldsymbol{w}_t\|_2 &= \max_{1\le t\le n}\left\|\sqrt{1+\boldsymbol{z}_t^\top \mathbf{V}_{t-1}\boldsymbol{z}_t}\cdot \mathbf{V}_t\boldsymbol{z}_t\right\|_2\\ &\le \max_{1\le t\le n}\left(\sqrt{1+\boldsymbol{z}_t^\top \mathbf{V}_{t-1}\boldsymbol{z}_t}\cdot \|\mathbf{V}_t^{\frac{1}{2}}\|_{\text{op}}\cdot \|\mathbf{V}_t^{\frac{1}{2}}\boldsymbol{z}_t\|_2\right)\\ &\le \sqrt{\left(1+\max_{1\le t\le n}\boldsymbol{z}_t^\top \mathbf{V}_{t-1}\boldsymbol{z}_t\right)\cdot \max_{1\le t\le n}\boldsymbol{z}_t^\top \mathbf{V}_t\boldsymbol{z}_t} = o_p(1).\end{aligned} \tag{66}$$

Combining equations (66) and (63) yields (6). Hence we complete the proof by applying Proposition 2.1.

Remark A.3. *The detailed proof of Lemma 3.7 can be found in the proof of Theorem 3.6.*

B Generalized Theorem 3.6

In Theorem 3.6, we impose the following condition (67) so that the ALEE estimator with weights specified in equation (28) achieves asymptotic normality:

$$\|\mathbf{V}_n\|_{\text{op}} = o_p(1). \tag{67}$$

However, it is typically difficult to directly verify the above condition in practice. To tackle this problem, in this section, we provide a modified version of ALEE estimator which achieves asymptotic normality without requiring condition (67). In this section, we use the same notations $\boldsymbol{\Sigma}_t$, $\boldsymbol{z}_t$, $\mathbf{V}_t$, and $\boldsymbol{w}_t$ as defined in equations (26), (27) and (28), respectively. Furthermore, we let $\lambda_1 \ge \ldots \ge \lambda_n$ be the eigenvalues of the matrix $\mathbf{V}_n^{-1}$ and $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n$ be the corresponding eigenvectors.

At a high level, we construct additional m_n vectors $\{\boldsymbol{z}_t\}_{n+1\le t\le n+m_n}$ so that the minimum eigenvalue of the resulting matrix $\mathbf{V}_{n+m_n}^{-1}$ is greater than a pre-specified constant κ_n, which satisfies $\lim_{n\to\infty}\kappa_n = \infty$. It is easy to see that by construction (see Algorithm 1), the matrix $\mathbf{V}_{n+m_n}$ satisfies

$$\|\mathbf{V}_{n+m_n}\|_{\text{op}} \le \frac{1}{\kappa_n} \xrightarrow{p} 0 \qquad \text{where} \quad m_n = \sum_{k=1}^{d} n_k. \tag{69}$$

Remark B.1. *Parameter κ_n is set to ensure condition (69) holds. In practice, we set $\kappa_n = d\log(n)$.*

Remark B.2. *It's worth mentioning that the number of extra $\{\boldsymbol{z}_t\}_{t>n}$ is a random variable. Therefore, in order to prove a similar asymptotic normality theorem to Theorem 3.6, we have to apply martingale central limit theorem with stopping times [11, Theorem 2.1].*

Theorem B.3 (Theorem 2.1 in [11])**.** *Let $\{\xi_{n,k}\}_{k\ge1,n\ge1}$ be an array of random variables defined on a probability space $(\Omega, \mathcal{F}, P)$ and let $\{\mathcal{F}_{n,k}\}_{n\ge1,k\ge0}$ be an array of σ-fields such that $\xi_{n,k}$ is $\mathcal{F}_{n,k}$-measurable and $\mathcal{F}_{n,k-1} \subset \mathcal{F}_{n,k} \subset \mathcal{F}$ for each n and $k \geqslant 1$. For each n, let k_n be a stopping time with respect to $\{\mathcal{F}_{n,k}\}_{k\ge0}$. Suppose that*

Algorithm 1: Modified ALEE estimate

Input: $\{(\boldsymbol{x}_t, y_t)\}_{t=1}^n$ and tuning parameter κ_n
Compute $\{(\boldsymbol{z}_t, \mathbf{V}_t, \boldsymbol{w}_t)\}_{t=1}^n$, $\{(\lambda_k, \boldsymbol{a}_k)\}_{k=1}^d$, and obtain a consistent estimate $\widehat{\sigma}^2$ of σ^2
Initiate $t = n$ and set $\tau_n = 1/\|\mathbf{\Sigma}_0^{-1/2}\|_{\text{op}}$
for $k = 1, \ldots, d$ **do**
    Compute $n_k = \lceil \max\{\kappa_n - \lambda_k, 0\} \cdot \tau_n \rceil$
    if $n_k > 0$ **then**
        for $i = 1, \ldots, n_k$ **do**
            Set $t = t + 1$
            Simulate $\epsilon_t \sim \mathcal{N}(0, \widehat{\sigma}^2)$
            Define $\boldsymbol{z}_t = \boldsymbol{a}_k / \tau_n$
            Compute

$$\mathbf{V}_t = \mathbf{V}_{t-1} - \frac{\mathbf{V}_{t-1}\boldsymbol{z}_t\boldsymbol{z}_t^\top \mathbf{V}_{t-1}}{1 + \boldsymbol{z}_t^\top \mathbf{V}_{t-1}\boldsymbol{z}_t} \qquad \text{and} \qquad \boldsymbol{w}_t = \sqrt{1 + \boldsymbol{z}_t^\top \mathbf{V}_{t-1}\boldsymbol{z}_t} \cdot \mathbf{V}_t \boldsymbol{z}_t$$

        end for
    end if
end for
Obtain $\widehat{\boldsymbol{\theta}}_{\text{ALEE}}$ from equation

$$\sum_{i=1}^n \boldsymbol{w}_i (y_i - \boldsymbol{x}_i^\top \widehat{\boldsymbol{\theta}}_{\text{ALEE}}) + \sum_{i=n+1}^t \boldsymbol{w}_i \epsilon_i = 0 \tag{68}$$

Output: $\widehat{\boldsymbol{\theta}}_{\text{ALEE}}$

$$\sum_{k=1}^{k_n} \mathbb{E}\left[\xi_{n,k} \mid \mathcal{F}_{n,k-1}\right] \xrightarrow{p} 0, \tag{70a}$$

$$\sum_{k=1}^{k_n} \text{Var}\left[\xi_{n,k} \mid \mathcal{F}_{n,k-1}\right] \xrightarrow{p} 1, \tag{70b}$$

$$\sum_{k=1}^{k_n} \mathbb{E}\left[|\xi_{n,k}|^{2+\delta} \mid \mathcal{F}_{n,k-1}\right] \xrightarrow{p} 0 \quad \textit{for some } \delta > 0, \tag{70c}$$

then $\sum_{k=1}^{k_n} \xi_{n,k} \xrightarrow{d} \mathcal{N}(0,1)$.

With this setup, we are now ready to prove the asymptotic normality of $\widehat{\boldsymbol{\theta}}_{\text{ALEE}}$ from (68).

Theorem B.4 (Generalized Theorem 3.6)**.** *Suppose condition* (3) *holds. Then, for any tuning parameters* $\mathbf{\Sigma}_0$ *and* κ_n *that satisfy* $\|\mathbf{\Sigma}_0^{-1}\|_{\text{op}} = o_p(1)$ *and* $\lim_{n\to\infty} \kappa_n = \infty$*, the ALEE estimator* $\widehat{\boldsymbol{\theta}}_{\text{ALEE}}$ *obtained from equation* (68) *satisfies*

$$\left(\sum_{t=1}^n \boldsymbol{w}_t \boldsymbol{x}_t\right) \cdot \frac{\widehat{\boldsymbol{\theta}}_{\text{ALEE}} - \boldsymbol{\theta}^*}{\widehat{\sigma}} \xrightarrow{d} \mathcal{N}(\mathbf{0}, \mathbf{I}_d),$$

where $\widehat{\sigma}$ *is a consistent estimator of* σ*.*

Remark B.5. *We would like to reiterate that the asymptotic variance of of the modified ALEE estimator obtained from* (68) *is the same as the one mentioned in Theorem 3.6. Additionally, this modified version does not require the condition* $\|\mathbf{V}_n\|_{\text{op}} = o_p(1)$ *hold and hence is more applicable in practice with theoretical guarantee.*

Proof. Rewriting equation (68), we have

$$\sum_{t=1}^n \boldsymbol{w}_t \boldsymbol{x}_t^\top (\widehat{\boldsymbol{\theta}}_{\text{ALEE}} - \boldsymbol{\theta}^*) = \sum_{t=1}^{n+m_n} \boldsymbol{w}_t \epsilon_t. \tag{71}$$

Therefore, by Cramér–Wold theorem, it suffices to show that for any unit vector $\boldsymbol{v}$,

$$\sum_{t=1}^{n+m_n} \boldsymbol{v}^\top \boldsymbol{w}_t \epsilon_t \xrightarrow{d} \mathcal{N}(0, \sigma^2). \tag{72}$$

The proof now follows by verifying the conditions (70a)-(70c) of Theorem B.3 with $\xi_{n,k} = \boldsymbol{v}^\top \boldsymbol{w}_k \epsilon_k$. We begin by verifying conditions (70a)-(70c). By Lemma 3.7, we have

$$\sum_{t=1}^{n+m_n} \boldsymbol{w}_t \boldsymbol{w}_t^\top = \mathbf{I}_d - \mathbf{V}_{n+m_n}. \tag{73}$$

Note that

$$\sum_{t=1}^{n+m_n} \mathrm{Var}[\boldsymbol{w}_t \epsilon_t \mid \mathcal{F}_{t-1}] = \sum_{t=1}^{n+m_n} \sigma^2 \boldsymbol{w}_t \boldsymbol{w}_t^\top + \sigma^2 \left(\frac{\widehat{\sigma}^2}{\sigma^2} - 1 \right) \sum_{t=n+1}^{n+m_n} \boldsymbol{w}_t \boldsymbol{w}_t^\top. \tag{74}$$

By equation (73) and the fact that $\widehat{\sigma}^2$ is consistent, we have

$$\sum_{t=n+1}^{n+m_n} \boldsymbol{w}_t \boldsymbol{w}_t^\top \preceq \mathbf{I}_d \qquad \text{and} \qquad \frac{\widehat{\sigma}^2}{\sigma^2} - 1 \xrightarrow{p} 0. \tag{75}$$

Combining equations (69), (73), (74) and (75), we conclude

$$\sum_{t=1}^{n+m_n} \mathrm{Var}[\boldsymbol{w}_t \epsilon_t \mid \mathcal{F}_{t-1}] \xrightarrow{p} \sigma^2 \mathbf{I}_d. \tag{76}$$

On the other hand, we have

$$\begin{aligned}
\max_{1 \le t \le n+m_n} \|\boldsymbol{w}_t\|_2 &\overset{(i)}{\le} \max_{1 \le t \le n+m_n} \left(\sqrt{1 + \boldsymbol{z}_t^\top \mathbf{V}_{t-1} \boldsymbol{z}_t} \cdot \|\mathbf{V}_t\|_{\text{op}} \cdot \|\boldsymbol{z}_t\|_2 \right) \\
&\overset{(ii)}{\le} \max_{1 \le t \le n+m_n} \sqrt{2} \|\boldsymbol{z}_t\|_2 \\
&\overset{(iii)}{\le} \sqrt{2} \|\boldsymbol{\Sigma}_0^{-1/2}\|_{\text{op}}.
\end{aligned}$$

Inequality (i) follows from the definition of $\boldsymbol{w}_t$. In inequality (ii), we use the assumption that $\boldsymbol{\Sigma}_0 \succeq \mathbf{I}_d$ and the fact that $\|\boldsymbol{z}_t\|_2 \le 1$ and $\|\mathbf{V}_t\|_{\text{op}} \le 1$. The last inequality (iii) follows from the definition of $\boldsymbol{z}_t$ and the condition that $\|\boldsymbol{\Sigma}_0^{-1}\|_{\text{op}} = o_p(1)$. Hence, we can see that

$$\max_{1 \le t \le n+m_n} \|\boldsymbol{w}_t\|_2 \xrightarrow{p} 0. \tag{77}$$

Therefore, we have

$$\max_{1 \le t \le n+m_n} |\boldsymbol{v}^\top \boldsymbol{w}_t| \xrightarrow{p} 0 \qquad \text{and} \qquad \sum_{t=1}^{n+m_n} \mathrm{Var}[\boldsymbol{v}^\top \boldsymbol{w}_t \epsilon_t \mid \mathcal{F}_{t-1}] \xrightarrow{p} \sigma^2. \tag{78}$$

Note that condition (70a) holds because $\{\boldsymbol{v}^\top \boldsymbol{w}_k \epsilon_k\}_{k \ge 1}$ is a martingale difference sequence by construction. Condition (70b) follows from statement (78). It remains to verify condition (70c). Observe that

$$\begin{aligned}
&\sum_{t=1}^{n+m_n} \mathbb{E}[|\boldsymbol{v}^\top \boldsymbol{w}_t \epsilon_t|^{2+\delta} \mid \mathcal{F}_{t-1}] = \sum_{t=1}^{n+m_n} |\boldsymbol{v}^\top \boldsymbol{w}_t|^{2+\delta} \mathbb{E}[|\epsilon_t|^{2+\delta} \mid \mathcal{F}_{t-1}] \\
\le\ & \left(\max_{1 \le t \le n+m_n} |\boldsymbol{v}^\top \boldsymbol{w}_t|^\delta \right) \cdot \left(\sup_{t \ge 1} \mathbb{E}[|\epsilon_t|^{2+\delta} \mid \mathcal{F}_{t-1}] \right) \cdot \max\{\frac{1}{\sigma^2}, \frac{1}{\widehat{\sigma}^2}\} \sum_{t=1}^{n+m_n} \mathrm{Var}[\boldsymbol{v}^\top \boldsymbol{w}_t \epsilon_t \mid \mathcal{F}_{t-1}] \\
\overset{(iv)}{=}\ & o_p(1) \cdot O_p(1) \cdot O_p(1) = o_p(1).
\end{aligned}$$

Equation (iv) follows from condition (3), equation (78) and the fact that $\widehat{\sigma}^2$ is a consistent estimator. Lastly, by applying Slutsky's theorem, we prove that

$$\frac{1}{\widehat{\sigma}} \sum_{t=1}^{n} \boldsymbol{w}_t \boldsymbol{x}_t^\top (\widehat{\boldsymbol{\theta}}_{\text{ALEE}} - \theta^*) \xrightarrow{d} \mathcal{N}(\mathbf{0}, \mathbf{I}_d). \tag{79}$$

□

C Simulation

In this section, we provide additional comparisons among the ALEE method, the OLS, the W-decorrelation [8], and the concentration inequality based bounds [1]. The code can be found at https://github.com/mufangying/ALEE.

C.1 Simulation details

Throughout our experiments, we utilize $\widehat{\sigma}^2$ from equation (9) as an (consistent) estimate of of σ^2 [19].

OLS: When data are i.i.d, the least squares estimator satisfies the following condition

$$\frac{1}{\sigma^2}(\widehat{\boldsymbol{\theta}}_{\mathrm{LS}} - \boldsymbol{\theta}^*)^\top \mathbf{S}_n (\widehat{\boldsymbol{\theta}}_{\mathrm{LS}} - \boldsymbol{\theta}^*) \xrightarrow{d} \chi^2_d.$$

Therefore, we consider $1-\alpha$ confidence region to be

$$\mathbf{C}_{\mathrm{LS}} = \left\{ \boldsymbol{\theta} \in \mathbb{R}^d : \frac{1}{\widehat{\sigma}^2}(\widehat{\boldsymbol{\theta}}_{\mathrm{LS}} - \boldsymbol{\theta})^\top \mathbf{S}_n (\widehat{\boldsymbol{\theta}}_{\mathrm{LS}} - \boldsymbol{\theta}) \le \chi^2_{d,1-\alpha} \right\}. \tag{80}$$

We point out that the above confidence region is not guaranteed to be accurate when the data is collected in an adaptive manner, as will also be highlighted in our experiments.

W-decorrelation: The W-decorrelation method is borrowed from Algorithm 1 in [8]. Specifically, the estimator takes the form

$$\widehat{\boldsymbol{\theta}}_{\mathrm{W}} = \widehat{\boldsymbol{\theta}}_{\mathrm{LS}} + \sum_{t=1}^{n} \boldsymbol{w}_t (y_t - \boldsymbol{x}_t^\top \widehat{\boldsymbol{\theta}}_{\mathrm{LS}}). \tag{81}$$

Given a parameter λ, weights $\{\boldsymbol{w}_t\}_{1\le t\le n}$ are set as follows

$$\boldsymbol{w}_t = \left(\mathbf{I}_d - \sum_{i=1}^{t-1} \boldsymbol{w}_t \boldsymbol{x}_t^\top \right) \boldsymbol{x}_t / (\lambda + \|\boldsymbol{x}_t\|_2^2). \tag{82}$$

Following the recommendations from the paper [8], in order to set λ appropriately, we first run the bandit algorithm or time series with N replications and record the corresponding minimum eigenvalues $\{\lambda_{\min}(\mathbf{S}_n^{(1)}), \ldots, \lambda_{\min}(\mathbf{S}_n^{(N)})\}$. We choose λ to be the 0.1-quantile of $\{\lambda_{\min}(\mathbf{S}_n^{(1)}), \ldots, \lambda_{\min}(\mathbf{S}_n^{(N)})\}$. Finally, we obtain a $1-\alpha$ confidence region for $\boldsymbol{\theta}^*$ as

$$\mathbf{C}_{\mathrm{W}} = \left\{ \boldsymbol{\theta} \in \mathbb{R}^d : \frac{1}{\widehat{\sigma}^2}(\widehat{\boldsymbol{\theta}}_{\mathrm{W}} - \boldsymbol{\theta})^\top \mathbf{W}^\top \mathbf{W} (\widehat{\boldsymbol{\theta}}_{\mathrm{W}} - \boldsymbol{\theta}) \le \chi^2_{d,1-\alpha} \right\}, \tag{83}$$

where $\mathbf{W} = (\boldsymbol{w}_1, \ldots, \boldsymbol{w}_n)^\top$.

Concentration based on self-normalized martingales: We consider [1, Theorem 1] for a single coordinate in two-armed bandit problem and AR(1) model. For contextual bandits, we apply [1, Theorem 2]. Applying concentration bounds requires a sub-Gaussian parameter, for which we use $\widehat{\sigma}$ from equation (9) as an estimate. We point out that this estimate of the sub-Gaussian parameter is conservative, as the sub-Gaussian parameter of a sub-Gaussian random variable is always lower bounded by its variance [33, Chapter 2]. This variance estimate is accurate for Gaussian noise random variables.

- For one dimensional examples, we have that for any $\lambda > 0$, with probability at least $1-\alpha$:

$$|\widehat{\theta}_{\mathrm{LS}} - \theta^*| \le \frac{\widehat{\sigma}\sqrt{\lambda + \sum_{t=1}^n x_t^2}}{\sum_{t=1}^n x_t^2} \sqrt{\log\left(\frac{\lambda + \sum_{t=1}^n x_t^2}{\lambda \alpha^2} \right)}. \tag{84}$$

  In two-armed bandit problem, x_t is simply $x_{t,1}$ for θ_1^* or $x_{t,2}$ for θ_2^*. Here we consider $\lambda = 1$.

- For the contextual bandit examples, we apply Theorem 2 from [1], and set $S = \sqrt{d}$; we set a small value of $\lambda = 0.01$ to mimic the performance of an OLS estimators. Specifically, we utilize the following $1-\alpha$ confidence region

$$\mathbf{C}_{\mathrm{con}} = \left\{ \boldsymbol{\theta} \in \mathbb{R}^d : (\widehat{\boldsymbol{\theta}}_r - \boldsymbol{\theta})^\top (\lambda \mathbf{I}_d + \mathbf{S}_n)(\widehat{\boldsymbol{\theta}}_r - \boldsymbol{\theta}) \le \left(\widehat{\sigma} \sqrt{\log\left(\frac{\det(\lambda \mathbf{I}_d + \mathbf{S}_n)}{\lambda^d \alpha^2} \right)} + \lambda^{\frac{1}{2}} S \right)^2 \right\}, \tag{85}$$

  where $\widehat{\boldsymbol{\theta}}_r = (\mathbf{X}_n^\top \mathbf{X}_n + \lambda \mathbf{I}_d)^{-1} \mathbf{X}_n^\top \mathbf{Y}_n$ and $\mathbf{Y}_n = (y_1, \ldots, y_n)^\top$.

C.2 Tables for contextual bandits

In all the contextual bandit simulations, we consider noises that are generated from a centered Poisson distribution (i.e. $Poisson(1) - 1$). We would like to highlight that the centered Poisson random variable is not sub-Gaussian. Therefore, it is important to note that concentration inequality-based bounds [1] may not be guaranteed to work. In the simulations of this section, we set the number of samples as $n = 1000$, and the tables below show results over 1000 replications. The tables below clearly show that the average log-volume of the confidence regions are smallest for ALEE among methods which yield valid confidence regions (empirical coverage is more than the target coverage). The volume of the confidence region obtained from the OLS estimate is the smallest, but they under-cover the true parameter. The confidence regions for ALEE are obtained from Theorem B.4 with $\mathbf{\Sigma_0} = \log(n) \cdot \mathbf{I}_d$ and $\kappa_n = d\log(n)$.

Table 2: Contextual bandit: d = 10

Method	Level of confidence					
	0.8		0.85		0.9	
	Avg. Coverage	Avg. log(Volumn)	Avg. Coverage	Avg. log(Volumn)	Avg. Coverage	Avg. log(Volumn)
ALEE	0.819 ($\pm$ 0.385)	-2.761 ($\pm$ 0.263)	0.872 ($\pm$ 0.334)	-2.370 ($\pm$ 0.263)	0.920 ($\pm$ 0.271)	-1.894 ($\pm$ 0.263)
OLS	0.807 ($\pm$ 0.395)	-7.306 ($\pm$ 0.262)	0.863 ($\pm$ 0.344)	-6.915 ($\pm$ 0.262)	0.905 ($\pm$ 0.293)	-6.439 ($\pm$ 0.262)
W-Decorrelation	0.785 ($\pm$ 0.411)	8.382 ($\pm$ 0.252)	0.827 ($\pm$ 0.378)	8.773 ($\pm$ 0.252)	0.868 ($\pm$ 0.338)	9.249 ($\pm$ 0.252)
Concentration	1.000 ($\pm$ 0.000)	2.517 ($\pm$ 0.252)	1.000 ($\pm$ 0.000)	2.548 ($\pm$ 0.252)	1.000 ($\pm$ 0.000)	2.591 ($\pm$ 0.252)

Table 3: Contextual bandit: d = 50

Method	Level of confidence					
	0.8		0.85		0.9	
	Avg. Coverage	Avg. log(Volumn)	Avg. Coverage	Avg. log(Volumn)	Avg. Coverage	Avg. log(Volumn)
ALEE	0.744 ($\pm$ 0.436)	72.759 ($\pm$ 1.403)	0.809 ($\pm$ 0.393)	73.680 ($\pm$ 1.403)	0.875 ($\pm$ 0.331)	74.822 ($\pm$ 1.403)
OLS	0.730 ($\pm$ 0.444)	44.640 ($\pm$ 1.370)	0.791 ($\pm$ 0.407)	45.560 ($\pm$ 1.370)	0.847 ($\pm$ 0.360)	46.703 ($\pm$ 1.370)
W-Decorrelation	0.192 ($\pm$ 0.394)	97.559 ($\pm$ 1.337)	0.225 ($\pm$ 0.418)	98.479 ($\pm$ 1.337)	0.276 ($\pm$ 0.447)	99.622 ($\pm$ 1.337)
Concentration	1.000 ($\pm$ 0.000)	90.964 ($\pm$ 1.312)	1.000 ($\pm$ 0.000)	91.004 ($\pm$ 1.312)	1.000 ($\pm$ 0.000)	91.060 ($\pm$ 1.312)

C.3 Asymptotic normality with centered Poisson noise variables

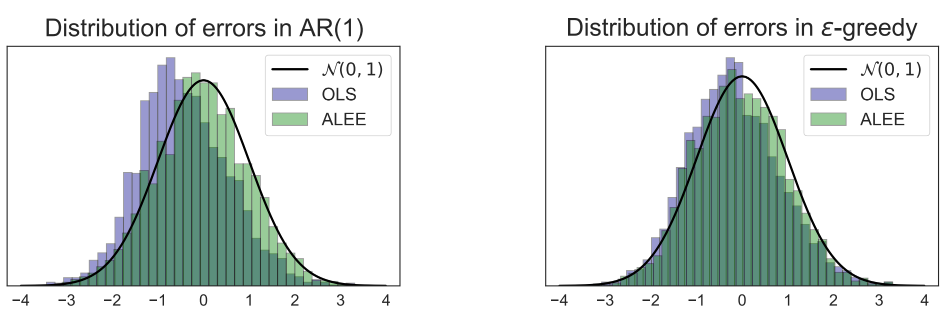

Figure 4: Same setting as Figure 1 but with noise variables $\{\epsilon_t\}$ distributed as centered $Poisson(1)$. We set $n = 3000$ and the number of replications is set to 1000. The simulations show that the asymptotic distribution of ALEE is in good accordance with the asymptotic normality proved in Corollary 3.5 and Theorem 3.1.