9 Appendix

Here, we provide the proof of Proposition 5. The proof makes use of the following two propositions, which we present and prove next.

Proposition 6. Let $\{\theta^t\}_{t\in\mathbb{N}}$ be a sequence generated by Algorithm 2. Then, for all $t \in \mathbb{N}$ and $0 \le k \le K-1$, we have

$$H(\theta^t, \theta^\star) - H(\theta^t, w^{t,k}) \le \frac{F_\theta^2}{2F_w}\|\theta^t - \theta^\star\|^2 - \left(\frac{1}{\alpha} - \frac{L}{2}\right)\|w^{t,k+1} - w^{t,k}\|^2.$$

Proof. Let $t \in \mathbb{N}$. From the F_w-strong convexity of $w \to H(\theta^t, w)$, we obtain that

$$H(\theta^t, w^{t,k+1}) \ge H(\theta^t, \theta^\star) + \langle \nabla_w H(\theta^t, \theta^\star), w^{t,k+1} - \theta^\star\rangle + \frac{F_w}{2}\|w^{t,k+1} - \theta^\star\|^2,$$

which means that

$$H(\theta^t, \theta^\star) - H(\theta^t, w^{t,k+1}) \le \langle \nabla_w H(\theta^t, \theta^\star), \theta^\star - w^{t,k+1}\rangle - \frac{F_w}{2}\|w^{t,k+1} - \theta^\star\|^2. \tag{8}$$

Moreover, in light of the fixed-point characterization of TD, namely that $\nabla_w H(\theta^\star, \theta^\star) = \mathbf{0}$, we can write

$$\begin{aligned}
\langle \nabla_w H(\theta^t, \theta^\star), \theta^\star - w^{t,k+1}\rangle &= \langle \nabla_w H(\theta^t, \theta^\star) - \nabla_w H(\theta^\star, \theta^\star), \theta^\star - w^{t,k+1}\rangle \\
&= \left(\nabla_w H(\theta^t, \theta^\star) - \nabla_w H(\theta^\star, \theta^\star)\right)^\top \left(\theta^\star - w^{t,k+1}\right) \\
&\le \frac{1}{2F_w}\|\nabla_w H(\theta^t, \theta^\star) - \nabla_w H(\theta^\star, \theta^\star)\|^2 + \frac{F_w}{2}\|w^{t,k+1} - \theta^\star\|^2 \\
&\le \frac{F_\theta^2}{2F_w}\|\theta^t - \theta^\star\|^2 + \frac{F_w}{2}\|w^{t,k+1} - \theta^\star\|^2. \qquad (9)
\end{aligned}$$

Here, the first inequality follows from the fact that for any two vectors a and b we have $a^\top b \le (1/2d)\|a\|^2 + (d/2)\|b\|^2$ for any $d > 0$. In this case, we chose $d = F_w$. Also the last inequality follows from the F_θ-Lipschitz property of $\nabla_w H$, which is our assumption.

Now, by combining (8) with (9) we obtain that

$$\begin{aligned}
H(\theta^t, \theta^\star) - H(\theta^t, w^{t,k+1}) &\le \frac{F_\theta^2}{2F_w}\|\theta^t - \theta^\star\|^2 + \frac{F_w}{2}\|w^{t,k+1} - \theta^\star\|^2 - \frac{F_w}{2}\|w^{t,k+1} - \theta^\star\|^2 \\
&= \frac{F_\theta^2}{2F_w}\|\theta^t - \theta^\star\|^2. \qquad (10)
\end{aligned}$$

From the Lipschitz assumption we can write, due to the Descent Lemma [19] applied to the function $w \to H(\theta^t, w)$, that:

$$H(\theta^t, w^{t,k+1}) - H(\theta^t, w^{t,k}) \le \langle \nabla_w H(\theta^t, w^{t,k}), w^{t,k+1} - w^{t,k}\rangle + \frac{L}{2}\|w^{t,k+1} - w^{t,k}\|^2$$

Now, notice that according to Algorithm 2 we have $w^{t,k+1} = w^{t,k} - \alpha\nabla_w H(\theta^t, w^{t,k})$, and so we can write:

$$\begin{aligned}
H(\theta^t, w^{t,k+1}) - H(\theta^t, w^{t,k}) &\le \frac{1}{\alpha}\langle w^{t,k} - w^{t,k+1}, w^{t,k+1} - w^{t,k}\rangle + \frac{L}{2}\|w^{t,k+1} - w^{t,k}\|^2 \\
&= -\left(\frac{1}{\alpha} - \frac{L}{2}\right)\|w^{t,k+1} - w^{t,k}\|^2 . \qquad (11)
\end{aligned}$$

Adding both sides of (10) with (11) yields:

$$H(\theta^t, \theta^\star) - H(\theta^t, w^{t,k}) \le \frac{F_\theta^2}{2F_w}\|\theta^t - \theta^\star\|^2 - \left(\frac{1}{\alpha} - \frac{L}{2}\right)\|w^{t,k+1} - w^{t,k}\|^2,$$

which proves the desired result. □

Now, we can prove the following result.

Proposition 7. Let $\{\theta^t\}_{t\in\mathbb{N}}$ be a sequence generated by Algorithm 2. Then, for all $t \in \mathbb{N}$ and $0 \le k \le K-1$, we have

$$\|w^{t,k+1} - \theta^\star\|^2 \le (1-\alpha F_w)\,\|w^{t,k} - \theta^\star\|^2 + \frac{\alpha F_\theta^2}{F_w}\|\theta^t - \theta^\star\|^2 - (2-\alpha L)\,\|w^{t,k+1} - w^{t,k}\|^2.$$

In particular, when $\alpha = 1/L$, we have

$$\|w^{t,k+1} - \theta^\star\|^2 \le \left(1 - \frac{F_w}{L}\right)\|w^{t,k} - \theta^\star\|^2 + \frac{F_\theta^2}{LF_w}\|\theta^t - \theta^\star\|^2.$$

Proof. Let $t \in \mathbb{N}$. From the definition of steps of Algorithm 2, that is, $w^{t,k+1} = w^{t,k} - \alpha\nabla_w H(\theta^t, w^{t,k})$, for any $0 \le k \le K-1$, we obtain that

$$\begin{aligned}
\|w^{t,k+1} - \theta^\star\|^2 &= \|(w^{t,k} - \theta^*) - \alpha\nabla_w H(\theta^t, w^{t,k})\|^2 \\
&= \|w^{t,k} - \theta^\star\|^2 + 2\alpha\langle \nabla_w H\left(\theta^t, w^{t,k}\right), \theta^\star - w^{t,k}\rangle + \|\alpha\nabla_w H(\theta^t, w^{t,k})\|^2 \\
&= \|w^{t,k} - \theta^\star\|^2 + 2\alpha\langle \nabla_w H\left(\theta^t, w^{t,k}\right), \theta^\star - w^{t,k}\rangle + \|w^{t,k+1} - w^{t,k}\|^2. \quad (12)
\end{aligned}$$

Using the F_w-strong convexity of $w \to H(\theta^t, w)$, we have

$$H\left(\theta^t, \theta^\star\right) \ge H\left(\theta^t, w^{t,k}\right) + \langle \nabla_w H\left(\theta^t, w^{t,k}\right), \theta^\star - w^{t,k}\rangle + \frac{F_w}{2}\|w^{t,k} - \theta^\star\|^2 . \quad (13)$$

Combining (12) and (13), we get:

$$\begin{aligned}
\|w^{t,k+1} - \theta^\star\|^2 \le \|w^{t,k} - \theta^\star\|^2 &+ 2\alpha\Big(H\left(\theta^t, \theta^\star\right) - H\left(\theta^t, w^{t,k}\right) - \frac{F_w}{2}\|w^{t,k} - \theta^\star\|^2\Big) \\
&+ \|w^{t,k+1} - w^{t,k}\|^2 \\
&= (1-\alpha F_w)\,\|w^{t,k} - \theta^\star\|^2 + 2\alpha\left(H\left(\theta^t, \theta^\star\right) - H\left(\theta^t, w^{t,k}\right)\right) + \|w^{t,k+1} - w^{t,k}\|^2.
\end{aligned}$$

Hence, from Proposition 6, we obtain

$$\begin{aligned}
\|w^{t,k+1} - \theta^\star\|^2 \le (1-\alpha F_w)\,\|w^{t,k} - \theta^\star\|^2 &+ 2\alpha\left(\frac{F_\theta^2}{2F_w}\|\theta^t - \theta^\star\|^2 - \left(\frac{1}{\alpha} - \frac{L}{2}\right)\|w^{t,k+1} - w^{t,k}\|^2\right) \\
&+ \|w^{t,k+1} - w^{t,k}\|^2 \\
&= (1-\alpha F_w)\,\|w^{t,k} - \theta^\star\|^2 + \frac{\alpha F_\theta^2}{F_w}\|\theta^t - \theta^\star\|^2 - (2-\alpha L)\,\|w^{t,k+1} - w^{t,k}\|^2,
\end{aligned}$$

which completes the first desired result.

Moreover, by specifically choosing the step-size $\alpha = 1/L$ we obtain that:

$$\begin{aligned}
\|w^{t,k+1} - \theta^\star\|^2 &\le \left(1 - \frac{F_w}{L}\right)\|w^{t,k} - \theta^\star\|^2 + \frac{F_\theta^2}{LF_w}\|\theta^t - \theta^\star\|^2 - \|w^{t,k+1} - w^{t,k}\|^2 \\
&\le \left(1 - \frac{F_w}{L}\right)\|w^{t,k} - \theta^\star\|^2 + \frac{F_\theta^2}{LF_w}\|\theta^t - \theta^\star\|^2 .
\end{aligned}$$

This concludes the proof of this proposition. □

Using these two results, we are ready to present the proof of Proposition 5

proof of Proposition 5. Let $t \in \mathbb{N}$. From Proposition 7 (recall that $\alpha = 1/L$) and the fact that $\theta^{t+1} = w^{t,K}$, we have

$$\begin{aligned}
\|\theta^{t+1} - \theta^\star\|^2 &= \|w^{t,K} - \theta^\star\|^2 \\
&\leq (1-\kappa)\,\|w^{t,K-1} - \theta^\star\|^2 + \eta^2\kappa\|\theta^t - \theta^\star\|^2 \\
&\leq (1-\kappa)\left[(1-\kappa)\,\|w^{t,K-2} - \theta^\star\|^2 + \eta^2\kappa\|\theta^t - \theta^\star\|^2\right] + \eta^2\kappa\|\theta^t - \theta^\star\|^2 \\
&= (1-\kappa)^2\,\|w^{t,K-2} - \theta^\star\|^2 + \eta^2\kappa\,(1 + (1-\kappa))\,\|\theta^t - \theta^\star\|^2 \\
&\leq \dots \\
&\leq (1-\kappa)^K\,\|w^{t,0} - \theta^\star\|^2 + \eta^2\kappa\sum_{k=0}^{K-1}(1-\kappa)^k\,\|\theta^t - \theta^\star\|^2 \\
&= (1-\kappa)^K\,\|\theta^t - \theta^\star\|^2 + \eta^2\kappa\sum_{k=0}^{K-1}(1-\kappa)^k\,\|\theta^t - \theta^\star\|^2,
\end{aligned}$$

where the last inequality follows from the fact that $w^{t,0} = \theta^t$. Because $\kappa \in [0,1]$, the geometric series on the right hand side is convergent, and so we can write:

$$(1-\kappa)^K + \eta^2\kappa\sum_{k=0}^{K-1}(1-\kappa)^k = (1-\kappa)^K + \eta^2\kappa\frac{1-(1-\kappa)^K}{1-(1-\kappa)} = (1-\kappa)^K + \eta^2\left(1-(1-\kappa)^K\right),$$

which completes the desired result.