Supplementary Material
Explainable and Efficient Randomized Voting Rules

In this supplementary material, we provide the details and proofs omitted from the main text. The structure of the supplementary material resembles the structure of the main text.

In Appendix A, we present the results on randomized positional scoring rules. Specifically, we first present novel insights (A.1), use them to prove distortion upper bounds for common randomized positional scoring rules (A.2), prove generic lower bounds that prove tightness of these results (A.3), and finally extend our results to randomized multi-level approval rules (A.4) as the first step towards characterizing the distortion of every randomized positional scoring rule.

In Appendix B, we present the results on random k-committee member rules, first presenting a lower bound (B.1) and then an algorithm achieving a non-trivial upper bound (B.2).

Finally, in Appendix C, we present additional experiments, such as analyzing how the distortion of explainable randomized rules depends on the Mallows noise parameter ϕ for fixed values of m (as opposed to the results in the main text, which analyze the dependence on m for fixed values of ϕ), and analyzing how the optimal k (and the distortion achieved at this optimal k for random k-committee member rules changes with m. We also present additional results for two other statistical models: the Polya-Eggenberger urn model and the Plackett-Luce model.

A Randomized Positional Scoring Rules

In this section, we expand on our discussion of randomized positional scoring rules.

A.1 High-Level Distortion Analysis and Novel Insights

Next, we elaborate on the high-level distortion analysis and novel insights presented in Section 3.1 as well as provide additional novel insights.

A.1.1 Absolute Welfare is Minimized at Dichotomous Utilities

As mentioned in Strategy 3 of Section 3.1, a novel insight of our work is that proving an absolute lower bound on the welfare achieved by a rule across all instances can be useful in bounding the distortion, even though the latter needs to compare the welfare achieved in each instance to the optimum welfare in that instance. First, recall the definition of minimum welfare.

Definition 1 (Minimum Welfare)**.** *Define the* minimum welfare *of a distribution over alternatives* $p \in \Delta(A)$ *on a preference profile* $\vec{\sigma}$ *as* $\textsf{min-sw}(p, \vec{\sigma}) = \inf_{\vec{u} \in \mathcal{C}(\vec{\sigma})} \textsf{sw}(p, \vec{u})$*, which is the minimum social welfare of* p *across all consistent utility profiles. The* minimum welfare *of a voting rule* f *is the minimum welfare of its output, minimized over all preference profiles:* $\textsf{min-sw}_{n,m}(f) = \min_{\vec{\sigma}} \textsf{min-sw}(f(\vec{\sigma}), \vec{\sigma})$*, where the minimum is taken over all preference profiles with* n *agents and* m *alternatives. We drop* n *and* m *when clear from the context.*

First, we show that this minimum welfare is in fact attained at a *dichotomous utility profile*.

Definition 5 (Dichotomous Utilities)**.** *For* $k \in [m]$*, define the* k*-dichotomous utility function for agent* i *to be*

$$\mathbb{1}_{i,k} = \begin{cases} 1/k & \textsf{rank}_i(a) \leqslant k, \\ 0 & \textit{o.w.}. \end{cases}$$

That is, agent i *is indifferent between her top* k *alternatives, but does not value any other alternative. We call* $\vec{u}$ *a* dichotomous utility profile *if, for each* $i \in N$*, we have* $u_i = \mathbb{1}_{i,k}$ *for some* $k \in [m]$*.*

Lemma 6. *For any preference profile* $\vec{\sigma}$ *and distribution over alternatives* $p \in \Delta(A)$*, there exists a dichotomous utility profile* $\vec{u}^* \in \arg\min_{\vec{u} \in \mathcal{C}(\vec{\sigma})} \textsf{sw}(p, \vec{u})$ *at which minimum welfare is achieved, and this minimum welfare is bounded as* $\textsf{min-sw}(p, \vec{\sigma}) = \textsf{sw}(p, \vec{u}^*) \leqslant n/m$*.*

Proof. Let $p = f(\vec{\sigma})$ be the distribution returned by the rule f. Take the utility profile $\vec{u} = \arg\min_{\vec{u} \in \mathcal{C}(\vec{\sigma})} \textsf{sw}(p, \vec{u})$. Suppose by contradiction that there exists an agent $i \in N$ such that u_i is not dichotomous. Since u_i is a unit-sum utility vector, u_i can be represented as $u_i = \sum_{k\in[m]} \alpha_k \mathbb{1}_{i,k}$ for a unique list of α_k's subject to $\sum_k \alpha_k = 1$. Then, we have

$$u_i(p) = \sum\nolimits_{k\in[m]} \alpha_k \cdot \mathbb{1}_{i,k}(p).$$

By the linearity of the expression above, we can assume, without loss of generality, that it is minimized at one of the $\mathbb{1}_{i,k}$'s.

To show that $\text{min-sw}(p, \vec{\sigma}) \leqslant \frac{n}{m}$, take the utility profile $u_i = \mathbb{1}_{i,m}$ for all i (all agents are indifferent). Then, $u_i(p) = \sum_{a\in A} \frac{p_a}{m} = \frac{1}{m}$ and $\text{sw}(p) = \frac{n}{m}$. ☐

Next, we show that the minimum welfare of any distribution that is returned by a randomized positional scoring rule on any preference profile is not much lower.

Lemma 7. *For any randomized positional scoring rule $f_{\vec{s}}^{\text{rand}}$ and preference profile $\vec{\sigma}$, the minimum welfare is bounded as* $\text{min-sw}(f_{\vec{s}}^{\text{rand}}(\vec{\sigma}), \vec{\sigma}) \geqslant n/(4m^2)$.

Proof. Normalize $\vec{s}$ such that $\|\vec{s}\|_1 = 1$. Following Lemma 11, for all $k \in [m]$, $\text{min-sw}(f_{\vec{s}_{k\text{-approval}}}^{\text{rand}}) \geqslant \frac{n}{4m^2}$. Furthermore, any scoring vector can be uniquely rewritten as $\vec{s} = \sum_{k\in[m]} \alpha_k \cdot \frac{\vec{s}_{k\text{-approval}}}{k}$ with $\alpha_k \geqslant 0$ and $\sum_{k\in[m]} \alpha_k = 1$. Note that scaling a scoring vector does not affect its distortion or obtained social welfare. Therefore,

$$\text{min-sw}(f_{\vec{s}}^{\text{rand}}) \geqslant \sum_{k\in[m]} \alpha_k \cdot \text{min-sw}(f_{\vec{s}_{k\text{-approval}}}^{\text{rand}}) \geqslant \frac{n}{4m^2} \cdot \sum_{k\in[m]} \alpha_k = \frac{n}{4m^2}.$$ ☐

From Lemmas 6 and 7, we have the following.

Corollary 6. *Every randomized positional scoring rule $f_{\vec{s}}^{\text{rand}}$ satisfies* $\text{min-sw}(f_{\vec{s}}^{\text{rand}}) \in [n/(4m^2), n/m]$.

See Table 1 for tight bounds for the rules induced by common scoring vectors.

A.1.2 Logarithmic Rounding of the Scores

Let us begin by providing a proof of the following result stated in Section 3.1.

Lemma 1 (Rounding Down Scores)**.** *Let $\alpha \geqslant 0$, and $\vec{s}, \vec{s'}$ be scoring vectors such that $s'_j \leqslant s_j \leqslant (1+\alpha)s'_j$ for all $j \in [m]$. Then, for every preference profile $\vec{\sigma}$ and consistent utility profile $\vec{u} \in \mathcal{C}(\vec{\sigma})$,*

$$\frac{1}{1+\alpha} \cdot \text{sw}(f_{\vec{s}'}^{\text{rand}}(\vec{\sigma}), \vec{u}) \leqslant \text{sw}(f_{\vec{s}}^{\text{rand}}(\vec{\sigma}), \vec{u}) \leqslant (1+\alpha) \cdot \text{sw}(f_{\vec{s}'}^{\text{rand}}(\vec{\sigma}), \vec{u}),$$

and consequently, $\frac{1}{1+\alpha} \cdot \text{dist}(f_{\vec{s'}}^{\text{rand}}) \leqslant \text{dist}(f_{\vec{s}}^{\text{rand}}) \leqslant (1+\alpha) \cdot \text{dist}(f_{\vec{s'}}^{\text{rand}})$.

Proof. Fix a preference profile $\vec{\sigma}$ and a consistent utility profile $\vec{u} \in \mathcal{C}(\vec{\sigma})$. Since $s'_j \leqslant s_j \leqslant (1+\alpha)s'_j$ for all $j \in [m]$, we have $\text{score}(a, \vec{s}) \geqslant \text{score}(a, \vec{s}')$ for all alternatives $a \in A$ as well as $\|\vec{s}\|_1 \leqslant (1+\alpha) \cdot \|\vec{s}'\|_1$. Hence,

$$\begin{aligned} \text{sw}(f_{\vec{s}}^{\text{rand}}(\vec{\sigma}), \vec{u}) = \sum_{a\in A} \text{sw}(a, \vec{u}) \cdot \frac{\text{score}(a, \vec{s})}{n\|\vec{s}\|_1} &\geqslant \sum_{a\in A} \text{sw}(a, \vec{u}) \cdot \frac{\text{score}(a, \vec{s}')}{n(1+\alpha)\|\vec{s}'\|_1} \\ &= \frac{1}{1+\alpha} \cdot \text{sw}(f_{\vec{s}'}^{\text{rand}}(\vec{\sigma}), \vec{u}). \end{aligned}$$

Hence,

$$\text{dist}(f_{\vec{s}}^{\text{rand}}(\vec{\sigma}), \vec{u}) \leqslant (1+\alpha) \cdot \text{dist}(f_{\vec{s}'}^{\text{rand}}(\vec{\sigma}), \vec{u}).$$

Since the above holds for all $\vec{\sigma}$ and $\vec{u} \in \mathcal{C}(\vec{\sigma})$, we have $\text{dist}(f_{\vec{s}}^{\text{rand}}) \leqslant (1+\alpha) \cdot \text{dist}(f_{\vec{s}'}^{\text{rand}})$.

The other direction follows similarly using $\text{score}(a, \vec{s}) \leqslant (1+\alpha) \cdot \text{score}(a, \vec{s}')$ and $\|\vec{s}\|_1 \geqslant \|\vec{s}'\|_1$. ☐

As stated in Section 3.1, applying this transformation to scores in the range $[\|\vec{s}\|_1/(4m^2), \|\vec{s}\|_1]$ allows us to reduce them to $O(\log m)$ distinct values. The next result shows that the remaining small scores can be ignored by reducing them to 0 while only changing the distortion by another factor of at most two.

Lemma 8 (Ignoring Small Scores)**.** *Let $\vec{s}$ be a scoring vector. Let $\vec{s'}$ be $\vec{s}$ except that $s_j \leqslant \frac{\|\vec{s}\|_1}{4m^2} \Rightarrow s'_j = 0$ for all $j \in [m]$. Then,*

$$\frac{1}{2} \cdot \mathsf{dist}(f^{\mathsf{rand}}_{\vec{s'}}) \leqslant \mathsf{dist}(f^{\mathsf{rand}}_{\vec{s}}) \leqslant 2 \cdot \mathsf{dist}(f^{\mathsf{rand}}_{\vec{s'}}).$$

Proof. Fix any preference profile $\vec{\sigma}$ and consistent utility profile $\vec{u} \in \mathcal{C}(\vec{\sigma})$. For each $a \in A$, define $\delta(a) = \mathsf{score}(a, \vec{s}) - \mathsf{score}(a, \vec{s}')$. Then,

$$\begin{aligned} \mathsf{sw}(f^{\mathsf{rand}}_{\vec{s}}(\vec{\sigma}), \vec{u}) &= \sum_{a \in A} \mathsf{sw}(a, \vec{u}) \cdot \left(\frac{\mathsf{score}(a, \vec{s}') + \delta(a)}{n\|\vec{s}\|_1} \right) \\ &= \frac{\|\vec{s}'\|_1}{\|\vec{s}\|_1} \cdot \mathsf{sw}(f^{\mathsf{rand}}_{\vec{s}'}(\vec{\sigma}), \vec{u}) + \sum_{a \in A} \mathsf{sw}(a, \vec{u}) \cdot \frac{\delta(a)}{n\|\vec{s}\|_1}. \end{aligned} \tag{1}$$

Note that $\|\vec{s}'\|_1 \geqslant \|\vec{s}\|_1 - (m-1) \cdot \frac{\|\vec{s}\|_1}{4m^2} \geqslant (1 - \frac{1}{4m+1}) \cdot \|\vec{s}\|_1$. Thus, in Equation (1), we have

$$\mathsf{sw}(f^{\mathsf{rand}}_{\vec{s}}(\vec{\sigma}), \vec{u}) \geqslant \left(1 - \frac{1}{4m+1}\right) \cdot \mathsf{sw}(f^{\mathsf{rand}}_{\vec{s}'}(\vec{\sigma}), \vec{u}).$$

This implies

$$\mathsf{dist}(f^{\mathsf{rand}}_{\vec{s}}(\vec{\sigma}), \vec{u}) \leqslant \frac{4m+1}{4m} \cdot \mathsf{dist}(f^{\mathsf{rand}}_{\vec{s}'}(\vec{\sigma}), \vec{u}).$$

Since this holds for all $\vec{\sigma}$ and $\vec{u} \in \mathcal{C}(\vec{\sigma})$, we have $\mathsf{dist}(f^{\mathsf{rand}}_{\vec{s}}) \leqslant (1 + \frac{1}{4m}) \cdot \mathsf{dist}(f^{\mathsf{rand}}_{\vec{s}'})$.

For the other direction, we use the facts that $\|\vec{s}'\|_1 \leqslant \|\vec{s}\|_1$ and $\frac{\delta(a)}{n\|\vec{s}\|_1} \leqslant \frac{1}{n\|\vec{s}\|_1} \cdot n \cdot \frac{\|\vec{s}\|_1}{4m^2} = \frac{1}{4m^2}$. Thus, in Equation (1), we have

$$\mathsf{sw}(f^{\mathsf{rand}}_{\vec{s}}(\vec{\sigma}), \vec{u}) \leqslant \mathsf{sw}(f^{\mathsf{rand}}_{\vec{s}'}(\vec{\sigma}), \vec{u}) + \sum_{a \in A} \mathsf{sw}(a, \vec{u}) \cdot \frac{1}{4m^2} = \mathsf{sw}(f^{\mathsf{rand}}_{\vec{s}'}(\vec{\sigma}), \vec{u}) + \frac{n}{4m^2} \leqslant 2 \cdot \mathsf{sw}(f^{\mathsf{rand}}_{\vec{s}'}),$$

where the last inequality is due to Lemma 7. Using the same argument as above, this results in $\mathsf{dist}(f^{\mathsf{rand}}_{\vec{s}'}) \leqslant 2 \cdot \mathsf{dist}(f^{\mathsf{rand}}_{\vec{s}})$. ☐

Using Lemma 1 with $\alpha = 1$ and combining with Lemma 8, we get the following.

Corollary 7 (Scoring Vector Reduction)**.** *Given any scoring vector $\vec{s}$, there exists a scoring vector $\vec{s}'$ with $O(\log m)$ distinct positive scores such that $(1/4) \cdot \mathsf{dist}(f^{\mathsf{rand}}_{\vec{s}}) \leqslant \mathsf{dist}(f^{\mathsf{rand}}_{\vec{s}'}) \leqslant 4 \cdot \mathsf{dist}(f^{\mathsf{rand}}_{\vec{s}})$.*

A.1.3 Helpful Technical Lemmas

Before proving Lemma 2, we show a weaker lemma which follows from a simpler proof. This is useful in particular for analyzing randomized k-approval rules with $k \in [1, m - \Omega(m)]$ (which includes the randomized dictatorship rule) and randomized approval mixtures rules where again we mix some k-approvals with $k \in [1, m - \Omega(m)]$.

Lemma 9. *Fix any preference profile $\vec{\sigma}$, subset of agents $T \subset N$, threshold $\tau \geqslant 0$, and rank $\ell \in [m]$. For a partial utility profile $\vec{u}$ in which every agent in T has utility at least τ for each of her top ℓ alternatives and all other utilities are 0, we have*

$$\mathsf{sw}(f^{\mathsf{rand}}_{\vec{s}_{k\text{-approval}}}) \geqslant \tau \cdot \frac{|T|^2}{nm} \cdot \frac{(\min\{k, \ell\})^2}{k}.$$

Proof. Let $t = \min\{k, \ell\}$. For all $a \in A$ let x_a denote the number of appearances of a among the top t votes of T. Then,

$$\mathsf{sw}(f^{\mathsf{rand}}_{\vec{s}_{k\text{-approval}}}) = \sum_{a \in A} \Pr[a] \cdot \mathsf{sw}(a)$$

$$
\begin{aligned}
&\geqslant \sum_{a\in A} \frac{x_a}{nk} \cdot (x_a \cdot \tau) \\
&= \frac{\tau}{nk} \cdot \sum_{a\in A} x_a^2 \\
&\geqslant \frac{\tau}{nk} \cdot \frac{\left(\sum_{a\in A} x_a\right)^2}{m} && \text{(by AM-QM inequality)} \\
&= \frac{\tau}{nk} \cdot \frac{(|T|\cdot t)^2}{m} = \tau \cdot \frac{|T|^2}{nm} \cdot \frac{(\min\{k,\ell\})^2}{k}. && \square
\end{aligned}
$$

The key limitation of the lemma above is that the probability of selecting alternative $\Pr[a]$ is lower bounded by $\frac{x_a}{nk}$ (the score a gets only from the top ℓ alternatives of T), while this could be higher due to the score that agents in $N \setminus T$ give to a. As a result, in the transition that uses the AM-QM inequality, we divide the top ℓ alternatives of T among all m alternatives. If the scores obtained from $N \setminus T$ is also considered, it may be the case that the top ℓ alternatives of T is divided among fewer number of alternatives (sublinear in m), which enables stronger lower bounds. To this end, we need a more complicated analysis provided in the following lemma.

Lemma 2. *Fix any scoring vector $\vec{s}$, preference profile $\vec{\sigma}$, subset of agents $T \subseteq N$, threshold $\tau \geqslant 0$, and rank $\ell \in [m]$. For a partial utility profile $\vec{u}$ in which every agent in T has utility at least τ for each of her top ℓ alternatives and all other utilities are 0, we have:*

$$
\mathsf{sw}_T(f^{\mathsf{rand}}_{\vec{s}}(\vec{\sigma}), \vec{u}) \geqslant \tau \cdot \frac{|T|\ell}{2n\|\vec{s}\|_1} \min_{h\in[m]} \frac{1}{h}\Big(2s_\ell \cdot |T|\ell + (n-|T|)\cdot \sum_{j=1}^{h} s_{m-j+1}\Big).
$$

Proof. For all alternative a, denote by x_a the number of appearances of a among the top ℓ votes of T. Note that $\sum_{a\in A} x_a = |T|\cdot\ell$. We have

$$
\begin{aligned}
\mathsf{sw}_T(f^{\mathsf{rand}}_{\vec{s}}) &= \sum_{a\in A} \mathsf{sw}_T(a) \cdot \frac{\mathsf{score}(a)}{n\|\vec{s}\|_1} \\
&\geqslant \sum_{a\in A} \tau \cdot x_a \cdot \frac{\mathsf{score}_T(a) + \mathsf{score}_{N\setminus T}(a)}{n\|\vec{s}\|_1} \\
&\geqslant \sum_{a\in A} \tau \cdot x_a \cdot \frac{x_a \cdot s_\ell + \mathsf{score}_{N\setminus T}(a)}{n\|\vec{s}\|_1}. \qquad (2)
\end{aligned}
$$

Rename the candidates such that $x_1 \geqslant x_2 \geqslant \cdots \geqslant x_m$.

Worst-case Preference Ranking of $N \setminus T$. The contribution of $N \setminus T$ to Equation (2) is $\sum_a x_a \cdot \mathsf{score}_{N\setminus T}(a)$ which can be decomposed to across agents, i.e. the contribution of each agent $i \in N \setminus T$ is $\sum_{a\in A} x_a \cdot \mathsf{score}_i(a)$. Since x_a's are only depends on agents in T, to obtain a lower bound, we may assume without loss of generality that their preference ranking is $m \succ_i m-1 \succ_i \ldots \succ_i 1$, since $x_m \leqslant x_{m-1} \leqslant \ldots \leqslant x_1$ and $\mathsf{score}_i(\sigma_i(1)) \geqslant \ldots \geqslant \mathsf{score}_i(\sigma_i(m))$.

Forming a Quadratic Program. Recall that we renamed the alternatives in decreasing order by x_a. Following the observation of worst-case preference ranking of $N \setminus T$, to obtain a lower bound, for $a \in [m]$, we have $\mathsf{score}_{N\setminus T}(a) = (n-|T|)\cdot s_{m-a+1}$. Now, we can rewrite Equation (2) as follows,

$$
\frac{\tau}{n\|\vec{s}\|_1 \cdot s_\ell} \cdot \sum_{a=1}^{m} s_\ell \cdot x_a \cdot (s_\ell \cdot x_a + (n-|T|)\cdot s_{m-i+1}). \qquad (3)
$$

Now, we simplify this expression to form a quadratic program and analyze its minimum. Since $\frac{\tau}{n\|\vec{s}\|_1\cdot s_\ell}$ is a constant value, we focus on the summation. For conciseness, define $y_a = s_\ell \cdot x_a$, $\gamma_a = (n-|T|)\cdot s_{m-a+1}$, and $\beta = s_\ell \cdot |T| \cdot \ell = \sum_{a=1}^{m} y_a$ (holds due to $\sum_a x_a = |T|\cdot\ell$). Then, we have the following quadratic program with variables $\mathbf{y} = \{y_a\}_{a\in A}$,

$$
\min \quad \sum_{a=1}^{m} y_a \cdot (y_a + \gamma_a)
$$

$$\text{s.t.} \quad \sum_{a=1}^{m} y_a = \beta$$
$$y_a \geqslant 0 \qquad\qquad \forall a \in [m]. \tag{4}$$

Applying the KKT Conditions. Our objective is convex in y_a's, and it is easy to check that this program satisfies the Slater's condition for $y_a = \beta/m$. Hence, we can apply the KKT conditions to find the minimizer of this program. The Lagrangian for $f(\mathbf{y}) = \sum_a y_a(y_a + \gamma_a)$, $g(\mathbf{y}) = \beta - \sum_a y_a$, and $h_a(\mathbf{y}) = -y_a$, is

$$\mathcal{L}(\mathbf{y}, \lambda, \boldsymbol{\mu}) = f(\mathbf{y}) + \lambda g(\mathbf{y}) + \sum_{a=1}^{m} \mu_a \cdot h_a(\mathbf{y}),$$

where, in the dual program, μ_a is the variable for $y_a \geqslant 0$ conditions and λ is for the single equality condition. From the KKT conditions, we have y_a's are the minimizer of this function if

C1. (stationarity) $\forall a \in [m],\ 2 \cdot y_a + \alpha_a - \mu_a - \lambda = 0$

C2. (primal feasibility) (1) $\forall a \in [m], y_a \geqslant 0$ and (2) $\sum_{a=1}^{m} = \beta$

C3. (dual feasibility) $\forall a \in [m], \mu_a \geqslant 0$

C4. (complementary slackness) $\forall a \in [m], \mu_a \cdot y_a = 0$.

By C1 and C3, we have

$$y_a \leqslant \frac{1}{2}\left(\lambda - \alpha_a\right).$$

For $y_a \neq 0$, we can multiply C1 by y_a and using C4 (i.e. $y_a \cdot \mu_a = 0$), we get

$$y_a \neq 0 \ \Rightarrow\ y_a = \frac{1}{2}\left(\lambda - \alpha_a\right)$$

Therefore, $y_a = \max\{0, \frac{1}{2}(\lambda - \alpha_a)\}$. Let $|C^*|$ be the number of non-zero y_a's (i.e. the number of candidates with non-zero appearances among the top-ℓ votes of T). Furthermore, by C2, we have

$$\sum_{a=1}^{m} y_a = \sum_{a=1}^{m} \frac{1}{2} \max\{0, \lambda - \alpha_a\} = \beta \quad \Rightarrow \quad \lambda = \frac{1}{|C^*|}\left(2\beta + \sum_{a=1}^{|C^*|} \alpha_a\right) \tag{5}$$

Since $\alpha_1 \leqslant \alpha_2 \leqslant \ldots \leqslant \alpha_m$, the final y_a's will form a decreasing series $y_1 \geqslant y_2 \geqslant \cdots \geqslant y_{|C^*|} > y_{|C^*|+1} = \cdots y_m = 0$ with $|C^*|$ many non-zero values. This is similar to a water-filling argument. Initial levels are α_a, and the water fills up from the bottom to level λ with a total water amount of 2β.

Deriving a Lower Bound. Now, we use the findings above to get a lower bound as follows

$$\begin{aligned}
\sum_{a=1}^{m} y_a(y_a + \alpha_a) &\geqslant \sum_{a=1}^{m} y_a(y_a + \alpha_a/2) \\
&\geqslant \sum_{a=1}^{|C^*|} y_a \left(\frac{1}{2}(\lambda - \alpha_a) + \alpha_a/2\right) \\
&\geqslant \frac{\lambda}{2} \sum_{a=1}^{|C^*|} y_a = \frac{\lambda \cdot \beta}{2} \\
&= \frac{\beta}{2} \cdot \frac{1}{|C^*|}\left(2\beta + \sum_{a=1}^{|C^*|} \alpha_a\right). \\
&\geqslant \min_{|C^*| \in [m]} \ \frac{\beta}{2|C^*|} \cdot \left(2\beta + \sum_{a=1}^{|C^*|} \alpha_a\right).
\end{aligned}$$

$$= \min_{|C^*|\in[m]} \frac{s_\ell \cdot |T|\ell}{2|C^*|} \cdot \left(2s_\ell \cdot |T|\ell + (n-|T|) \cdot \sum_{a=1}^{|C^*|} s_{m-i+1}\right).$$

Combined with Equation (2), we have

$$\mathsf{sw}_T(f^{\mathsf{rand}}_{\vec{s}}) \geqslant \tau \cdot \frac{|T|\ell}{2n\|s\|_1} \min_{|C^*|\in[m]} \frac{1}{|C^*|}\left(2s_\ell \cdot |T|\ell + (n-|T|) \cdot \sum_{a=1}^{|C^*|} s_{m-i+1}\right). \qquad \square$$

A.1.4 Proof of Lemma 4

Lemma 4. *For any preference profile $\vec{\sigma}$, consistent utility profile $\vec{u} \in \mathcal{C}(\vec{\sigma})$, and alternative $a \in A$, we have $\mathsf{sw}(a) \leqslant (\mathsf{Borda}(a)+n)/m$.*

Proof. Fix an alternative a and agent i. By the unit-sum assumption, we have

$$u_i(a) \leqslant \frac{1}{\mathsf{rank}_i(a)} \leqslant \frac{m-\mathsf{rank}_i(a)+1}{m} = \frac{\mathsf{Borda}(a,i)+1}{m}.$$

By summing over all agents, we get $\mathsf{sw}(a) \leqslant \frac{\mathsf{Borda}(a)+n}{m}$. $\square$

A.2 Distortion of Common Scoring Rules

Here, we use the insights and high-level strategies laid out above to analyze common randomized positional scoring rules. At first, we derive only a lower bound on their minimum welfare and an upper bound on their distortion, and later we show that our bounds are tight.

A.2.1 Randomized Plurality

Lemma 10. *The minimum welfare of the randomized plurality (randomized dictatorship) rule is $\mathsf{min\text{-}sw}(f^{\mathsf{rand}}_{\vec{s}_{\mathsf{plu}}}) \geqslant \frac{n}{m^2}$.*

Proof. For $a \in A$, let $\mathsf{plu}(a) = \mathsf{score}(a, \vec{s}_{\mathsf{plu}})$ be the score of alternative a. Then, $\mathsf{sw}(a) \geqslant \mathsf{plu}(a) \cdot \frac{1}{m}$, since each voter deems a utility of at least $\frac{1}{m}$ for their top alternative. Then,

$$\mathsf{sw}(f_{\mathsf{plu}}) = \sum_{a\in A} \frac{\mathsf{plu}(a)}{n} \cdot \mathsf{sw}(a) \geqslant \frac{1}{n}\sum_{a\in A} \frac{\mathsf{plu}(a)^2}{m} \overset{(1)}{\geqslant} \frac{1}{n}\left(\sum_{a\in A}\frac{\mathsf{plu}(a)}{m}\right)^2 \geqslant \frac{n}{m^2},$$

where inequality (1) holds by the AM-QM inequality,and we used $\sum_{a\in A} \mathsf{plu}(a) = n$ in the last inequality.[1] $\square$

Theorem 8. *The distortion of the randomized plurality rule is $O(m\sqrt{m})$.*

Proof. For an alternative $a \in A$, let $N_a^+ \subseteq N$ be the set of agents whose top alternative is a. Then, we have $|N_a^+| = \mathsf{plu}(a)$, and

$$\mathsf{sw}_{N_a^+}(a^*) = \sum_{i\in N_a^+} u_i(a^*) \leqslant |N_a^+| = \mathsf{plu}(a),$$

where the inequality comes from the unit-sum assumption that utilities are at most 1. Moreover, $\mathsf{sw}_{N_a^+}(a) \geqslant \mathsf{sw}_{N_a^+}(a^*)$, since voters in N_a^+ prefer a to a^*. Thus,

$$\begin{aligned}\mathsf{sw}(f^{\mathsf{rand}}_{\vec{s}_{\mathsf{plu}}}) = \sum_{a\in A}\frac{\mathsf{plu}(a)}{n}\cdot \mathsf{sw}(a) &\geqslant \frac{1}{n}\sum_{a\in A}\mathsf{sw}_{N_a^+}(a^*)\cdot \mathsf{sw}_{N_a^+}(a^*)\\ &\geqslant \frac{1}{n}\cdot\frac{1}{m}\left(\sum_{a\in A}\mathsf{sw}_{N_a^+}(a^*)\right)^2 = \frac{\mathsf{sw}(a^*)^2}{n\cdot m},\end{aligned}$$

[1] This is an instantiation of Lemma 9 for $k = \ell = 1$, $\tau = \frac{1}{m}$, $|T| = n$.

where the second inequality holds by the AM-QM inequality. Consequently,

$$\mathsf{dist}(f_{\mathsf{plu}}) \leqslant \frac{n \cdot m}{\mathsf{sw}(a^*)}.$$

Furthermore, by Lemma 10 we have $\mathsf{dist}(f^{\mathsf{rand}}_{\vec{s}_{\mathsf{plu}}}) \leqslant \frac{\mathsf{sw}(a^*)}{n/m^2}$. Combining the two bounds, we have

$$\mathsf{dist}(f^{\mathsf{rand}}_{\vec{s}_{\mathsf{plu}}}) \leqslant \min\left\{\frac{m^2 \cdot \mathsf{sw}(a^*)}{n}, \frac{n \cdot m}{\mathsf{sw}(a^*)}\right\} \leqslant \sqrt{\frac{m^2 \cdot \mathsf{sw}(a^*)}{n} \cdot \frac{n \cdot m}{\mathsf{sw}(a^*)}} \leqslant m\sqrt{m}. \qquad \square$$

A.2.2 Randomized k-Approval

Lemma 11. *The minimum welfare of the randomized k-approval rule is,*

- *when $k \leqslant \sqrt{m}$,* $\mathsf{min\text{-}sw}(f^{\mathsf{rand}}_{\vec{s}_{k\text{-approval}}}) \geqslant \frac{n}{4m} \cdot \frac{k}{m}$,
- *and when $k > \sqrt{m}$,* $\mathsf{min\text{-}sw}(f^{\mathsf{rand}}_{\vec{s}_{\mathsf{Borda}}}) \geqslant \frac{n}{4(m-k+1)} \cdot \frac{1}{k}$.

Proof. When $k = m$, $f^{\mathsf{rand}}_{\vec{s}_{k\text{-approval}}}$ is equivalent to selecting an alternative uniformly at random, which achieves a social welfare of exactly $\frac{n}{m}$. Now, suppose $k \leqslant m - 1$.

Fix any preference profile $\vec{\sigma}$ and consistent utility profile $\vec{u} \in \mathcal{C}(\vec{\sigma})$. First, similar to the analysis of randomized Borda in Lemma 3, we make a few modifications to the preference profile that are guaranteed to not increase the welfare, and then invoke Lemma 2. For conciseness, let $\vec{s} = \vec{s}_{k\text{-approval}}$, and $f = f^{\mathsf{rand}}_{\vec{s}_{k\text{-approval}}}$.

Simplify the preference and utility profiles. By Lemma 6, to obtain a lower bound, we assume without loss of generality that each agent i has a dichotomous utility, i.e. $u_i \in \{\mathbb{1}_{i,\ell}\}_{\ell\in[m]}$. Partition $N = N_1 \cup N_2$, where $N_1 = \{u_i = \mathbb{1}_{i,\ell} \mid i \in N, \ell \leqslant k\}$ and $N_2 = \{u_i = \mathbb{1}_{i,\ell} \mid i \in N, \ell > k\}$.

Case $|N_1| \geqslant |N_2|$. Fix an agent $i \in N_1$. Let A_i be the set of top ℓ alternatives in σ_i and $a_i \in \arg\min_{a\in A_i} \mathsf{score}(a, \vec{s})$ be the alternative in A_i with the lowest score (equivalently, with the lowest probability of selection under f). Note that $\sum_{a\in A_i} u_i(a) = 1$. Now,

$$u_i(f) \geqslant \sum_{a\in A_i} \frac{\mathsf{score}(a, \vec{s})}{n\|\vec{s}\|_1} \cdot u_i(a) \overset{(1)}{\geqslant} \frac{\mathsf{score}(a_i, \vec{s})}{n\|\vec{s}\|_1} \cdot \left(\sum_{a\in A_i} u_i(a)\right) \geqslant \frac{\mathsf{score}(a_i, \vec{s})}{n\|\vec{s}\|_1} \cdot 1,$$

where (1) follows from the definition of a_i. This can be rewritten as $u'_i(f(\vec{\sigma}'))$, where $\vec{\sigma}'$ is a preference profile in which each agent $i \in N_1$ ranks a_i first, and $\vec{u}'$ is a partial utility profile in which each agent $i \in N_1$ has utility of 1 for her top alternative and 0 for the rest. For $i \in N_2$, we assume they have 0 utility for all alternatives. Summing the above for all agents, we have $\mathsf{sw}(f(\vec{\sigma}), \vec{u}) \geqslant \mathsf{sw}(f(\vec{\sigma}'), \vec{u}')$. Now, to lower bound it, we invoke Lemma 2 with $\vec{s} \leftarrow \vec{s}_{k\text{-approval}}$, $\vec{\sigma} \leftarrow \vec{\sigma}'$, $\vec{u} \leftarrow \vec{u}'$, $T = N_1$, $\tau \leftarrow 1$, and $\ell \leftarrow 1$. Using $\|\vec{s}\|_1 = k$, this gives us

$$\begin{aligned}
\mathsf{sw}(f(\vec{\sigma}'), \vec{u}') &\geqslant 1 \cdot \frac{\frac{n}{2} \cdot 1}{2n \cdot k} \cdot \min_{h\in[m]} \frac{1}{h}\left(2 \cdot \frac{n}{2} \cdot 1 + \frac{n}{2} \cdot \max\{0, h - (m-k)\}\right) \\
&= \frac{n}{8k} \cdot \min_{h\in[m]} \left(\frac{2 + \max\{0, h-(m-k)\}}{h}\right) \\
&\geqslant \frac{n}{8k} \cdot \min\left\{\frac{2}{m-k}, \frac{3}{m-k+1}, \frac{4}{m-k+2}, \dots, \frac{k+2}{m}\right\} \\
&\overset{(1)}{\geqslant} \frac{n}{8k} \cdot \frac{2}{m-k+1} = \frac{n}{4k(m-k+1)},
\end{aligned}$$

where (1) holds for $k \in [m-1]$.

Case $|N_1| \leqslant |N_2|$. Any agent $i \in N_2$ has a utility of at least $\frac{1}{\ell} \geqslant \frac{1}{m}$ for each of her top k alternatives. This can be considered as a utility profile $\vec{u}'$ where agents $i \in N_2$ have a utility of $\frac{1}{m}$ for their top k alternatives and 0 for the rest, and agents $i \in N_1$ have a 0 utility for all alternatives. Now, we invoke Lemma 9 with $k \leftarrow k$, $\vec{\sigma} \leftarrow \vec{\sigma}$, $\vec{u} \leftarrow \vec{u}'$, $T \leftarrow N_2$, $\tau \leftarrow \frac{1}{m}$, $\ell \leftarrow k$. Since $\|\vec{s}\| = k$, we have

$$\mathsf{sw}(f(\vec{\sigma}), \vec{u}) \geqslant \frac{1}{m} \cdot \frac{\left(\frac{n}{2}\right)^2}{nm} \cdot \frac{k^2}{k} = \frac{nk}{4m^2}.$$

Since either of the two cases must hold,

$$\mathsf{sw}(f(\vec{\sigma}), \vec{u}) \geqslant \min\left\{\frac{n}{4k(m-k+1)}, \frac{nk}{4m^2}\right\},$$

which completes the proof. $\square$

Lemma 12. *The distortion of the randomized k-approvals rule is*

- $O\left(\frac{m\sqrt{m}}{k\sqrt{k}}\right)$ *when* $k \leqslant m^{1/3}$,
- $O(m)$ *when* $m^{1/3} \leqslant k \leqslant \sqrt{m}$,
- *and* $O(k\sqrt{m-k+1})$ *when* $k > \sqrt{m}$.

Proof. Fix any preference profile $\vec{\sigma}$ and consistent utility profile $\vec{u} \in \mathcal{C}(\vec{\sigma})$. Let $a^* \in \arg\max_{a \in A} \mathsf{sw}(a, \vec{u})$ be an optimal alternative. Partition the agents $N = N^+ \cup N^-$ based on whether they approve a^* or not, i.e. $N^+ = \{i \in N \mid \mathsf{rank}_i(a^*) \leqslant k\}$ and $N^- = \{i \in N \mid \mathsf{rank}_i(a^*) > k\}$. Furthermore, $\mathsf{sw}(a^*) = \mathsf{sw}_{N^+}(a^*) + \mathsf{sw}_{N^-}(a^*)$. We derive distortion bounds for two cases based on the comparison of $\mathsf{sw}_{N^+}(a^*)$ and $\mathsf{sw}_{N^-}(a^*)$, and report the maximum of the two bounds as the upper bound. Before the case analysis, recall that by following strategy 3 and Lemma 11, we have

$$\begin{aligned} &\mathsf{sw}(f^{\mathsf{rand}}_{\vec{s}_{k\text{-approval}}}) \geqslant \mathsf{min\text{-}sw}(f^{\mathsf{rand}}_{\vec{s}_{k\text{-approval}}}) \geqslant \min\left\{\frac{nk}{4m^2}, \frac{n}{4k(m-k+1)}\right\} = g(n,m) \\ \Rightarrow\quad &\mathsf{dist}(f(\vec{\sigma}), \vec{u}) \leqslant \frac{\mathsf{sw}(a^*)}{g(n,m)}. \end{aligned} \tag{6}$$

Case $\mathsf{sw}_{N^+}(a^*) \geqslant \mathsf{sw}_{N^-}(a^*)$. Since $\mathsf{sw}_{N^+}(a^*) \leqslant |N^+|$, it follows that $\mathsf{sw}(a^*) \leqslant 2 \cdot \mathsf{sw}_{N^+}(a^*) \leqslant 2|N^+|$. By Equation (6), we have

$$\mathsf{dist}(f(\vec{\sigma}), \vec{u}) \leqslant \frac{2|N^+|}{g(n,m)}.$$

By strategy 2 and that $\Pr[a^* \in f(\vec{\sigma})] = \frac{|N^+|}{nk}$, we get

$$\mathsf{dist}(f(\vec{\sigma}), \vec{u}) \leqslant \frac{nk}{|N^+|}.$$

Putting the two together we have

$$\mathsf{dist}(f(\vec{\sigma}), \vec{u}) \leqslant \min\left\{\frac{2|N^+|}{g(n,m)}, \frac{nk}{|N^+|}\right\} \overset{(1)}{\leqslant} \sqrt{\frac{2|N^+|}{g(n,m)} \cdot \frac{nk}{|N^+|}} = \sqrt{\frac{2nk}{g(n,m)}},$$

where (1) follows from the inequality $\min\{a,b\} \leqslant \sqrt{a \cdot b}$ for $a, b \geqslant 0$. By expanding $g(n,m)$, we have

$$\mathsf{dist}(f(\vec{\sigma}), \vec{u}) \leqslant \begin{cases} \sqrt{\frac{2nk}{nk/(4m^2)}} = \sqrt{8} \cdot m, & \text{if } k \in [1, \sqrt{m}], \\ \sqrt{\frac{2nk}{n/(4k(m-k+1))}} = \sqrt{8} \cdot k\sqrt{m-k+1} & \text{if } k \in [\sqrt{m}, m]. \end{cases} \tag{7}$$

Case $\mathsf{sw}_{N^+}(a^*) < \mathsf{sw}_{N^-}(a^*)$. Similar to the previous case, $\mathsf{sw}(f(\vec{\sigma}), \vec{u}) \leqslant 2 \cdot \mathsf{sw}_{N^-}(f(\vec{\sigma}), \vec{u})$. The key insight in this case is to apply strategy 2 (analyzing the welfare above a^* in N^-). First, construct a new utility profile by rounding down the utilities to the closest power of two, i.e., $u'_i(a^*) = 2^{\lfloor \log_2 u_i(a^*) \rfloor}$ and replace utilities less than $\frac{1}{4m^2}$ with 0. This way,

$$\mathsf{sw}(a^*) \leqslant 2 \cdot \mathsf{sw}_{N^-}(a^*, \vec{u}), \quad \text{and} \quad \mathsf{sw}_{N^-}(a^*, \vec{u}) \leqslant 2 \cdot \mathsf{sw}_{N^-}(a^*, \vec{u}') + \frac{n}{m^2}$$

Now, we subdivide N^- based on their utility for a^*. Since $\mathsf{rank}_i(a^*) \geqslant k+1$ for all $i \in N^-$, $u_i(a^*) \leqslant \frac{1}{k+1}$ (otherwise agent's total utility for her top $k+1$ alternatives exceeds one). For $z \in \left[\lfloor \log_2 \frac{1}{k+1} \rfloor, \lfloor \log_2 \frac{1}{m^2} \rfloor\right]$, let $N^-_z = \{i \in N_i \mid u'_i(a^*) = 2^z\}$. For each group, let $\vec{u}'_z$ be the utility profile where agents $i \in N^-_z$ have utility of 2^z for their top k votes and value the rest at 0, and agents $i \in N \setminus N_z^-$ have utility of 0 for all agents. Note that this is a decomposition of the $\vec{u}'$, and each nonzero utilities is only considered in one of the $\vec{u}'_z$'s. Now, we invoke Lemma 9 with $\vec{s} \leftarrow \vec{s}_{k\text{-approval}}$, $\vec{\sigma} \leftarrow \vec{\sigma}$, $\vec{u} \leftarrow \vec{u}'_z$, $\ell \leftarrow k$, $k \leftarrow k$, $T \leftarrow N_z^-$, $\tau \leftarrow 2^z$, and we have

$$\mathsf{sw}(f(\vec{\sigma}), \vec{u}'_z) \geqslant 2^z \cdot \frac{|T|^2}{nm} \cdot \frac{k^2}{k} = 2^z \cdot \frac{|N_z^-|^2 \cdot k}{nm}.$$

Since the utilities above are disjoint, we can combine the bounds above and derive

$$\mathsf{sw}(f(\vec{\sigma}), \vec{u}') \geqslant \sum_z 2^z \cdot \frac{|N_z^-|^2 \cdot k}{nm}.$$

We use the above combined with the absolute welfare guarantee in Equation (6), to derive the distortion guarantee as follows.

$$\begin{aligned}
\mathsf{dist}(f(\vec{\sigma}), \vec{u}) &= \frac{\mathsf{sw}(a^*, \vec{u})}{\mathsf{sw}(f(\vec{\sigma}), \vec{u}))} \\
&\leqslant \frac{4 \cdot \mathsf{sw}_{N^-}(a^*) + \frac{2n}{m^2}}{\mathsf{sw}(f(\vec{\sigma})} \\
&\leqslant \frac{4 \cdot \sum_z 2^z \cdot |N_z^-| + \frac{2n}{m^2}}{\max\left\{\sum_z 2^z \cdot \frac{k}{nm} \cdot |N_z^-|^2 \ , \ g(n,m)\right\}} \\
&\leqslant \frac{2n}{m^2 \cdot g(n,m)} + 4 \sum_z \frac{2^z \cdot |N_z^-|}{\max\{2^z \cdot \frac{k}{nm} \cdot |N_z^-|^2, g(n,m)\}} \\
&\leqslant 8 + 4 \sum_z \frac{2^z \cdot |N_z^-|}{\max\{2^z \cdot \frac{k}{nm} \cdot |N_z^-|^2, g(n,m)\}} && (\text{since } g(n,m) \geqslant \frac{n}{4m^2}) \\
&\leqslant 8 + 4 \sum_z \min\left\{\frac{nm}{|N_z^-| \cdot k}, \frac{2^z |N_z^-|}{g(n,m)}\right\} \\
&\leqslant 8 + 4 \sum_z \sqrt{\frac{nm}{|N_z^-| \cdot k} \cdot \frac{2^z |N_z^-|}{g(n,m)}} && (\min\{a,b\} \leqslant \sqrt{ab}) \\
&\overset{(1)}{\leqslant} 8 + 4 \sqrt{\frac{nm}{k \cdot g(n,m)}} \cdot \sum\nolimits_{z=-\lceil \log_2(k+1) \rceil}^{\lceil \log_2(m^2) \rceil} (2^z)^{\frac{1}{2}} \\
&\leqslant 8 + \frac{4}{\sqrt{k}} \cdot \sqrt{\frac{nm}{g(n,m)}} \cdot \frac{4}{\sqrt{k+1}} && (2^z \leqslant \frac{1}{k+1}),
\end{aligned}$$

where (1) holds due to the following

$$\sum\nolimits_{z=-\lceil \log_2(k+1) \rceil}^{\lceil \log_2(m^2) \rceil} (2^z)^{\frac{1}{2}} \leqslant \frac{1}{\sqrt{k+1}} \cdot \sum_{j=0}^{\infty} 2^{-\frac{j}{2}} \leqslant \frac{1}{\sqrt{k+1}} \cdot 2 \sum_{j=0}^{\infty} 2^{-j} \leqslant \frac{4}{\sqrt{k+1}}.$$

By expanding $g(n,m)$, we have

$$\mathsf{dist}(f(\vec{\sigma}), \vec{u}) \leqslant 8 + \begin{cases} \frac{16}{k} \cdot \sqrt{\frac{nm}{nk/(m^2)}} = 16 \cdot \frac{m\sqrt{m}}{k\sqrt{k}}, & \text{if } k \in [1, \sqrt{m}], \\ \frac{16}{k} \cdot \sqrt{\frac{nm}{n/(4k(m-k+1))}} = 32 \cdot \frac{\sqrt{m(m-k+1)}}{\sqrt{k}}, & \text{if } k \in [\sqrt{m}, m]. \end{cases} \tag{8}$$

Now, by taking the pairwise maximum of Equations (7) and (8) we derive the following distortion upper bounds,

$$\mathsf{dist}(f(\vec{\sigma}), \vec{u}) \leqslant \begin{cases} O(\frac{m\sqrt{m}}{k\sqrt{k}}) & \text{if } k \in [1, m^{1/3}], \\ O(m) & \text{if } k \in [m^{1/3}, \sqrt{m}], \\ O(k\sqrt{m-k+1}) & \text{if } k \in [\sqrt{m}, m]. \end{cases}$$

□

A.2.3 Randomized Harmonic

Boutilier et al. [22] proposed the rule that executes the randomized harmonic rule with probability $1/2$ and selects an alternative uniformly at random with the remaining probability $1/2$. They main contribution was to show that this rule achieves $O(\sqrt{m \log m})$ distortion (tightness of this bound was shown by Ebadian et al. [19]), very close to their lower bound of $\Omega(\sqrt{m})$. We show that the latter half (uniform selection) of this rule, which is often critized as impractical, is largely unnecessary. Simply executing the randomized harmonic rule achieves a distortion of $\Theta(\sqrt{m} \log m)$, which is only a $\Theta(\sqrt{\log m})$ factor larger. Let us first prove a lower bound on its minimum welfare.

Lemma 13. *The minimum welfare of the randomized harmonic rule is* $\mathsf{min\text{-}sw}(f^{\mathsf{rand}}_{\vec{s}_{\mathsf{harmonic}}}) \geqslant \frac{n}{mH_m}$.

Proof. Fix a preference profile $\vec{\sigma}$ and utility profile $\vec{u}$. Since all agents give a score of at least $\frac{1}{m}$ to all the alternatives, $\mathsf{score}(a) \geqslant \frac{n}{m}$. Therefore, by $\|\vec{s}_{\mathsf{harmonic}}\|_1 = H_m$, we have $\Pr[a \in f^{\mathsf{rand}}_{\vec{s}_{\mathsf{harmonic}}}] = \frac{\mathsf{score}(a)}{n\|\vec{s}\|_1} \geqslant \frac{1}{mH_m}$, and $\mathsf{sw}(f(\vec{\sigma}), \vec{u}) \geqslant \frac{1}{mH_m} \cdot \sum_{a \in A} \mathsf{sw}(a) = \frac{n}{mH_m}$. □

Now, we show a well-known useful fact about harmonic scores.

Lemma 14 ([22])**.** *For any preference profile $\vec{\sigma}$, consistent utility profile $\vec{u} \in \mathcal{C}(\vec{\sigma})$, and alternative $a \in A$, we have* $\mathsf{sw}(a, \vec{u}) \leqslant \mathsf{score}(a, \vec{s}_{\mathsf{harmonic}})$.

Proof. Fix an agent $i \in N$. Then,

$$\mathsf{score}_i(a) = \frac{1}{\mathsf{rank}_i(a)} \geqslant u_i(a),$$

where the inequality is due to the unit-sum assumption, i.e. otherwise agent i's total utility for her top $\mathsf{rank}_i(a)$ alternatives exceeds one. Summing above for all agents yields the sought goal. □

Lemma 15. *The distortion of the randomized harmonic rule is $O(\sqrt{m}H_m)$.*

Proof. Fix a preference profile $\vec{\sigma}$ and utility profile $\vec{u}$. Let $a^* \in \arg\max_{a \in A} \mathsf{sw}(a, \vec{u})$ be an optimal alternative. By strategy 3 (absolute welfare guarantee), we have

$$\mathsf{dist}(f^{\mathsf{rand}}_{\vec{s}_{\mathsf{harmonic}}}(\vec{\sigma}), \vec{u}) \leqslant \frac{\mathsf{sw}(a^*)}{n/(mH_m)} \leqslant \frac{mH_m \cdot \mathsf{score}(a^*)}{n},$$

where the last inequality holds due to Lemma 14. Furthermore, by strategy 2 (probability of a^*), we have

$$\mathsf{dist}(f^{\mathsf{rand}}_{\vec{s}_{\mathsf{harmonic}}}(\vec{\sigma}), \vec{u}) \leqslant \frac{nH_m}{\mathsf{score}(a^*)}.$$

Putting the two together, we have

$$\begin{aligned} \mathsf{dist}(f^{\mathsf{rand}}_{\vec{s}_{\mathsf{harmonic}}}(\vec{\sigma}), \vec{u}) &\leqslant \min\left\{\frac{mH_m \cdot \mathsf{score}(a^*)}{n}, \frac{nH_m}{\mathsf{score}(a^*)}\right\} \\ &\leqslant \sqrt{\frac{mH_m \cdot \mathsf{score}(a^*)}{n} \cdot \frac{nH_m}{\mathsf{score}(a^*)}} = H_m\sqrt{m}, \end{aligned}$$ □

where the last inequality is due to $\min\{a, b\} \leqslant \sqrt{ab}$ for $a, b \geqslant 0$. Since this holds for all preference and utility profiles, $\mathsf{dist}(f^{\mathsf{rand}}_{\vec{s}_{\mathsf{harmonic}}}) \leqslant H_m \cdot \sqrt{m}$.

A.3 Lower Bounds

Next, we prove tightness of the distortion upper bounds obtained above. We do this by deriving a few general bounds.

Theorem 9. *The distortion of any randomized positional scoring rule $f^{\mathsf{rand}}_{\vec{s}}$ with $s_m \leqslant s_1/\sqrt{m}$ is $\Omega\big(\frac{\|\vec{s}\|_1}{s_1} \cdot \sqrt{t^*}\big)$, where $t^* = \arg\max_{t \in [m]} \left\{t \mid \sum_{j=1}^{t} s_{m-j+1} \leqslant s_1\right\}$.*

Proof. Consider the preference profile $\vec{\sigma}$ where alternative a^* appears as the top choice of $n/\sqrt{t^*}$ of the agents and the bottom of the list of the rest of them. Now consider set A_1 of t^* alternatives. Each member of A_1 appears as the top choice of $(n - n/\sqrt{t^*})/t^*$ agents. Whenever a member of A_1 is not the top choice of an agent, she appears in the bottom t^* places of his ranking. We create a symmetric setting among these alternatives. Now think of the utility profile $\vec{u}$ in which each agent has utility of 1 for his top choice and zero for the rest. We have $\mathsf{sw}(a^*, \vec{u}) = \frac{n}{\sqrt{t^*}}$, for $a \in A_1$, $\mathsf{sw}(a, \vec{u}) = \frac{n}{t^*}$ and the rest of the alternatives have zero social welfare. In addition for any positional scoring voting rule $f^{\mathsf{rand}}_{\vec{s}}$, we have

$$\Pr[f^{\mathsf{rand}}_{\vec{s}}(\vec{\sigma}) = a^*] = \frac{s_1 n/\sqrt{t^*} + s_m(n - n/\sqrt{t^*})}{n\|\vec{s}\|_1} \leqslant \frac{s_1 + s_m\sqrt{t^*}}{\|\vec{s}\|_1\sqrt{t^*}} \leqslant \frac{2s_1}{\|\vec{s}\|_1\sqrt{t^*}},$$

where the last inequality is due to the fact that $s_m\sqrt{t^*} \leqslant s_m\sqrt{m} \leqslant s_1$. In addition, for $a \in A_1$,

$$\Pr[f^{\mathsf{rand}}_{\vec{s}}(\vec{\sigma}) = a] = \frac{n\left(s_1 + \sum_{j=1}^{t} s_{m-j+1}\right)}{n\|\vec{s}\|_1|A_1|} \leqslant \frac{2s_1}{\|\vec{s}\|_1 t^*}.$$

That implies

$$\begin{aligned}
\mathbb{E}_{a\sim f^{\mathsf{rand}}_{\vec{s}}(\vec{\sigma})}[\mathsf{sw}(a,\vec{u})] &= \sum_{a\in A} \Pr[f^{\mathsf{rand}}_{\vec{s}}(\vec{\sigma}) = a]\cdot \mathsf{sw}(a,\vec{u}) \\
&= \Pr[f^{\mathsf{rand}}_{\vec{s}}(\vec{\sigma}) = a^*]\cdot \mathsf{sw}(a^*,\vec{u}) + \sum_{a\in A_1} \Pr[f^{\mathsf{rand}}_{\vec{s}}(\vec{\sigma}) = a]\cdot \mathsf{sw}(a,\vec{u}) \\
&\leqslant \frac{2s_1}{\|\vec{s}\|_1\sqrt{t^*}}\cdot\frac{n}{\sqrt{t^*}} + t^*\cdot\frac{2s_1}{\|\vec{s}\|_1 t^*}\cdot\frac{n}{t^*} \\
&\leqslant \frac{4n}{\|\vec{s}\|_1 t^*}.
\end{aligned}$$

$$\Rightarrow \mathsf{dist}(f^{\mathsf{rand}}_{\vec{s}}) \geqslant \mathsf{dist}(f^{\mathsf{rand}}_{\vec{s}}, \vec{\sigma}) \geqslant \frac{n/\sqrt{t^*}}{4ns_1/\|\vec{s}\|_1 t^*} \geqslant \frac{\|\vec{s}\|_1\sqrt{t^*}}{4s_1} = \Omega\left(\frac{\|\vec{s}\|_1}{s_1}\cdot\sqrt{t^*}\right). \qquad \square$$

Table 2: Lower bounds on the distortion of common randomized positional scoring rules, achieved by Theorem 9.

Rule name	**Scoring vector** $\vec{s}$	$\|\vec{s}\|_1$	s_1	t^*	**Lower bound**
Harmonic	$(1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{m})$	H_m	1	$> \frac{m}{2}$	$\Omega\left(\sqrt{m}\log m\right)$
Veto	$(1, 1, \ldots, 1, 0)$	$m-1$	1	2	$\Omega\left(m\right)$
Half harmonic, half uniform	$(1+\frac{H_m}{m}, \frac{1}{2}+\frac{H_m}{m}, \ldots, \frac{H_m+1}{m})$	$2H_m$	$1+\frac{H_m}{m}$	$> \frac{m}{2H_m}$	$\Omega\left(\sqrt{m}\log m\right)$
k-approval	$(\underbrace{1, \ldots, 1}_{k \text{ ones}}, 0, \ldots, 0)$	k	1	$m-k+1$	$\Omega\left(k\sqrt{m-k+1}\right)$
Plurality	$(1, 0, \ldots, 0)$	1	1	m	$\Omega\left(\sqrt{m}\right)$
Borda	$(m-1, m-2, \ldots, 0)$	$\frac{m(m-1)}{2}$	$m-1$	$> \sqrt{m}$	$\Omega\left(m^{\frac{5}{4}}\right)$

Corollary 10. *The lower bounds on the distortion of common randomized positional scoring rules implied by Theorem 9 are shown in Table 2. These are tight for the randomized harmonic rule, the randomized k-approval rule with $k = \Omega(\sqrt{m})$, and the randomized veto rule.*

Next, we derive another general lower bound, which would help us establish the tightness for some more randomized positional scoring rules.

Theorem 11. *Let $\vec{s}$ be a scoring vector with at most k non-zero values, i.e. $\forall j \in [k+1, m]$, $s_j = 0$. Then, $f^{\mathsf{rand}}_{\vec{s}}$ incurs a distortion of at least $\mathsf{dist}(f^{\mathsf{rand}}_{\vec{s}}) = \Omega\left(\frac{m\sqrt{m}}{k\sqrt{k}}\right)$.*

Proof. Assume that $m-1$ is divisible by 3. Let a^* be some alternative and partition the rest of the alternatives into three sets A_1, A_2, A_3, each of size $(m-1)/3$. In addition we partition the agents into two sets N_1 of size $n\sqrt{k/m}$ and $N_2 = N \setminus N_1$. Consider the preference profile where members of N_1 fill the top k positions of their rankings with members of A_1 and the rest of the agents have members of A_2 in the top k position of their rankings (the preference profile is symmetric among the members of each set, i.e., each member of A_i appears in the j-th position of $|N_i|/(|A_i| \cdot k)$ agents). Every agent has a^* in the $k+1$-th position, and all the members of A_3 after that (up to rank $k+1+m/3$). We do not care about the rest of the preference profile.

Now consider the utility profile $\vec{u}$ where members of N_1 have a utility of $1/(k+1)$ for their top $k+1$ alternatives and members of N_2 have a utility of $1/(k+1+m/3)$ for each of their top $k+1+m/3$ alternatives. We have

$$\mathsf{sw}(a^*, \vec{u}) = |N_1|\frac{1}{k+1} + |N_2|\frac{1}{k+1+m/3} \geqslant \frac{n\sqrt{\frac{k}{m}}}{k+1} > \frac{n}{\sqrt{km}},$$

$$a_1 \in A_1 \implies \mathsf{sw}(a_1, \vec{u}) = \frac{|N_1|}{|A_1|}\frac{k}{k+1} = \frac{nk\sqrt{\frac{k}{m}}}{m(k+1)/3} < \frac{3n\sqrt{k}}{m\sqrt{m}},$$

$$a_2 \in A_2 \implies \mathsf{sw}(a_2, \vec{u}) = \frac{|N_2|}{|A_2|}\frac{k}{k+1+m/3} = \frac{nk\left(1-\sqrt{\frac{k}{m}}\right)}{m(k+1+m/3)/3} < \frac{9nk}{m^2}.$$

On the other hand, if we consider the probability given to each candidate we have:

$$\begin{aligned}
&\Pr_{a\sim f^{\mathsf{rand}}_{\vec{s}}(\vec{\sigma})}[a = a^*] = 0,\\
&\Pr_{a\sim f^{\mathsf{rand}}_{\vec{s}}(\vec{\sigma})}[a \in A_1] = \frac{|N_1|}{n} = \sqrt{\frac{k}{m}},\\
&\Pr_{a\sim f^{\mathsf{rand}}_{\vec{s}}(\vec{\sigma})}[a \in A_2] < 1,
\end{aligned}$$

which means

$$\mathbb{E}_{a\sim f^{\mathsf{rand}}_{\vec{s}}(\vec{\sigma})}[\mathsf{sw}(a, \vec{u})] \leqslant \sqrt{\frac{k}{m}} \cdot \frac{3n\sqrt{k}}{m\sqrt{m}} + 1 \cdot \frac{9nk}{m^2} = \frac{12nk}{m^2},$$

and that implies:

$$\mathsf{dist}(f^{\mathsf{rand}}_{\vec{s}}(\vec{\sigma}), \vec{u}) \geqslant \frac{\mathsf{sw}(a^*, \vec{u})}{\mathbb{E}_{a\sim f^{\mathsf{rand}}_{\vec{s}}(\vec{\sigma})}[\mathsf{sw}(a, \vec{u})]} \geqslant \frac{\frac{n}{\sqrt{km}}}{\frac{12nk}{m^2}} = \frac{m\sqrt{m}}{12k\sqrt{k}} \in \Omega\left(\frac{m\sqrt{m}}{k\sqrt{k}}\right).$$ □

The theorem above immediately shows that our analysis for randomized k-approvals rules with $k \in [m^{1/3}]$ in Lemma 12 is in fact tight.

Corollary 12. *The randomized k-approval rule with $k \in [m^{1/3}]$ incurs a distortion of $\Omega(\frac{m\sqrt{m}}{k\sqrt{k}})$.*

The results above do not match our upper bound of $O(m)$ for the randomized k-approval rule when $k \in [m^{1/3}, \sqrt{m}]$. In the following, we separately establish a matching lower bound of $\Omega(m)$ for this case as well.

Lemma 16. *The randomized k-approval rule with $k \in [m^{1/3}, \sqrt{m}]$ incurs a distortion of $\Omega(m)$.*

Proof. Suppose $m-1$ is divisible by 2. Let a^* be an alternative. Partition the rest of the alternatives into two subsets of A_1, A_2 each of size $(m-1)/2$. Then, construct a preference profile as follows. Let $\frac{nk}{m}$ agents N_1 have a^* as their top vote. Divide the other top $k-1$ top alternatives of N_1 and all the top k alternatives of $N \setminus N_1$ among A_1. This way,

$$\forall a \in A_1,\ \mathsf{score}(a) \leqslant \frac{nk}{|A_1|} \quad \Rightarrow \quad \Pr[a] \leqslant \frac{1}{|A_1|},$$

and $\Pr[a^*] = \frac{|N_1|}{nk}$. Divide the $k+1$ to $m/2$-th ranks of all voters among alternatives A_2 and fill the bottom of the ranking arbitrarily. This way, for all $a \in A_2$, $\Pr[a] = 0$ since they have a score of 0.

Furthermore, suppose N_1 have a utility of 1 for a^* and 0 for the rest, and $N \setminus N_1$ have a utility of $\frac{2}{m}$ for their top $\frac{m}{2}$ alternatives. This way,

$$\forall a \in A_1 \ \mathsf{sw}(a) \leqslant \frac{nk}{|A_1|} \cdot \frac{2}{m}.$$

Then,

$$\begin{aligned} \mathsf{sw}(f^{\mathsf{rand}}_{\vec{s}_{k\text{-approval}}}) &\leqslant \Pr[a^*] \cdot \mathsf{sw}(a^*) + \sum_{a \in A_1} \Pr[a] \cdot \mathsf{sw}(a) + \sum_{a \in A_1} \Pr[a] \cdot \mathsf{sw}(a) \\ &\leqslant \frac{|N_1|}{nk} \cdot |N_1| + |A_1| \cdot \frac{1}{|A_1|} \cdot \frac{2nk}{|N_1| \cdot m}. \end{aligned}$$

Hence,

$$\begin{aligned} \mathsf{dist}(f^{\mathsf{rand}}_{\vec{s}_{k\text{-approval}}}(\vec{\sigma})) &\geqslant \frac{|N_1|}{\frac{1}{nk} \cdot |N_1|^2 + \frac{2nk}{|N_1| \cdot m}} \\ &\geqslant \min\left\{\frac{nk}{|N_1|}, \frac{|N_1| m}{2nk}\right\}, \end{aligned}$$

which for the choice of $|N_1| = \frac{nk}{m}$ gives a lower bound of $\Omega(m)$. $\square$

Finally, it remains to show that our lower bounds on minimum welfare of these randomized positional scoring rules are also tight. This follows easily because our distortion upper bounds are essentially derived as a function of the minimum welfare bounds, and one can check that in each case an asymptotically better lower bound on minimum welfare would translate to an asymptotically better upper bound on distortion, which is not possible because we have already established tightness of our distortion bounds for the common randomized positional scoring rules.

Corollary 13. *The minimum welfare bounds presented in Table 1 for the randomized versions of plurality, Borda, harmonic, veto, and k-approval rules are asymptotically tight.*

A.4 Randomized Approval Mixture Rules

As a step towards analyzing the distortion of randomized positional scoring rules for any scoring vector, we present our distortion bounds for *approval mixture scores*, which are tight up to logarithmic factors.

Definition 14 (Approval Mixture Scores)**.** *For $k_1 < k_2 < \ldots < k_R \in [m]$, the* approval mixture score *denoted by $\{k_1, \ldots, k_R\}$-$\mathsf{mix\text{-}approval}$ is defined as*

$$\vec{s}_{\{k_1,\ldots,k_R\}\text{-mix-approval}} = \frac{1}{R} \sum_{r=1}^{R} \frac{\vec{s}_{k_r\text{-approval}}}{\|\vec{s}_{k_r\text{-approval}}\|_1},$$

that is the uniform mixture of the k_r-$\mathsf{approval}$ scores.

This class of scores generalizes our results for randomized k-approvals (hence, plurality, veto).

A.4.1 Minimum Welfare Analysis

Lemma 17. *Fix any constant $\epsilon > 0$ and $k_1 < k_2 \ldots < k_R \in [(1-\epsilon)m]$. The minimum welfare of the randomized approval mixture rule with $\vec{s}_{\{k_1,\ldots,k_R\}\text{-mix-approval}}$ scoring vector is*

$$\mathsf{min\text{-}sw}(f^{\mathsf{rand}}_{\vec{s}_{\{k_1,\ldots,k_R\}\text{-mix-approval}}}) = \Omega\left(\frac{n}{m} \cdot \frac{1}{R \log^2 m} \min\left\{\frac{1}{k_1}, \sqrt{\frac{k_1}{k_2}}, \sqrt{\frac{k_2}{k_3}}, \cdots, \sqrt{\frac{k_{R-1}}{k_R}}, \frac{k_R}{m}\right\}\right).$$

Proof. Fix any preference profile $\vec{\sigma}$ and consistent utility profile $\vec{u} \in \mathcal{C}(\vec{\sigma})$. For conciseness, we use $\vec{s} = \vec{s}_{\{k_1,\ldots,k_R\}\text{-mix-approval}}$.

Similar to the analysis in Lemma 11, to obtain a lower bound, by Lemma 6 we may assume without loss of generality that agents have dichotomous utilities, i.e. $u_i \in \{\mathbb{1}_{i,\ell}\}_{\ell \in [m]}$. To obtain a lower bound, we construct a new partial utility profile $\vec{u}'$ by rounding down the utilities to the nearest power of two, i.e. $u'_i \in \frac{1}{2} \cdot \{\mathbb{1}_{i,2^z}\}_{z \in [\lfloor \log_2 m \rfloor]}$. Now, we partition agents by their utility vectors to $\log m$ many groups, i.e. for $z \in [\lfloor \log_2 m \rfloor]$, define $N_z = \{i \mid u'_i = \mathbb{1}_{i,2^z}\}$. Furthermore, decompose $\vec{u}'$ to $\{\vec{u}'_z\}_{z \in [\log m]}$ where $\vec{u}'_z$ is $\vec{u}'$ except that agents in $N \setminus N_z$ have 0 utility for all alternatives. Now, we derive welfare lower bounds for each group by invoking Lemma 9 with $\vec{\sigma} \leftarrow \vec{\sigma}$, $\vec{u} \leftarrow \vec{u}'_z$, $\ell \leftarrow \frac{1}{2} \cdot \frac{1}{2^z}$, $T \leftarrow N_z$, $\tau \leftarrow 2^z$. We assign parameter k based on the different cases below.

Case $2^z \in [k_1]$. Invoking Lemma 9 with above and $k \leftarrow k_1$, we have

$$\mathsf{sw}(f^{\mathsf{rand}}_{\vec{s}_{k_1\text{-approval}}}, \vec{u}'_z) \geqslant \frac{1}{2} \cdot \frac{1}{2^z} \cdot \frac{|N_z|^2}{nm} \cdot \frac{(2^z)^2}{k_1} = \frac{|N_z|^2 \cdot 2^z}{2nmk_1}.$$

Since $2^z \geqslant 1$,

$$\mathsf{sw}(f_{\vec{s}}) \geqslant \frac{1}{R} \cdot \mathsf{sw}(f^{\mathsf{rand}}_{\vec{s}_{k_1\text{-approval}}}) \geqslant \frac{|N_z|^2}{2R \cdot nm} \cdot \frac{1}{k_1}.$$

Case $2^z \in [k_r, k_{r+1}]$. Invoking Lemma 9 with above and $k \leftarrow k_r$, we have

$$\mathsf{sw}(f^{\mathsf{rand}}_{\vec{s}_{k_r\text{-approval}}}, \vec{u}'_z) \geqslant \frac{1}{2} \cdot \frac{1}{2^z} \cdot \frac{|N_z|^2}{nm} \cdot \frac{(\min\{2^z, k_r\})^2}{k_r} = \frac{|N_z|^2}{2nm} \cdot \frac{k_r}{2^z}.$$

By invoking Lemma 9 for $k \leftarrow k_{r+1}$, we have

$$\mathsf{sw}(f^{\mathsf{rand}}_{\vec{s}_{k_{r+1}\text{-approval}}}, \vec{u}'_z) \geqslant \frac{1}{2} \cdot \frac{1}{2^z} \cdot \frac{|N_z|^2}{nm} \cdot \frac{(\min\{2^z, k_{r+1}\})^2}{k_{r+1}} = \frac{|N_z|^2}{2nm} \cdot \frac{2^z}{k_{r+1}}.$$

Thus,

$$\begin{aligned}\mathsf{sw}(f^{\mathsf{rand}}_{\vec{s}}, \vec{u}'_z) &\geqslant \frac{1}{R} \cdot \left(\mathsf{sw}(f^{\mathsf{rand}}_{\vec{s}_{k_r\text{-approval}}}, \vec{u}'_z) + \mathsf{sw}(f^{\mathsf{rand}}_{\vec{s}_{k_{r+1}\text{-approval}}}, \vec{u}'_z)\right) \\ &\geqslant \frac{1}{R} \cdot \frac{|N_z|^2}{2nm} \left(\frac{k_r}{2^z} + \frac{2^z}{k_{r+1}}\right) \overset{(1)}{\geqslant} \frac{|N_z|^2}{R \cdot nm} \cdot \sqrt{\frac{k_r}{k_{r+1}}},\end{aligned}$$

where (1) follows from the AM-GM inequality.

Case $2^z \in [k_R, m]$. Invoking Lemma 9 with above and $k \leftarrow k_R$, we have

$$\mathsf{sw}(f^{\mathsf{rand}}_{\vec{s}_{k_1\text{-approval}}}, \vec{u}'_z) \geqslant \frac{1}{2} \cdot \frac{1}{2^z} \cdot \frac{|N_z|^2}{nm} \cdot \frac{(\min\{2^z, k_R\})^2}{k_R} = \frac{|N_z|^2}{2nm} \cdot \frac{k_R}{2^z}.$$

Since $2^z \leqslant m$, we have

$$\mathsf{sw}(f_{\vec{s}}) \geqslant \frac{1}{R} \cdot \mathsf{sw}(f^{\mathsf{rand}}_{\vec{s}_{k_2\text{-approval}}}) \geqslant \frac{|N_z|^2}{2R \cdot nm} \cdot \frac{k_R}{m}.$$

For at least one value of $z \in [\log_2 m]$ we have $|N_z| \geqslant \frac{n}{\log m}$, by the pigeon-hole principle. Thus, we can take the minimum of the three cases above with $|N_z| \geqslant \frac{n}{\log m}$ to obtain a lower bound as follow

$$\begin{aligned}\mathsf{sw}(f^{\mathsf{rand}}_{\vec{s}}) &\geqslant \frac{\left(\frac{n}{\log m}\right)^2}{2R \cdot nm} \min\left\{\frac{1}{k_1}, \min_{r \in [R-1]}\left\{\sqrt{\frac{k_r}{k_{r+1}}}\right\}, \frac{k_R}{m}\right\} \\ &= \frac{n}{2R \cdot m \log^2 m} \cdot \min\left\{\frac{1}{k_1}, \sqrt{\frac{k_1}{k_2}}, \sqrt{\frac{k_2}{k_3}}, \ldots, \sqrt{\frac{k_{R-1}}{k_R}}, \frac{k_R}{m}\right\}.\end{aligned}$$

□

A.4.2 Distortion Analysis

Theorem 15. *Fix any constant $\epsilon > 0$ and $k_1 < k_2 \ldots < k_R \in [(1-\epsilon)m]$. The distortion of the randomized approval mixture rule with $\vec{s}_{\{k_1,\ldots,k_R\}\text{-mix-approval}}$ scoring vector is*

$$O\left(R \log m \cdot \sqrt{\frac{n}{g}} \cdot \sqrt{\max\left\{k_1, \min\left(\frac{k_2}{k_1}, \frac{m}{(k_1)^2}\right), \ldots, \min\left(\frac{k_R}{k_{R-1}}, \frac{m}{(k_{R-1})^2}\right), \frac{m}{(k_R)^2}\right\}}\right),$$

where $g = \mathsf{min\text{-}sw}(f^{\mathsf{rand}}_{\vec{s}_{\{k_1,\ldots,k_R\}\text{-mix-approval}}})$.

Proof. Fix any preference profile $\vec{\sigma}$ and consistent utility profile $\vec{u} \in \mathcal{C}(\vec{\sigma})$. Let $a^* \in \arg\max_{a \in A} \mathsf{sw}(a, \vec{u})$ be an optimal alternative.

Partition the agents based on their score to a^*, i.e., for $r \in [1, R]$ define $N_r = \{i \in N \mid \mathsf{rank}_i(a^*) \in [k_{r-1}, k_r]\}$ ($k_0 = 1$) and let $N_{R+1} = \{i \in N \mid \mathsf{rank}_i(a^*) \in [k_R, m]\}$ be the agents who give score of 0 to a^*. Furthermore,

$$\mathsf{sw}(a^*) = \sum_{r \in [R]} \mathsf{sw}_{N_r}(a^*) \leqslant R \cdot \max_{r \in [R]} \mathsf{sw}_{N_r}(a^*). \tag{9}$$

Suppose the maximum above is achieved at N_{r^*}. Next, we show upper bounds on distortion based on the value of r^*, and report the maximum of all as a distortion upper bound. Before doing so, to obtain an upper bound on the distortion, we round down agents utility to the nearest power of two, ignore utilities less than $\frac{1}{m^2}$ (replace with 0). Call the new utility profile $\vec{u}'$. Then,

$$\mathsf{sw}(a^*, \vec{u}) \leqslant 2 \cdot \mathsf{sw}(a^*, \vec{u}') + \frac{n}{m^2} \quad \text{and} \quad \mathsf{sw}(f(\vec{\sigma}), \vec{u}) \leqslant \mathsf{sw}(f(\vec{\sigma}), \vec{u}')$$

$$\Rightarrow \quad \mathsf{dist}(f(\vec{\sigma}), \vec{u}) \leqslant \frac{n/m^2}{\mathsf{sw}(f(\vec{\sigma}), \vec{u})} + 2 \cdot \mathsf{dist}(f(\vec{\sigma}), \vec{u}') \leqslant 4 + 2 \cdot \mathsf{dist}(f(\vec{\sigma}), \vec{u}')$$

where the last inequality is due to Lemma 7.

Strategy 1 (Welfare above a^*). Now, we subdivide each group N_r based on their utility for a^* and derive welfare lower bounds by invoking Lemma 9. Fix $r \in [2, R+1]$ and for $z \in \left[\lfloor \log_2 \frac{1}{k_{r-1}+1} \rfloor, \lfloor \log_2 \frac{1}{m^2} \rfloor\right]$, let $N_{r,z} = \{i \in N_r \mid u'_i(a^*) = 2^z\}$. Since for agents $i \in N_r$, $\mathsf{rank}_i(a^*) \geqslant k_{r-1} + 1$, $u'_i(a^*) \leqslant \frac{1}{k+1}$ (otherwise, the unit-sum assumption is violated since her total utility for her top $k_{r-1} + 1$ alternatives exceeds one). Now, for each subgroup, we invoke Lemma 9 with $\vec{s} \leftarrow \vec{s}_{k_{(r-1)}\text{-approval}}$, $T \leftarrow N_{r,z}$, $\vec{\sigma} \leftarrow \vec{\sigma}$, $\tau \leftarrow 2^z$, $\ell \leftarrow k_{r-1}$, $k \leftarrow k_{r-1}$, and we have

$$\mathsf{sw}(f(\vec{\sigma}), \vec{u}'_z) \geqslant 2^z \cdot \frac{|T|^2}{nm} \cdot \frac{(k_{r-1})^2}{k_{r-1}} = 2^z \cdot \frac{|N_{r,z}|^2 \cdot k_{r-1}}{nm} \tag{10}$$

Strategy 2 (Probability of a^*). Following strategy 2, for $r \in [1, R]$, we have

$$\Pr[a^* \in f(\vec{\sigma})] \geqslant \frac{1}{R} \cdot \frac{|N_r|}{nk_r} \quad \Rightarrow \quad \mathsf{dist}(f(\vec{\sigma}), \vec{u}') \leqslant R \cdot \frac{nk_r}{|N_r|}. \tag{11}$$

Strategy 3 (Absolute Welfare Guarantee). Following strategy 3 and by Lemma 17, we have

$$\mathsf{sw}(f) \geqslant \text{min-}\mathsf{sw}_{n,m}(f) = g(n, m) \quad \Rightarrow \quad \mathsf{dist}(f(\vec{\sigma}), \vec{u}') \leqslant \frac{\mathsf{sw}(a^*)}{g(n, m)}. \tag{12}$$

We are ready to show distortion upper bounds based on the choice of r^*.

Case $r^* = 1$. In this case, we only apply strategies 2 and 3 to N_{r^*} (not the subgroups). By Equation (9) we have $\mathsf{sw}(a, \vec{u}) \leqslant R \cdot \mathsf{sw}_{N_1}(a, \vec{u})$. Furthermore, $\mathsf{sw}_{N_1}(a^*) \leqslant |N_1|$. Then, by Equations (11) and (12),

$$\mathsf{dist}(f(\vec{\sigma}), \vec{u}') \leqslant \min\left\{\frac{R \cdot |N_1|}{g(n, m)}, R \cdot \frac{nk_1}{|N_1|}\right\} \leqslant R \cdot \sqrt{\frac{nk_1}{g(n, m)}}. \tag{13}$$

Case $r^* \in [2, R]$. We use all the three strategies here. By the pigeonhole principle, there exists a $z^* \in [\log m^2]$ such that $\mathsf{sw}_{N_{r^*}}(a^*) \leqslant 2 \log m \cdot \mathsf{sw}_{N_{r^*,z^*}}(a^*) \leqslant 2 \log m \cdot 2^z \cdot |N_{r^*,z^*}|$. Thus, by Equation (10) in strategy 1,

$$\mathsf{dist}(f(\vec{\sigma}), \vec{u}') \leqslant 2R \log m \cdot \frac{2^z \cdot |N_{r^*,z^*}|}{2^z \cdot \frac{|N_{r^*,z^*}|^2 \cdot k_{(r^*-1)}}{nm}} = 2R \log m \cdot \frac{nm}{|N_{r^*,z^*}| \cdot k_{(r^*-1)}}$$

Combined with Equation (11) in strategy 2 we get

$$\mathsf{dist}(f(\vec{\sigma}), \vec{u}') \leqslant 2R \log m \cdot \min\left\{\frac{nm}{|N_{r^*,z^*}| \cdot k_{(r^*-1)}}, \frac{nk_{r^*}}{|N_{r^*,z^*}|}\right\}$$

Putting together with Equation (12) in strategy 3 we get

$$\mathsf{dist}(f(\vec{\sigma}), \vec{u}') \leqslant \min\left\{2R\log m \cdot \frac{2^{z^*}\cdot |N_{r^*,z^*}|}{g(n,m)}, 2R\log m \cdot \min\left\{\frac{nm}{|N_{r^*,z^*}|\cdot k_{(r^*-1)}}, \frac{nk_{r^*}}{|N_{r^*,z^*}|}\right\}\right\}.$$
$$\leqslant 2R\log m \cdot \sqrt{\frac{n}{g(n,m)}\cdot\frac{1}{k_{r-1}}\cdot\min\left\{\frac{m}{k_{r^*-1}}, k_{r^*}\right\}}$$

Case $r^* = R+1$. This case follows exactly like the previous case except that we cannot apply strategy 2, since probability of selection and the score of a^* from ranks below k_R is 0. Thus,

$$\mathsf{dist}(f(\vec{\sigma}), \vec{u}') \leqslant 2R\log m \cdot \sqrt{\frac{n}{g(n,m)}\cdot\frac{1}{k_{r-1}}\cdot\frac{m}{k_R}}.$$

Now, we report an upper bound on distortion by taking the maximum of all cases. Hence,

$$\mathsf{dist}(f(\vec{\sigma}), \vec{u}) \leqslant 4 + 2\cdot \mathsf{dist}(f(\vec{\sigma}), \vec{u}')$$
$$\leqslant 4 + 4\cdot R\log m\cdot\sqrt{\frac{n}{g(n,m)}}\cdot$$
$$\sqrt{\max\left\{k_1, \min\left\{\frac{k_2}{k_1}, \frac{m}{(k_2)^2}\right\}, \ldots \min\left\{\frac{k_R}{k_{R-1}}, \frac{m}{(k_R)^2}\right\}, \frac{k_R}{m}\right\}}. \qquad \square$$

Deriving Distortion of the Randomized t-Truncated Harmonic Rules. Next, we apply the result above to a natural extension of the harmonic scores, which was recently used by Gkatzelis et al. [27]. Define the *t-truncated harmonic* scoring vector as follows

$$\vec{s}_{t\text{-harmonic}} = (1, 1/2, \ldots, 1/t, 0, \ldots, 0),$$

for which the first t scores is equal to the $\vec{s}_{\mathsf{harmonic}}$ and the rest is 0. For simplicity, suppose t is a power of two. By rounding the scores, we have

$$\vec{s}'_{t\text{-harmonic}} = (1, 1/2, 1/4, 1/4, 1/8, \ldots, 1/t, 0, \ldots, 0),$$

Now, take the $\{1, 2, \ldots, \log_2 t\}$-mix-approval scoring vector. Since $s_i = \frac{1}{\log t}\sum_{j=\lceil\log i\rceil}^{\log t}\frac{1}{2^j}$ It holds that

$$s'_i \leqslant s_i \leqslant \log t \cdot s'_i,$$

by Lemma 1 we can approximate $\mathsf{dist}(f^{\mathsf{rand}}_{\vec{s}_{t\text{-harmonic}}})$ up to $O(\log m)$ factor by analyzing the randomized $\{1, 2, \ldots, \log_2 t\}$-mix-approval rule. To analyze this rule, we utilize the bounds from Theorem 15 and Lemma 17. Note that $\frac{k_r}{k_{r-1}} = 2$ for all $r \in [2, R]$, $k_1 = 1$, and $k_R = t$. Then,

$$\mathsf{min\text{-}sw}(f^{\mathsf{rand}}_{\{1,2,\ldots,\log_2 t\}\text{-mix-approval}}) = \Omega\left(\frac{n}{m}\cdot\frac{1}{\log t\cdot\log^2 m}\cdot\min\left\{1, \sqrt{\frac{1}{2}}, \sqrt{\frac{1}{2}}, \ldots, \sqrt{\frac{1}{2}}, \frac{t}{m}\right\}\right)$$
$$= \Omega\left(\frac{n}{m}\cdot\frac{1}{\log t\cdot\log^2 m}\cdot\frac{t}{m}\right) = g$$

Furthermore, $\mathsf{dist}(f^{\mathsf{rand}}_{\{1,2,\ldots,\log_2 t\}\text{-mix-approval}})$ is at most

$$O\left(\log t\cdot\log m\cdot\sqrt{\frac{n}{g}}\cdot\sqrt{\max\left\{1, \min(2, m), \min(2, \frac{m}{2^2}), \min(2, \frac{m}{4^2}), \ldots, \min(2, \frac{m}{t^2}), \frac{m}{t^2}\right\}}\right)$$
$$= O\left(\log t\cdot\log m\cdot\sqrt{\frac{n}{g}}\cdot\max\left\{1, \sqrt{\frac{m}{t^2}}\right\}\right)$$
$$= O(\mathrm{polylog}(t, m))\cdot\sqrt{\frac{m^2}{t}}\cdot\max\left\{1, \sqrt{\frac{m}{t^2}}\right\})$$
$$= \begin{cases} O(\frac{m\sqrt{m}}{t\sqrt{t}}\cdot\mathrm{polylog}(t, m)) & \text{if } t \leqslant \sqrt{m}, \\ O(\frac{m}{\sqrt{t}}\cdot\mathrm{polylog}(t, m)) & \text{if } t \in [\sqrt{m}, m]. \end{cases}$$

B Random k-Committee Member Rules

B.1 Lower Bound

We begin by proving a lower bound on the distortion of *any* randomized rule with a support size of at most k, including, but not limited to, the ones that pick the winner uniformly at random from this support.

Theorem 16. *For $k \in [m/3]$, any randomized voting rule that has a support size of at most k incurs a distortion of at least $\Omega(m^2/k^2)$.*

Proof. Consider voting rule f, which has a support of size at most k. We create a preference profile $\vec{\sigma}$ and argue that the distortion of any such voting rule on this preference profile is at least ${}^{m^2}/_{6k^2}$. Partition the alternatives into two sets, A_1 with size $m-k$, and A_2 with size k. In $\vec{\sigma}$ each member of A_1 appears as the top choice of $n/(m-k)$ agents, and each member of A_2 appears as the second choice of n/k agents, in a way that each pair appears in the top-2 positions of $n/(k(m-k))$ agents. If the rule gives positive probability to all members of A_2, then consider the utility profile where each agent has utility 1 for her top choice and 0 for the rest. The social welfare of all alternatives with positive probability is zero, hence the distortion of this voting rule is unbounded. Now we can assume that there exists $a^* \in A_2$, that has probability zero in the output. Let $A_1^+ \subseteq A_1$ be the set of alternatives in A_1 that get positive probability in $f(\vec{\sigma})$, and $A_1^- = A_1 \setminus A_1^+$. Consider the utility profile agents that have members of A_1^+ as their top choice has the utility of $1/m$ for all the alternatives, agents that have members of A_1^- as their top choice and a^* as their second choice have utility $1/2$ for their top 2 alternatives and 0 for the rest, and other agents have utility 1 for their top choice and 0 for the rest. In this utility profile, we have

$$\mathsf{sw}(a^*) \geqslant \frac{1}{2} \cdot \frac{|A_1^-|n}{k(m-k)} \geqslant \frac{(m-2k)n}{2k(m-k)} \geqslant \frac{n}{4k},$$

and for $a' \in A_1^+ \cup A_2 \setminus \{a^*\}$:

$$\mathsf{sw}(a') \leqslant \frac{1}{m} \cdot \frac{|A_1^+|n}{m-k} \leqslant \frac{3k}{2m^2}.$$

That gives a bound on the social welfare of any alternative with positive probability and implies that $\mathbb{E}_{a\sim f(\vec{\sigma})}[\mathsf{sw}(c)] \leqslant \frac{3k}{2m^2}$ which means $\mathsf{dist}(f(\vec{\sigma}), \vec{\sigma}) \geqslant \frac{n/4k}{3k/2m^2} = \frac{m^2}{6k^2}$. □

We are now ready to prove the desired lower bound for random k-committee member rules.

Theorem 3. *For $k \in [m]$, any random k-committee member rule incurs $\Omega(\max(k, m^2/k^2))$ distortion. This lower bound is at least $\Omega(m^{2/3})$ for all k.*

Proof. First, consider the preference profile where all the agents have the same ranking with a^* as their top choice. Now consider the utility profile where each agent has utility 1 for his top choice and zero for the rest. In this utility profile, if the selected committee does not include a^* then the distortion is unbounded, and if a^* is part of the committee then the distortion is k. On the other hand, by Theorem 16 we have the lower bound of $\Omega(m^2/k^2)$ which gives us the desired bound of $\Omega\left(\max\left(k, \frac{m^2}{k^2}\right)\right)$. □

B.2 Upper Bound

Our algorithm uses the notion of *Approximately Stable Committees* introduced by Jiang et al. [48] and used by Ebadian et al. [19] in the design of the *Stable Committee Rule*.

Definition 17 (Approximately Stable Committee[48])**.** *A committee $X \subseteq A$ of size k is α-stable w.r.t. preference profile $\vec{\sigma}$ if for any candidate $a \in A$ we have $|i \in N : a \succ_i X| \leqslant \alpha \cdot \frac{n}{k}$, where $a \succ_i X$ means that voter i prefers a to every member of X.*

Theorem 18 ([48])**.** *Given any preference profile and $k \in [m]$, a 16-stable committee of size k exists.*

ALGORITHM 1: Top-Biased Stable k-Committee

Input: Preference profile $\vec{\sigma}$, Committee size k
Output: Shortlisted Committee of size k
$A_{\mathsf{stable}} \leftarrow$ an approximately stable committee of size $k/3$
$A_{\mathsf{plu}} \leftarrow$ top $k/3$ alternatives with the highest plurality score
$N_1 \leftarrow$ voters whose top vote is among A_{plu}
$A_{\mathsf{greedy}} \leftarrow \emptyset$
for $i \in N_1$ **do** $\mathsf{hits}(i) = 1$
for $t = 1$ *to* $k/3$ **do**
  $\overline{A} \leftarrow A \setminus (A_{\mathsf{plu}} \cup A_{\mathsf{greedy}} \cup A_{\mathsf{stable}})$
  for $i \in N_1$ **do**
    $S_i \leftarrow$ top $m/(\mathsf{hits}(i)+1)$ alternatives of i among $\overline{A}$
  $a^* \leftarrow \arg\max_{a \in \overline{A}} |\{i \in N_1 \mid a \in S_i\}|$
  $A_{\mathsf{greedy}} \leftarrow A_{\mathsf{greedy}} \cup \{a^*\}$
  for $i \in N_1$ *and* $a^* \in S_i$ **do**
    $\mathsf{hits}(i) \leftarrow \mathsf{hits}(i) + 1$
return $A_{\mathsf{plu}} \cup A_{\mathsf{greedy}} \cup A_{\mathsf{stable}}$

Algorithm 1 starts with selecting an approximately stable committee of size $k/3$ and $k/3$ alternatives with the highest plurality scores. Then, to gain more social welfare from the alternatives N_1 whom top alternative is selected by the algorithm, it proceeds as follows. Initialize $\mathsf{hits}(i)$ to one. The point of the hits count is that we can ensure a welfare of $\frac{\mathsf{hits}(i)}{m}$ by when the procedure ends. Initially, we can ensure a welfare of $\frac{1}{m}$ for all agents in N_1. Next, the sets S_i are set to be the top $m/2$ alternatives of agents in N_1. We will show that picking any of these alternatives will guarantee a welfare of at least $\frac{2}{m}$ for a user. The algorithm makes a greedy choice a^* that hits the highest number of agents and adds that alternative to the selected committee and increases the hit number of the agents hit by a^*. After the second hit, it updates the sets S_i to be the top $m/3$ remaining alternatives of the hit agent. Then $m/4$ after the thirds, and so forth. The algorithm selects the remaining $k/3$ alternatives of the committee as described.

B.2.1 Absolute Welfare Lower Bound

For the absolute welfare lower bound analysis, we will show that for each voter $i \in N_1$ we can guarantee a utility of at least $\mathsf{hits}(i)/m$. First, we present a helpful technical lemma.

Lemma 18. *Consider agent $i \in N$ with preference profile σ_i and utility function u_i, and a set $A' \subseteq A$ of alternatives. If A' includes the top choice of i, and for any $2 \leqslant \ell \leqslant t$ at least $t - \ell + 2$ members of A' appear in the top m/ℓ choices of i, then the total utility of i for members of A' is at least t/m, i.e. $\sum_{a \in A'} u_i(a) \geqslant t/m$.*

Proof. We can write u_i as a weighted sum of m dichotomous utility functions, i.e. $u_i = \sum_{j=1}^{m} \alpha_j \mathbb{1}_{i,j}$, where $\sum_{j=1}^{m} \alpha_j = 1$. For $j \in [m]$ we define

$$g(j) = \begin{cases} t - \lceil m/j \rceil + 2 & j \geqslant m/t, \\ 1 & o.w., \end{cases}$$

as a lower bound on the number of members of A' that appear in the top j positions of this agent's preference ranking. Each of these alternatives gets $1/j$ utility in $\mathbb{1}_{i,j}$. If we sum it up for all values of j, we have

$$u_i(a) \geqslant \sum_{j \geqslant \mathsf{rank}_i(a)} \alpha_j / j \implies \sum_{a \in A'} u_i(a) \geqslant \sum_{j=1}^{m} \alpha_j \frac{g(j)}{j}.$$

For $j < m/t$, $g(j)/j \geqslant t/m$, and after that $\frac{t - \lceil m/j \rceil + 2}{j}$ is increasing up to $j = {}^{2m}/_{t+2}$, decreasing afterwards, and is minimized at $j = m$. That means $g(j)/j$ is always greater than t/m, which implies

$$\sum_{a \in A'} u_i(a) \geqslant \sum_{j=1}^{m} \alpha_j \frac{g(j)}{j} \geqslant \frac{t}{m} \sum_{j=1}^{m} \alpha_j = \frac{t}{m}.$$

□

Theorem 19. *For $k \in [m]$, there exists a deterministic k-committee selection voting rule f_k^*, that for any pair of $\vec{\sigma}, \vec{u}$ guarantees $\sum_{a \in f_k^*(\vec{\sigma})} \mathsf{sw}(a, \vec{u}) \geqslant \frac{nk\sqrt{k}}{6m^2}$.*

Proof. Let $\widehat{A}$ be the set of alternatives returned by Algorithm 1. First, we show by Lemma 18 that for all agents $i \in N_1$, $\sum_{a \in \widehat{A}} u_i(a) \geqslant \mathsf{hits}(i)/m$. That is because A_{plu} includes the top choice of members of N_1. In addition by changing S_i in line 9 of the Algorithm 1, we make sure that the i-th hit in an agent's preference ranking is among his m/i top alternatives. This means that at the end if there are t hits in an agent's ranking, then for any $2 \leqslant \ell \leqslant t$ at least $t - \ell + 2$ members of $A_{\mathsf{greedy}} \cup A_{\mathsf{plu}}$ appear in that agent's top m/ℓ positions.

Consequently, we have

$$\sum_{a \in \widehat{A}} \mathsf{sw}(a) \geqslant \sum_{i \in N_1} \frac{\mathsf{hits}(i)}{m}. \tag{14}$$

Lower bound on the total number of hits. Next, we show $\sum_{i \in N_1} \mathsf{hits}(i) \geqslant |N_1| \cdot \sqrt{k}$. Let $a_1, a_2, \ldots, a_{k/3}$ be the sequence of alternatives greedily picked in the algorithm (lines 6-13). For $t \in [k/3]$, let $\mathsf{hits}^t(i)$ be the number of hits of voter i at the beginning of iteration t and h_t be the number of voters that were hit by a_t during the t-th iteration. Let $h_{\min} = \min_{t \in [k/3]} h_t$. Indeed we have

$$\sum_{i \in N_1} \mathsf{hits}(i) = \sum_{t \in [k/3]} h_t \geqslant h_{\min} \cdot k/3. \tag{15}$$

Moreover, since at iteration t we pick the candidate which hits the highest number of agents, h_t is at least as much as the average total $|S_i|$'s, i.e.

$$h_t \geqslant \frac{1}{m} \cdot \sum_{i \in N_1} |S_i| \geqslant \frac{1}{m} \cdot \sum_{i \in N_1} \frac{m}{\mathsf{hits}^t(i) + 1} \geqslant \frac{|N_1|^2}{\sum_{i \in N_1} \mathsf{hits}^t(i) + |N_1|},$$

where the last transition follows from the AM-HM inequality. Since the RHS is minimized at time $t = k/3$ (the sum in the denominator is non-decreasing), and by Equation (15), we have

$$\frac{3}{k} \cdot \sum_{i \in N_1} \mathsf{hits}(i) \geqslant h_{\min} \geqslant \frac{|N_1|^2}{\sum_{i \in N_1} \mathsf{hits}(i) + |N_1|}$$

Denote $\alpha = \sum_{i \in N_1} \mathsf{hits}(i)$. Then, we have $\alpha \cdot (\alpha + |N_1|) \geqslant \frac{k}{3}|N_1|^2$, which holds only if

$$\sum_{i \in N_1} \mathsf{hits}(i) = \alpha \geqslant \frac{1}{2} \cdot |N_1| \left(\sqrt{1 + 4k/3} - 1\right) \geqslant |N_1|\sqrt{k/3}, \tag{16}$$

where the last transition holds for $k \geqslant 1$.

Deriving the bound. Moreover, $|N_1| \geqslant nk/3m$, since the $k/3$ alternatives with the highest number of top votes must have a total of at least $k/3m$ fraction of the n top votes. This observation combined with Equations (14) and (16) yields

$$\sum_{a \in \widehat{A}} \mathsf{sw}(a) \geqslant \frac{1}{3\sqrt{3}} \cdot \frac{nk\sqrt{k}}{m^2}.$$

□

B.2.2 Distortion Analysis

The goal of this section is to prove the following theorem.

Theorem 4. *There is a polynomial-time computable random k-committee member rule with distortion $O(\max\{k, m^2/(k\sqrt{k})\})$. This is minimized at $k = m^{4/5}$, where the bound becomes $O(m^{4/5})$.*

Proof. We select $\frac{k}{3}$ members of the committee using $f_{\frac{k}{3}}^*$ from Theorem 19 rule and for the rest, we select a 16-stable committee of size $\frac{k}{3}$.

For a preference profile $\vec{\sigma}$ and utility profile $\vec{u}$, let a^* be the optimal alternative, and $X \subseteq A$ be a committee of size k selected by our rule. By Theorem 18 we know that $|i \in N\colon a^* \succ_i X| \leqslant \frac{48n}{k}$. That means for at least $n - \frac{48n}{k}$ agents, at least one member of the selected committee gets as much utility as a^*. The maximum utility that a^* can get from the rest of the agents is $\frac{48n}{k}$. That means:

$$\mathsf{sw}(a^*, \vec{u}) \leqslant \sum_{a \in X} \mathsf{sw}(a, \vec{u}) + \frac{48n}{k}.$$

Let $\mathbb{U}[X]$ be a uniform distribution over the members of X, we have

$$\begin{aligned}
\mathsf{dist}(\mathbb{U}[X], \vec{u}) &= \frac{\mathsf{sw}(a^*, \vec{u})}{\frac{1}{|X|}\sum_{a \in X} \mathsf{sw}(a, \vec{u})} \\
&= \frac{2k \cdot \mathsf{sw}(a^*, \vec{u})}{2\sum_{a \in X} \mathsf{sw}(a, \vec{u})} \\
&\leqslant \frac{2k\mathsf{sw}(a^*, \vec{u})}{\max(0, \mathsf{sw}(a^*, \vec{u}) - \frac{48n}{k}) + \frac{n\frac{k}{3}\sqrt{\frac{k}{3}}}{6m^2}} && \text{(by Theorem 19)} \\
&\leqslant \frac{2k\mathsf{sw}(a^*, \vec{u})}{\max(0, \mathsf{sw}(a^*, \vec{u}) - \frac{48n}{k}) + \frac{nk\sqrt{k}}{32m^2}}.
\end{aligned}$$

Now we consider two cases, first if $\mathsf{sw}(a^*, \vec{u}) \geqslant \frac{96n}{k}$. In this case we have

$$\begin{aligned}
\mathsf{dist}(\mathbb{U}[X], \vec{u}) &\leqslant \frac{2k\mathsf{sw}(a^*, \vec{u})}{\max(0, \mathsf{sw}(a^*, \vec{u}) - \frac{48n}{k}) + \frac{nk\sqrt{k}}{32m^2}} \leqslant \frac{2k\mathsf{sw}(a^*, \vec{u})}{\mathsf{sw}(a^*, \vec{u}) - \frac{48n}{k}} \\
&\leqslant \frac{4k\mathsf{sw}(a^*, \vec{u})}{\mathsf{sw}(a^*, \vec{u})} = 4k = O\left(k\right).
\end{aligned}$$

Then we consider the other case where $\mathsf{sw}(a^*, \vec{u}) < \frac{96n}{k}$. Here we have

$$\begin{aligned}
\mathsf{dist}(\mathbb{U}[X], \vec{u}) &\leqslant \frac{2k\mathsf{sw}(a^*, \vec{u})}{\max(0, \mathsf{sw}(a^*, \vec{u}) - \frac{48n}{k}) + \frac{nk\sqrt{k}}{32m^2}} \leqslant \frac{2k\mathsf{sw}(a^*, \vec{u})}{\frac{nk\sqrt{k}}{32m^2}} \\
&\leqslant \frac{2k\frac{96n}{k}}{\frac{nk\sqrt{k}}{32m^2}} = O\left(\frac{m^2}{k\sqrt{k}}\right).
\end{aligned}$$

These two bounds together give us the desired lower bound of $O(\max\{k, m^2/(k\sqrt{k})\})$. □

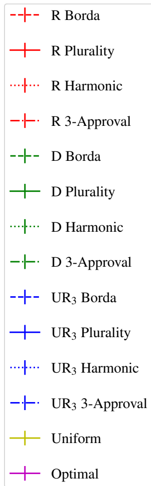
C Additional Experimental Results

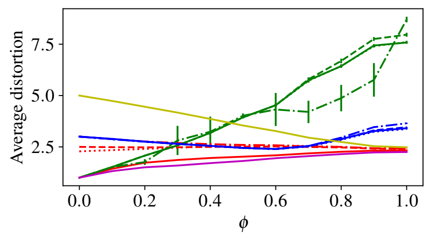
In this section, we provide some complementary results that give a better perspective on the empirical efficiency of the rules we study. The setup of these experiments is the same as described in Section 5.

C.1 Additional Results for the Mallows Model

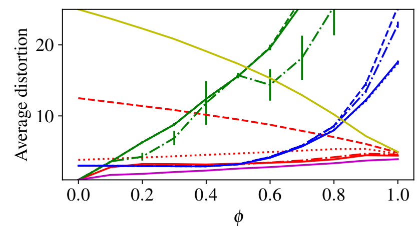
Figures 2a to 2c show the average distortion of different rules for different values of ϕ with $m \in \{5, 25, 50\}$, respectively. For $m = 5$, we find that randomized plurality is better than every other rule, regardless of the value of ϕ. But as m grows larger, we can see that random committee member rules begin to perform almost as well as randomized positional scoring rules for $\phi \leqslant 0.5$. In addition, as we have seen in Section 5, when ϕ grows large (moving towards the impartial culture), randomized positional scoring rules outperform deterministic and random committee member rules as well as the uniform benchmark. Overall, these plots reinforce the claim we made in Section 5 that there almost always seems to be an explainable randomized rule that achieves better efficiency than deterministic rules.

Figure 2d is the counterpart of Figure 1d presented in Section 5, where instead of plotting the (average) best value of k against ϕ, we plot the (average) distortion achieved at the said best value of k against ϕ, for various random committee member rules. In a sense, this shows the limit of how well each

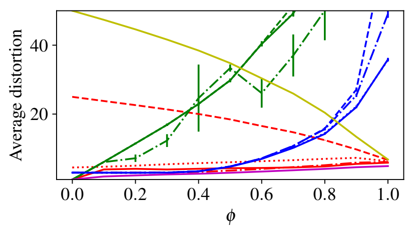




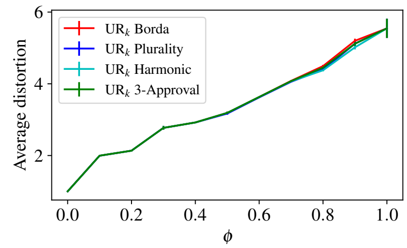
(a) Average distortion with $m = 5$.



(b) Average distortion with $m = 25$.



(c) Average distortion with $m = 50$.



(d) Average distortion with best k based on ϕ, for $m = 25$.

Figure 2: All figures show results averaged over 150 runs along with the standard error. Figures 2a to 2c share the legend on the left.

random committee member rule can perform, when paired with its corresponding optimal k. As we can see, all the four rules we consider achieve approximately the same average distortion under these optimized conditions, which increases almost linearly with ϕ.

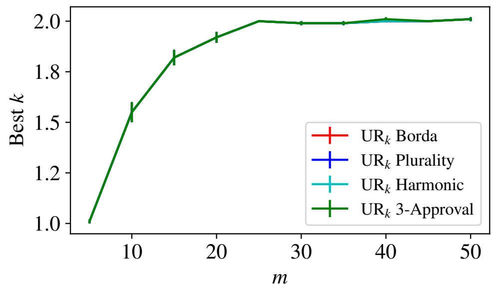
Figure 3 shows the value of k that yields the minimum distortion for different scoring vectors as a function of m, averaged over 100 runs and fixing the value of $\phi \in \{0.1, 0.5, 1\}$. Recall that theoretically, the value of k that optimizes the worst-case distortion is between $\Omega(m^{2/3})$ and $O(m^{4/5})$. For optimizing the average distortion, it turns out that the best k is close to 1 when ϕ is small, but as ϕ grows the best k gets closer to m. The growth as a function of m is highly sublinear for small ϕ, but almost linear for large ϕ.

We have also presented the (average) distortion achieved at these best values of k. Once again, we can notice that while the average distortion certainly grows with m, the different random committee member rules we consider perform about the same when paired with their optimal k. This average distortion is small when ϕ is small, and is approximately $\sqrt{m}$ for $\phi = 1$. An interesting observation is that for relatively small values of ϕ (i.e., $\phi = 0.1$ and $\phi = 0.5$), the distortion is very similar to the (average) best value of k. This indicates that for these values of ϕ, it may be the case that the optimal alternative is almost always part of the committee, for all four of the employed voting rules. Further, since k is small, not much preference information is observed. Hence, the worst case social welfare of any other alternatives included in the committee can be very small, bringing the distortion of choosing a random committee member close to the inverse of the probability of choosing the optimal alternative from the committee, which is k.

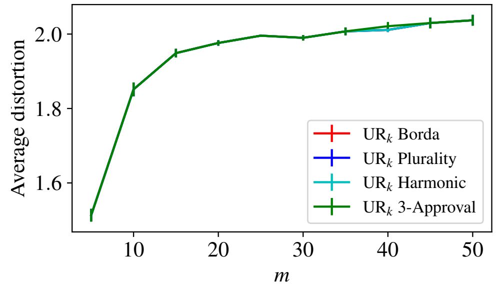
C.2 Results for Polya-Eggenberger Urn and Plackett-Luce Models

In addition to the Mallows model, we generate preference profiles from the Polya-Eggenberger Urn model [51] and the Plackett-Luce model [52, 53].

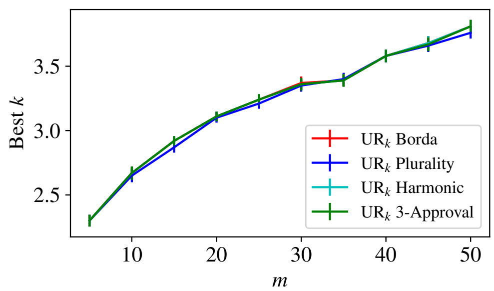
Polya-Eggenberger Urn Model. The urn model takes a non-negative parameter α which determines the extent to which rankings are correlated. To generate a preference profile from the urn model with parameter α, we start with a pool of all $m!$ possible rankings. At step i, we randomly pick the ranking of the ith voter from the pool and put it back in the pool together with $\alpha m!$ copies of that ranking. For $\alpha = 0$, the model picks a random ranking for each agent which is equivalent to the impartial culture, and the Mallows model with $\phi = 1$. For each combination of $n = 100$ agents,



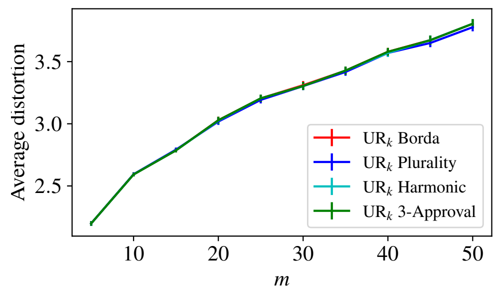
(a) Best value of k based on m, for $\phi = 0.1$.



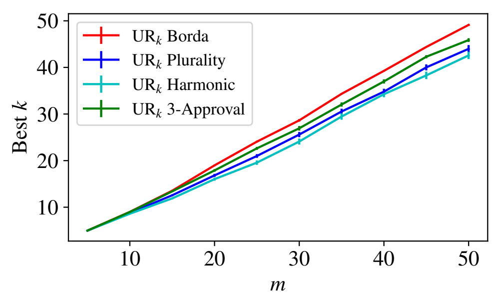
(b) Average distortion with best k based on m, for $\phi = 0.1$.



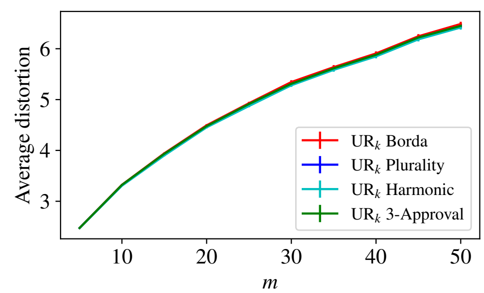
(c) Best value of k based on m, for $\phi = 0.5$.



(d) Average distortion with best k based on m, for $\phi = 0.5$.



(e) Best value of k based on m, for $\phi = 1$.



(f) Average distortion with best k based on m, for $\phi = 1$.

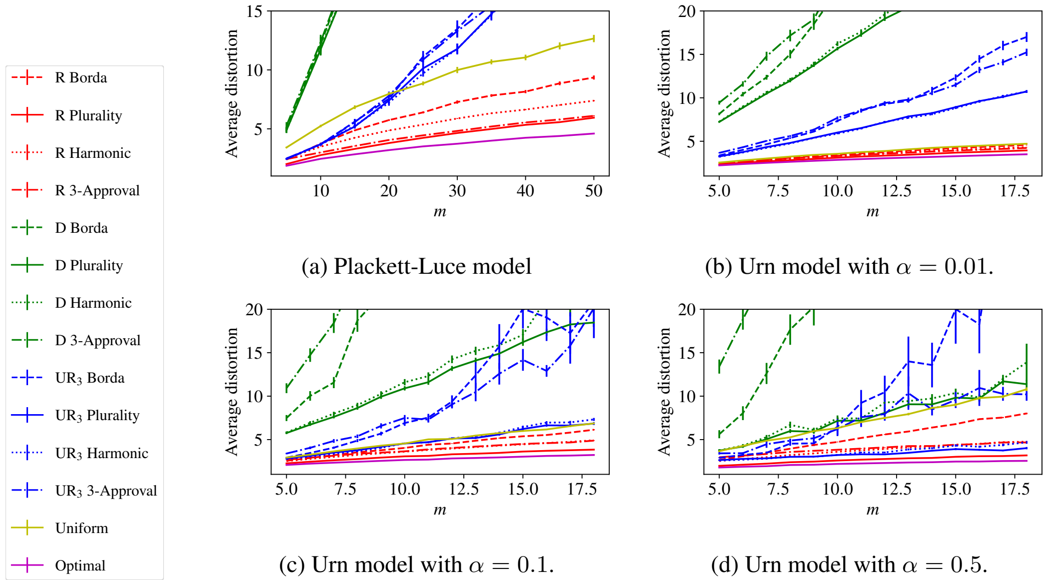
Figure 3: All figures show results averaged over 100 runs along with the standard error.

$\alpha \in \{0.01, 0.1, 0.5\}$ and $m \in [5, 18]$ alternatives, we generate 150 different instances. We used the PrefLib tools package[2] [54], which has a limitation of at most $m < 20$ alternatives.

Plackett-Luce Model. For experiments with the Plackett-Luce model, we generate preference profiles as follows. First, we sample a distribution over candidates $p \sim \text{Dirichlet}(1, \ldots, 1)$ from the symmetric Dirichlet distribution where the concentration parameter is 1 for all candidates. Then from a subset $A' \subseteq A$, we draw alternative $a \in A'$ with probability proportional to p_a, that is $\Pr(a \mid A', p) = \frac{p_a}{\sum_{a' \in A'} p_{a'}}$. We generate the preference ranking of each agent i separately in an iterative manner from the top alternative to the bottom one as follows. The kth ranked alternative of agent i is selected among $A_{i,k} = A \setminus \{\sigma_i(1), \ldots, \sigma_i(k-1)\}$ with probabilities $\Pr(a \mid A_{i,k},\ p)$. For $n = 100$ and $m \in \{5, 10, \ldots, 50\}$ we generate 150 instances.

Results. The results for the two models described are presented in Figure 4. For the Plackett-Luce model and the urn model with smaller values of $\phi = 0.01$, we observe results similar to those of the Mallows model with $\alpha = 1$ (the impartial culture). The randomized positional scoring rules

[2] https://github.com/PrefLib/preflibtools



(a) Plackett-Luce model

(b) Urn model with $\alpha = 0.01$.

(c) Urn model with $\alpha = 0.1$.

(d) Urn model with $\alpha = 0.5$.

Figure 4: Average distortion of different rules under other statistical models. All figures show distortion averaged over 150 runs along with the standard error.

outperform the randomized committee member rules, which in turn outperform the deterministic rules. The urn model with $\alpha = 0$ is exactly the impartial culture, thus the result for $\alpha = 0.01$ is as expected.

Results for the urn model with higher values of α are more intriguing. The order among the three families of rules is similar to the Mallows results and another similarity between these two models is that randomized plurality generally performs very well. However, there are crucial differences. For instance, in the Mallows experiments, there are no significant differences among different deterministic rules or between different randomized committee member rules. In contrast, clear distinctions surface in the urn experiments, with plurality-based rules demonstrating superior performance.