Appendix

This appendix contains the remaining proofs, additional simulations, some background on the RKHS and the connections between different settings in the literature on kernel ridge regression.

A Verifying reproducing property

To see the reproducing property of the kernel $\widetilde{\mathbb{K}}$, defined in (5), for the space $\widetilde{\mathbb{H}}$, note that for any $f = \sum_\ell \alpha_\ell \psi_\ell \in \widetilde{\mathbb{H}}$, we have

$$\langle f, \widetilde{\mathbb{K}}(\cdot, y)\rangle_{\widetilde{\mathbb{H}}} = \sum_{\ell=1}^{r}\sum_{k=1}^{r} \alpha_\ell \mu_k \psi_k(y) \langle \psi_\ell, \psi_k\rangle_{\widetilde{\mathbb{H}}} = \sum_{k=1}^{r} \alpha_k \psi_k(y) = f(y).$$

B Remaining proofs

Proof of Proposition 1. We write $y = (y_1, \dots, y_n)$ and $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n)$. Then, the model can be compactly written as $y = \sqrt{n} S_{\boldsymbol{x}}(f^*) + \varepsilon$. Let $\widetilde{y} = y/\sqrt{n}$ and $\widetilde{\varepsilon} = w/\sqrt{n}$ so that

$$\widetilde{y} = S_{\boldsymbol{x}}(f^*) + \widetilde{\varepsilon}.$$

By the representer theorem, a general TKRR solution is $\widetilde{f}_{r,\lambda} = \widetilde{S}_{\boldsymbol{x}}^*(\widetilde{\omega})$ where $\widetilde{\omega}$ is a solution of

$$\min_{\omega \in \mathbb{R}^n} \frac{1}{n} \|y - \sqrt{n}\widetilde{K}\omega\|^2 + \lambda \omega^T \widetilde{K} \omega. \tag{18}$$

The first-order optimality condition gives $\widetilde{K}[(\widetilde{K}\widetilde{\omega} - \widetilde{y}) + \lambda \widetilde{\omega}] = 0$. Let us write $K = U\Lambda U^T$ for the eigen-decomposition of the full kernel matrix, where $\Lambda = \operatorname{diag}(\mu_1, \dots, \mu_n)$ and U has columns $u_1, \dots, u_n$. Let $U_1 = [u_1 \mid u_2 \mid \cdots \mid u_r] \in \mathbb{R}^{n\times r}$, $U_2 = [u_{r+1} \mid \cdots \mid u_n] \in \mathbb{R}^{n \times (n-r)}$ and $\Lambda_1 = \operatorname{diag}(\mu_1, \dots, \mu_r) \in \mathbb{R}^{r\times r}$. Then, $\widetilde{K} = U_1 \Lambda_1 U_1^T$ and $\widetilde{\omega} = U_1 \alpha + U_2 \beta$ for some vectors α and β. Substituting into the first-order condition and noting $U_1^T U_1 = I_r$ and $U_1^T U_2 = 0$, we have

$$U_1 \Lambda_1 [(\Lambda_1 \alpha - U_1^T \widetilde{y}) + \lambda \alpha] = 0.$$

Let $\xi_{(1)} = U_1^T \widetilde{y}$. Multiplying both sides of the above by $\Lambda_1^{-1} U_1^T$, we obtain $(\Lambda_1 \alpha - \xi_{(1)}) + \lambda \alpha = 0$. Letting $A_\lambda = \Lambda_1 + \lambda I_r$, we have $\alpha = A_\lambda^{-1} \xi_{(1)}$. Thus all the solutions $\widetilde{\omega}$ of (18) are of the form

$$\widetilde{\omega} = U_1 A_\lambda^{-1} \xi_{(1)} + U_2 \beta \tag{19}$$

for an arbitrary $\beta \in \mathbb{R}^{n \times (n-r)}$.

Next, combining (5) and (7), we have for any $\omega \in \mathbb{R}^n$,

$$\widetilde{S}_{\boldsymbol{x}}(\omega) = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} \omega_j \sum_{k=1}^{r} \mu_k \psi_k \psi_k(x_j) = \sum_{j=1}^{n} \omega_j \sum_{k=1}^{r} \mu_k \psi_k u_{kj} = \sum_{k=1}^{r} (u_k^T \omega) \mu_k \psi_k. \tag{20}$$

Since $u_k^T U_2 \beta = 0$ for all $k = 1, \dots, r$, it follows that $\widetilde{f}_{r,\lambda} = \widetilde{S}_{\boldsymbol{x}}^*(\widetilde{\omega})$ is the same regardless of the value of β, proving the uniquness. In fact, noting that $u_k^T \widetilde{\omega} = (\mu_k + \lambda)^{-1} u_k^T \widetilde{y}$ for all $k \in [r]$,

$$\widetilde{f}_{r,\lambda} = \sum_{k=1}^{r} \frac{\mu_k}{\mu_k + \lambda} (u_k^T \widetilde{y}) \psi_k. \tag{21}$$

For future reference, $\widetilde{f}_{r,\lambda} = \widetilde{S}_{\boldsymbol{x}}^*(\widetilde{\omega})$ implies $\widetilde{f}_{r,\lambda}(x_i) = \frac{1}{\sqrt{n}} \sum_{j=1}^n \widetilde{\omega}_j \widetilde{\mathbb{K}}(x_i, x_j) = \sqrt{n}[\widetilde{K}\widetilde{\omega}]_i$, hence

$$S_{\boldsymbol{x}}(\widetilde{f}_{r,\lambda}) = \widetilde{K}\widetilde{\omega} = (U_1 \Lambda_1 U_1^T)(U_1 A_\lambda^{-1} \xi_{(1)} + U_2 \beta) = U_1 \Lambda_1 A_\lambda^{-1} \xi_{(1)}. \tag{22}$$

The proof is complete. □

Proof of Theorem 1. Using the previous notation and recalling that $\xi^* = U^T S_{\boldsymbol{x}}(f^*)$, we have $\xi := U^T\widetilde{y} = \xi^* + \boldsymbol{z}$ and $\boldsymbol{z} := U^T\widetilde{\varepsilon}$. Writing $\xi = U^T\widetilde{y} = (\xi_{(1)}, U_2^T\widetilde{y})$, we can rewrite (22) as $S_{\boldsymbol{x}}(\widetilde{f}_{r,\lambda}) = U\Gamma_\lambda\xi$. It follows that

$$\begin{aligned}\|\widetilde{f}_{r,\lambda} - f^*\|_n^2 &= \|S_{\boldsymbol{x}}(\widetilde{f}_{r,\lambda}) - S_{\boldsymbol{x}}(f^*)\|_2^2 \\ &= \|U\Gamma_\lambda\xi - U\xi^*\|_2^2 \\ &= \|\Gamma_\lambda\xi - \xi^*\|_2^2 = \|(\Gamma_\lambda - I_n)\xi^* + \Gamma_\lambda \boldsymbol{z}\|_2^2.\end{aligned}$$

Expanding and using $\mathbb{E}[\boldsymbol{z}] = 0$, we get

$$\mathbb{E}\|\widetilde{f}_{r,\lambda} - f^*\|_n^2 = \|(I_n - \Gamma_\lambda)\xi^*\|_2^2 + \operatorname{tr}\big(\Gamma_\lambda^2\mathbb{E}[\boldsymbol{z}\boldsymbol{z}^T]\big).$$

Noting that $\mathbb{E}[\boldsymbol{z}\boldsymbol{z}^T] = \operatorname{cov}(\boldsymbol{z}) = U^T\operatorname{cov}(\widetilde{\varepsilon})U = \frac{\sigma^2}{n}U^TU = \frac{\sigma^2}{n}I_n$ gives

$$\mathbb{E}\|\widetilde{f}_{r,\lambda} - f^*\|_n^2 = \|(I_n - \Gamma_\lambda)\xi^*\|_2^2 + \frac{\sigma^2}{n}\operatorname{tr}(\Gamma_\lambda^2)$$

which is the desired result. The expression (10) is obtained by writing $\sum_{i=r+1}^n(\xi_i^*)^2 = \|\xi^*\|_2^2 - \sum_{i=1}^r(\xi_i^*)^2$ and noting that $\|f\|_n = \|\xi^*\|_2$. □

Proof of Proposition 2. The expression for $\overline{\mathrm{MSE}}$ follows by taking the expectation of both sides of (10) and noting that $\mathbb{E}(\xi_i^*)^2 = 1/b$ when nonzero and $\mathbb{E}\|f^*\|_n^2 = 1$.

For part (a), first we note that $\overline{\mathrm{MSE}}$ is increasing in intervals $[1,\ell]$ and $[\ell+b, n]$. This is immediate from the expression, since in both cases, the middle term in (12) remains constant as a function of r, while the the estimation error (the third term) contributes positive terms to the $\overline{\mathrm{MSE}}$ when increasing r. For the middle interval $[\ell, \ell+b]$, we consider the two intervals $[\ell, j^*) = [\ell, j^*-1]$ and $[j^*, \ell+b]$ separately.

Assume first that $j^* \in [\ell+1, \ell+b)$. Since $i \mapsto \mu_i$ is decreasing, we have

$$\begin{aligned} 1 + \frac{2\lambda}{\mu_i} &< \frac{\sigma^2}{n}b \quad \text{for } i \in [\ell+1, j^*-1], \\ 1 + \frac{2\lambda}{\mu_i} &\le \frac{\sigma^2}{n}b \quad \text{for } i = j^*, \\ 1 + \frac{2\lambda}{\mu_i} &> \frac{\sigma^2}{n}b \quad \text{for } i \in [j^*+1, \ell+b]. \end{aligned} \tag{23}$$

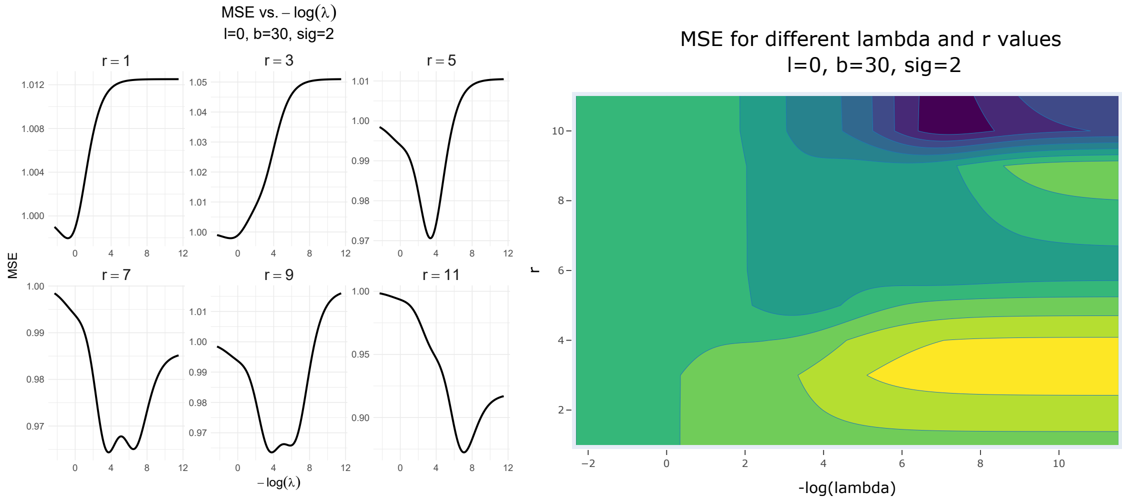
The first two lines above follow since $i \mapsto \mu_i$ is an strictly decreasing sequence by the distinctness of $\{\mu_i\}$, hence the inequality can potentially turn into an equality only at the endpoint $i = j^*$. The third line, (23), follows by the maximally of j^*. Inequality (23) is equivalent to $\frac{1}{b}a_i(\lambda) > \frac{\sigma^2}{n}\mu_i^2$ showing that the combined contribution to the $\overline{\mathrm{MSE}}$ by the ith terms of the two sums in (12) is negative for $i \in [j^*+1, \ell+b]$, hence the $\overline{\mathrm{MSE}}$ is decreasing on $[j^*, \ell+b]$. Similarly, the first inequality shows that the combined contribution by the ith terms of the two sums in (12) is positive for $i \in [\ell+1, j^*-1]$, hence the $\overline{\mathrm{MSE}}$ is increasing in $[\ell, j^*-1]$. This completes the proof of part (a) when $j^* \in [\ell+1, \ell+b)$. Note that in this case, the $\overline{\mathrm{MSE}}$ is possibly flat only on $[j^*-1, j^*]$. When $j^* = \ell+b$, the assertion about the interval $[j^*, \ell+b]$ is vacuous. When, $j^* = \ell$, by definition, inequality (23) holds for all $i \in [\ell+1, \ell+b]$, hence the $\overline{\mathrm{MSE}}$ is decreasing in $[\ell, \ell+b]$ by the previous argument. The proof of part (a) is complete.

For part (b), we note that the variable term of the $\overline{\mathrm{MSE}}$ for $\ell \in [0, r-b]$ is

$$\frac{1}{b}\sum_{i=\ell+1}^{\ell+b}\frac{-a_i(\lambda)}{(\mu_i+\lambda)^2} = \frac{1}{b}\sum_{i=\ell+1}^{\ell+b}\frac{\lambda^2}{(\mu_i+\lambda)^2} - 1. \tag{24}$$

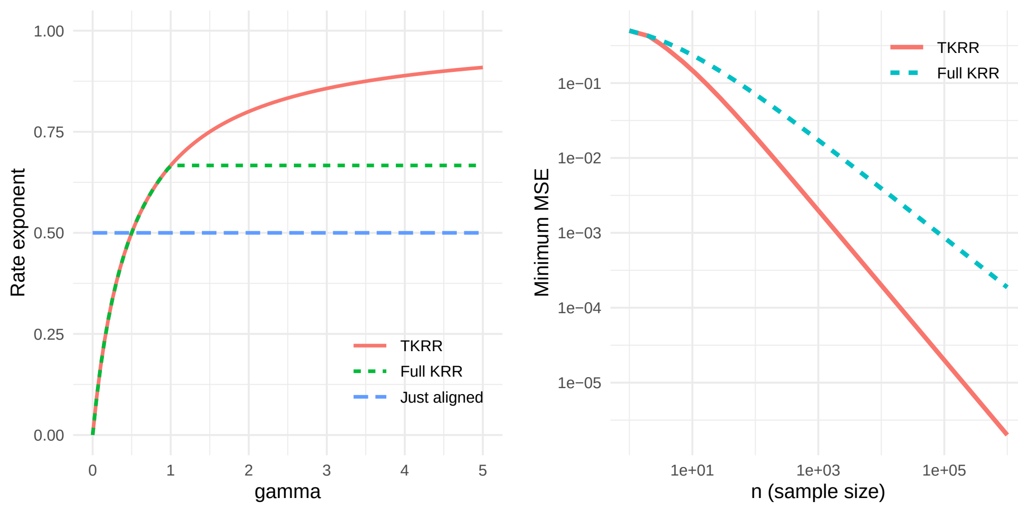
Since $i \mapsto \frac{\lambda^2}{(\mu_i+\lambda)^2}$ is an increasing function, summing it over a sliding window of length b starting at $\ell+1$, produces larger values as ℓ increases.

For part (c), the variable term of $\overline{\mathrm{MSE}}$ is again (24) for $b \in [1, r-\ell]$. The variable part is the average of the sequence $i \mapsto \frac{\lambda^2}{(\mu_i+\lambda)^2}$ over a window of length b. Increasing the window length then increases the average since the sequence is increasing. □



(a) Panel plot (b) Contour plot

Figure 3: Multiple-descent and phase transition of λ-regularization curve: (a) Expected MSE as a function of $-\log(\lambda)$ for different values of r, and (b) overall contour plot of expected MSE for r vs. $-\log(\lambda)$.



(a) Rate exponents (b) Minimum MSE vs. sample size

Figure 4: (a) The rate exponent function $s(\gamma)$ for the TKRR, Eq. (17), compared with that of full KRR $s(\delta) = s(1 \wedge \gamma)$ and the minimax exponent over the RKHS ball $s(1/2)$. (b) The minimum achievable MSE by TKRR and full KRR, as a function of the sample size, when $\alpha = 1$ and $\gamma = 10$.

C Additional simulations

Multiple-descent and phase transition We study the behavior of the λ-regularization curves for different values of the truncation parameter r given a fixed noise level σ. The panel plot and corresponding contour plot are shown in Figure 3a and 3b, respectively. The plots show multiple-descent and phase transition as demonstrated in the λ-regularization curves for different values of the noise level and r-regularization curves shown in Section 5.

Rate of TKRR vs. KRR We perform some experiments to coroborate the results of Theorem 2. We let the eigenvalues and TA scores decay polynomially with rates specified as in (13), and take the truncation parameter r to be as derived in Theorem 2(a). For the full KRR and TKRR, we calculate the respective minimum value of MSE among 1000 values of the regularization parameter λ, evenly

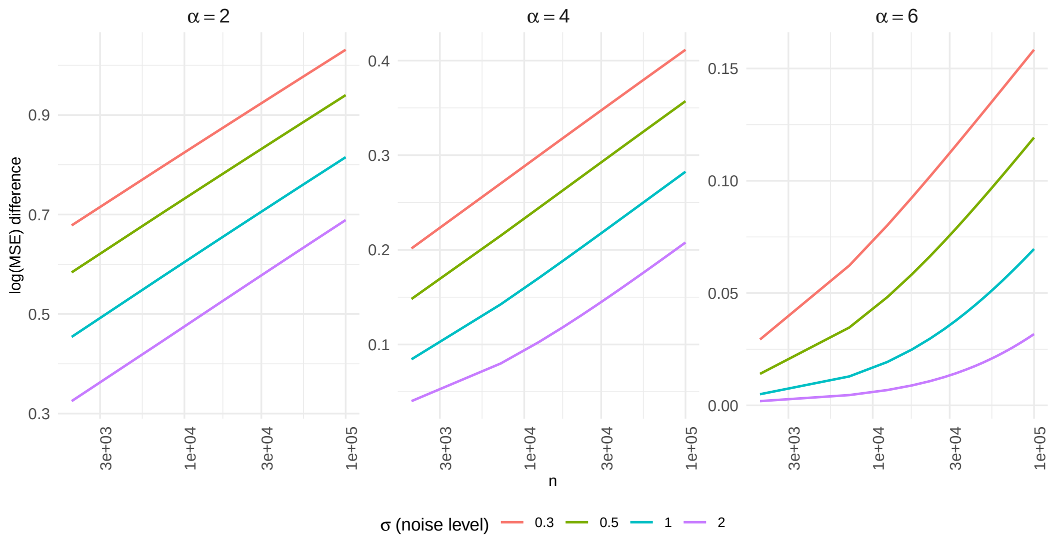


Figure 5: The difference of log(MSE) between the full KRR and TKRR versus the sample size n for $\gamma = 5$, and various values of α and the noise level σ.

distributed between 10^{-10} and 10^2. Figure 4(b) shows the MSE for the two methods, when $\alpha = 1$ and $\gamma = 10$, as a function of the sample size, on a log-log scale,. The difference in slope clearly shows the difference in rate between the two approaches.

The plots in Figure 5 show the difference in minimum log(MSE) between the full KRR and TKRR versus the sample size (on the log-scale), for different combinations of decay rate α and the noise level σ. For all the plots, we have $\gamma = 5$. According to Theorem 2, for sufficiently large n, the difference in minimum log(MSE) between the full KRR and TKRR should follow a line with positive slope when plotted as a function of $\log n$. This is clearly shown in Figure 5, where the positivity of the slope signifies the difference in rates between the two methods.

D RKHS background

Assume that $\mathcal{X}$ is a measurable space with a σ-finite measure μ and $\mathbb{H}$ is a separable RKHS over $\mathcal{X}$ with a measurable kernel $\mathbb{K} : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$. We write $L^2 := L^2(\mu)$ for the L^2 space of functions from $\mathcal{X}$ to $\mathbb{R}$. For simplicity, we write $\|\mathbb{K}\|_{L^2}$ for the L^2 norm of the function $x \mapsto \sqrt{\mathbb{K}(x,x)}$. We assume

$$\|\mathbb{K}\|_{L^2} < \infty. \tag{25}$$

Then $\mathbb{H}$ is a subset of L^2 and the inclusion map $J : \mathbb{H} \to L^2$ is continuous. The adjoint of this map $J^* : L^2 \to \mathbb{H}$ is the following integral operator

$$J^* f(x) = \int \mathbb{K}(\cdot, x) f d\mu = \langle \mathbb{K}(\cdot, x), f \rangle_{L^2} \quad f \in L^2.$$

Let $T = JJ^* : L^2 \to L^2$. This can be thought of a the same integral operator acting on L^2 with output in L^2. The decomposition $T = JJ^*$ shows that T is self-adjoint and positive. Condition (25) implies that T is a Hilbert-Schmidt, and hence a compact, operator.

The spectral theorem for self-adjoint compact operators on L^2 implies that

$$Tf = \sum_{i \in I} \lambda_i e_i \langle f, e_i \rangle_{L^2} \quad \text{for all } f \in L^2$$

where $\{\lambda_i\}_{i \in I}$ are the non-zero eigenvalues of T order in decreasing fashion and $\{e_i\}_{i \in I} \subset L^2$ a corresponding sequence of eigenvectors (at most countable), forming an orthonormal system (ONS) in L^2. That is, $Te_i = \lambda_i e_i$ and $\langle e_i, e_j \rangle_{L^2} = 1\{i = j\}$.

One can also view $\{e_i\}$ as functions in $\mathbb{H}$, and it is not hard to see that $\{e_i\}_{i\in I}$ is an orthogonal sequence in $\mathbb{H}$ with $\|e_i\|_{\mathbb{H}}^2 = 1/\lambda_i$. That is, $\langle e_i, e_j\rangle_{\mathbb{H}} = 1\{i = j\}/\lambda_i$. In other words, $\{\sqrt{\lambda_i} e_i\}_{i\in I}$ is an ONS in $\mathbb{H}$.

Assume from now on that we are dealing with a Mercer kernel $\mathbb{K}$, that is, $\mathcal{X}$ is a compact space and $\mathbb{K}$ is a continuous function. Then, we have the Mercer decomposition of the kernel function

$$K(x, y) = \sum_i \lambda_i e_i(x) e_i(y), \quad \forall x, y \in \mathcal{X} \tag{26}$$

where the convergence is uniform and absolute. It then follows that $\{\sqrt{\lambda_i} e_i\}$ is an orthonormal basis (ONB) of $\mathbb{H}$ and we have

$$\mathbb{H} = \Big\{ \sum_i \alpha_i e_i \mid \sum_i \frac{\alpha_i^2}{\lambda_i} < \infty \Big\}.$$

The treatment up to this point follows more or less the treatment in [23, Chapter 4].

From now on, we patch the sequence $\{e_i\}_{i\in I}$ to a complete orthonormal basis for the entire L^2 namely $\{e_i\}_{i\in I'}$ where I' is a proper subset of I. Let $I_0 := I' \setminus I$. Then, $e_i, i \in I_0$ span the orthogonal complement of the image of T (i.e., the null space of T). We let $\lambda_i = 0$ for $i \in I_0$, so that

$$Tf = \sum_{i\in I'} \lambda_i e_i \langle f, e_i\rangle_{L^2} \quad \text{for all } f \in L^2$$

still holds. The statement $\|e_i\|_{\mathbb{H}}^2 = \frac{1}{\lambda_i}$ also hold over $i \in I'$, interpreting $1/0$ as ∞. That is, $\|e_i\|_{\mathbb{H}} = \infty$ when $i \in I_0$, consistent with the fact that such e_i are not in $\mathbb{H}$ (or more precisely do not have a version that is in $\mathbb{H}$).

D.1 Target alignment

With this notation, every function in L^2 has a decomposition of the form $f = \sum_{i\in I'} \alpha_i e_i$ where $\alpha_i = \langle f, e_i\rangle_{L^2}$. Then, the RKHS $\mathbb{H}$ consists of those f for which

$$\|f\|_{\mathbb{H}}^2 = \sum_{i\in I'} \frac{\alpha_i^2}{\lambda_i} < \infty. \tag{27}$$

One can think of either $\{\alpha_i\}_{i\in I}$ or $\{\alpha_i\}_{i\in I'}$ as the population level kernel alignment spectrum (that is, the population counterpart of Definition 1). Note that if α_i is nonzero for any $i \in I_0$, then $\|f\|_{\mathbb{H}} = \infty$ and that f is not in $\mathbb{H}$. Even if $\alpha_i = 0$ for all $i \in I_0$, $\{\alpha_i\}_{i\in I}$ needs to decay as imposed in (27) for the function to belong to the RKHS. For example, a necessary condition is $\alpha_i = o(\sqrt{\lambda_i})$ for $i \in I_0$. In other words, belonging to the RKHS itself implies some amount of alignment between the target and the kernel (i.e. some level of decay for $\{\alpha_i\}$.)

To summarize, we can write

$$\mathbb{H} = \Big\{ f \in L^2 \mid \sum_{i\in I'} \frac{\langle f, e_i\rangle_{L^2}^2}{\lambda_i} < \infty \Big\}.$$

Let us connect to the setup of [9] and [12]. In short, these two papers impose the following condition

$$f = \sum_{i\in I'} \alpha_i e_i, \quad \sum_{i\in I'} \frac{\alpha_i^2}{\lambda_i^c} < \infty \tag{28}$$

for some $c \in [1, 2]$. If $c = 1$ this just means that $f \in \mathbb{H}$. If $c > 1$ it means that it is in a proper subset of $\mathbb{H}$. The c here is the same as the c in [9] and we have $c = 2r$ for parameter r used in [12]. In our notation in this paper, $c = 2\gamma$. (Note that in our paper, r is reserved to the spectral truncation level and is a different parameter.)

In addition [9] assumes $\lambda_i \asymp i^{-b}$ which is the same as the condition in [12], that is, $\lambda_i \asymp i^{-\alpha}$, for $\alpha = b$. Here, our notation matches that of [12]; see (13) which is the empirical counterpart of $\lambda_i \asymp i^{-\alpha}$. Also, [9] consider the case where λ_i drop to zero exactly after some point (finite RKHS) which they refer to as the case $b = \infty$.

D.2 Details of matching the setups

The conditions [9, 12] are not stated as cleanly as (28). Let us see how they can be reformulated in this equivalent fashion. The paper [9] which seems to be the origin of this condition works in the abstract setting of vector-valued RKHSs. We adapt the notation to the scalar-valued RKHSs. They work with operator $K_x : \mathbb{R} \to \mathbb{H}$ whose adjoint $K_x^* : \mathbb{H} \to \mathbb{R}$ is given by $K_x^* f = f(x)$ for every $f \in \mathbb{H}$. We then have $aK_x^* f = \langle K_x a, f\rangle_{\mathbb{H}}$ for any $a \in \mathbb{R}$ by the definition of an adjoint operator. Since $aK_x^* f = af(x) = \langle a\mathbb{K}(\cdot, x), f\rangle_{\mathbb{H}}$, it follows that

$$K_x a = a\mathbb{K}(\cdot, x), \quad a \in \mathbb{R}.$$

Then, they define the operator $T_x := K_x K_x^* : \mathbb{H} \to \mathbb{H}$ and $T = \int_{\mathcal{X}} T_x d\mu(x)$. We have

$$\begin{aligned}
\langle e_i, T_x e_j\rangle_{L^2} = \langle e_i, K_x K_x^* e_j\rangle_{L^2} &= \langle e_i, K_x e_j(x)\rangle_{L^2} \\
&= \langle e_i, e_j(x) K(\cdot, x)\rangle_{L^2} \\
&= e_j(x)\langle e_i, K(\cdot, x)\rangle_{L^2} = e_j(x)\lambda_i e_i(x)
\end{aligned}$$

where the last step is since e_i is an eigenvector of the integral operator $f \mapsto (x \mapsto \langle f, K(\cdot, x)\rangle_{L^2})$ from L^2 to L^2, and that the range of this operator is in fact in $\mathbb{H}$ (so evaluations make sense). This also follows from the Mercer decomposition. It then follows that

$$\langle e_i, Te_j\rangle_{L^2} = \int \langle e_i, T_x e_j\rangle_{L^2} d\mu(x) = \lambda_i \int e_i(x) e_j(x) d\mu(x) = \lambda_i \langle e_i, e_j\rangle_{L^2} = \lambda_i 1\{i = j\}.$$

That is T can be viewed as a diagonal matrix $T = \mathrm{diag}(\lambda_i, i \in I')$ in the basis $\{e_i\}_{i\in I'}$.

The condition in [9] is $f = T^{(c-1)/2} g$ where $g \in \mathbb{H}$ (or more precisely $\|g\|_{\mathbb{H}}^2 \le R$). Let us write $g = \sum_{i\in I'} \beta_i e_i$ and $f = \sum_{i\in I'} \alpha_i e_i$. Since $T^{(c-1)/2}$ is a diagonal matrix in this basis, we have $\alpha_i = \lambda_i^{(c-1)/2}\beta_i$ or equivalently $\beta_i = \lambda_i^{(1-c)/2}\alpha_i$. Then, $g \in \mathbb{H}$ iff $\sum_i \beta_i^2/\lambda_i < \infty$ which is equivalent to

$$\sum_i \frac{1}{\lambda_i}(\lambda_i^{(1-c)/2})^2 \alpha_i^2 < \infty \iff \sum_i \frac{\alpha_i^2}{\lambda_i^c} < \infty$$

and this is the desired condition.

Now to see that the condition in [12] is the same with $2r = c$, note that they require $\|\Sigma^{1/2-r}\theta^*\|_{\mathbb{H}} < \infty$ in their equation (7) which is a typo and is meant to be $\|\Sigma^{1/2-r}\theta^*\|_{\ell^2} < \infty$ in the ℓ^2 sequence norm. Here $\Sigma = \mathrm{diag}(\lambda_i)$ in our notation.

As for $\theta^* = (\theta_i^*)$ which is a sequence in ℓ^2, it is defined by the expansion $f^* = \sum_i \theta_i^* \psi_i$ where $\psi_i = \sqrt{\lambda_i} e_i$ in our notation. Thus, if we let $f = \sum_i \alpha_i e_i$, then $\alpha_i = \sqrt{\lambda_i}\theta_i^*$. So the condition imposed in [12] is

$$\sum_i (\lambda_i^{1/2-r}\theta_i^*)^2 < \infty \iff \sum_i (\lambda_i^{1/2-r}\lambda_i^{-1/2}\alpha_i)^2 < \infty \iff \sum_i \lambda_i^{-2r}\alpha_i^2 < \infty$$

which is the desired condition with $c = 2r$.

Immediately after stating this condition in [12], it is abandoned in favor of the condition $\lambda_i^{-2r}\alpha_i^2 \asymp i^{-1}$ which gives, together with $\lambda_i \asymp i^{-b}$

$$\alpha_i \asymp i^{-1/2}\lambda_i^r \asymp i^{-\frac{1+2rb}{2}}.$$

or equivalently $\theta_i^* \asymp \lambda_i^{-1/2}\alpha_i \asymp O(i^{-\frac{1+b(2r-1)}{2}})$. This is condition (8) in [12].

D.3 Minimax rates

Theorem 1 and 2 in [9] together establish that the minimax rate for the signal model (28) when $c \in (1, 2]$ is given by $(1/\ell)^{bc/(bc+1)}$, where ℓ is the sample size. Moreover, the same rate is minimax for $c = 1$ up to logarithmic factors. Translating to our notation with $c = 2\gamma$, $\ell = n$ and $b = \alpha$, the minimax rate in our model is $(1/n)^{2\gamma\alpha/(2\gamma\alpha+1)}$ when $\gamma \in (1/2, 1]$, as claimed in Section 4.2.