Evidently, this result implies that if we can determine a suitable bound for \(\|e_i^T A (I - QQ^T)\|^2\) then we automatically get a proper bound for the element-wise approximations of Algorithm 1. If \(A\) has a fast decaying spectrum and \(Q\) captures the dominant eigenspace of \(A\) we can expect that our approximations are very accurate, even for small \(l\). For the general case, however, the following Lemma 3 as well as the optimality of the JL lemma [31] already hint that this is not possible (see also Appendix II, Limitations of low-rank projections).

**Lemma 3.** Let \(A \in \mathbb{R}^{n \times d}\). For \(1 \leq k < d\), it holds that \(\|e_i^T (A - A_k)\|^2 \leq \sigma_{k+1}^2(A) \leq \frac{\|A_k\|_F^2}{k}\).

**Proof.** Clearly, \(\|e_i^T (A - A_k)\|^2 \leq \max_{\|x\|=1} \|x^T (A - A_k)\|^2 = \sigma_{k+1}^2(A)\). For the second part we have that \(\sigma_{k+1}^2(A) \leq \frac{1}{k} \sum_{i=1}^k \sigma_i^2(A) = \frac{\|A_k\|_F^2}{k}\). 

### 2.1 Projecting rows on randomly chosen subspaces

To proceed further with the analysis, we show some length-preserving properties of the orthogonal projector \(QQ^\top\), which is an orthogonal projector on a random subspace as obtained in line 3 of Algorithm 1. Note that Corollary 1 is stated for constant factor approximations. Here we provide a brief proof sketch. For the main result we refer to Lemma 8 in Appendix III.

**Corollary 1** (Projection on rowspace(\(SA^\top A\))). *(Proof in the Appendix)* Let \(\delta \in (0, \frac{1}{2})\), \(\tilde{A}_k = A - A_k\), and \(S\) be such that

(i) \(S \sim \mathcal{D}\), where \(\mathcal{D}\) is an \((1/3, \delta, \bar{A})\)-OSE for any fixed \(k\)-dimensional subspace;

(ii) \(S\) is a \((1/3, \delta, \bar{A})\)-JLT.

If \(Q\) is a matrix that forms an orthonormal basis for rowspace(\(SA^\top A\)), then, with probability at least \(1 - 2\delta\), for all \(i \in [n]\) simultaneously, it holds that

\[
\|e_i^T A (I - QQ^\top)\|^2 \leq \|e_i^T (\tilde{A}_k)\|^2 + 2\sigma_{k+1}^2(A) \|e_i^T A_k\| \|e_i^T (\tilde{A}_k)\| \leq 3 \|e_i^T A\| \|e_i^T (\tilde{A}_k)\|.
\]

**Proof sketch.** To prove the result it suffices to find a projector within rowspace(\(SA^\top A\)) with the desired properties. To do this, we consider the matrix \(\Pi_k = V_k (\Sigma_k \Sigma_k^\top) \dagger SA^\top A\), where \(V_k\), \(\Sigma_k\) originate from the SVD of \(A_k = U_k \Sigma_k V_k^\top\). Clearly, this \(\Pi_k\) is a rank-\(k\) matrix within rowspace(\(SA^\top A\)). After some algebra, the problem reduces to get a bound for the quantities

\[
\|e_i^T AV_k (\Sigma_k \Sigma_k^\top) \dagger \Sigma_k \Sigma_k^\top V_k^\top A e_i\|
\]

for all \(i \in [n]\). This is achieved by using Cauchy-Schwarz and by applying the OSE and JLT properties of \(S\).

Having all pieces in-place, we can finally bound the element-wise approximations of Algorithm 1.

**Theorem 1.** *(Proof in the Appendix)* Let \(A \in \mathbb{R}^{n \times d}\) and \(n \geq d\). If we use Algorithm 1 with \(m\) matrix-vector queries to estimate the Euclidean lengths of the rows of \(A\), then there exists a global constant \(C\) such that, as long as

(i) \(m \geq l \geq O(\log(n/\delta))\), such that \(G\) satisfies Lemma 1 and \(S\) forms an \((1/3, \delta, \bar{A})\)-JLT,

(ii) \(m \geq O(k + \log(1/\delta))\), such that \(S\) forms an \((1/3, \delta)\)-OSE for a \(k\)-dimensional subspace,

then it holds that

\[
|\tilde{x}_i - \|e_i^T A\|^2| \leq C \sqrt{\frac{\log(\frac{2}{1 - \delta})}{l}} \|e_i^T (A - A_k)\| \|e_i^T A\| \|A_k\|_F \|e_i^T A\|,
\]

for all \(i \in [n]\) with probability at least \(1 - 3\delta\).