A Proofs for Fat-Tailed Federated Learning

A.1 Proof of FAT-Clipping-PR

For notional clarity, we have the following update:

$$\text{Local update: } \mathbf{x}_{t,i}^{k+1} = \mathbf{x}_{t,i}^{k} - \eta_L \nabla f_i(\mathbf{x}_{t,i}^{k}, \xi_{t,i}^{k}), k \in [K],$$

$$\text{Clipping: } \mathbf{x}_{t,i}^{K+1} = \mathbf{x}_{t,i}^{k} - \eta_L \text{clipping}(\sum_{k\in[K]} \nabla f_i(\mathbf{x}_{t,i}^{k}, \xi_{t,i}^{k})),$$

$$\Delta_{t,i} = \sum_{k\in[K]} \nabla f_i(\mathbf{x}_{t,i}^{k}, \xi_{t,i}^{k}), \tilde{\Delta}_{t,i} = \text{clipping}(\sum_{k\in[K]} \nabla f_i(\mathbf{x}_{t,i}^{k}, \xi_{t,i}^{k}), \lambda),$$

$$\Delta_t = \frac{1}{m}\sum_{i\in[m]} \Delta_{t,i}, \tilde{\Delta}_t = \frac{1}{m}\sum_{i\in[m]} \tilde{\Delta}_{t,i}$$

$$\mathbf{x}_{t+1} = \mathbf{x}_t - \eta\eta_L \tilde{\Delta}_t.$$

Lemma 1 (Bounded Variance of Stochastic Local Updates for FAT-Clipping-PR)**.** *Assume $f_i(\mathbf{x}, \xi)$ satisfies the Bounded $\alpha-$Moment assumption 3, then for* FAT-Clipping-PR *we have:*

$$\mathbb{E}[\|\tilde{\Delta}_t\|^2] \leq K^2 G^\alpha \lambda^{2-\alpha},$$

$$\mathbb{E}\|\tilde{\Delta}_t - \mathbb{E}[\tilde{\Delta}_t]\|^2 \leq \frac{K^2}{m} G^\alpha \lambda^{2-\alpha},$$

$$\|\frac{1}{K}\mathbb{E}[\tilde{\Delta}_t] - \nabla f(\mathbf{x}_t)\|^2 \leq L^2\eta_L^2 K^2 G^2 + K^2 G^{2\alpha}\lambda^{-2(\alpha-1)} + L\eta_L K^2 G^{1+\alpha}\lambda^{1-\alpha}.$$

Note here the expectation is on the random samples $\xi_{t,i}^k$.

Proof.

$$\begin{aligned}
\mathbb{E}[\|\tilde{\Delta}_t\|^2] &= \max_{i\in[m]} \mathbb{E}[\|\tilde{\Delta}_{t,i}\|^2] \\
&\leq \mathbb{E}[\|\tilde{\Delta}_{t,j}\|^\alpha]\lambda^{2-\alpha} \\
&\leq K\sum_{k\in K} \mathbb{E}[\|\nabla f(\mathbf{x}_{t,j}^k, \xi_{t,j}^k)\|^\alpha]\lambda^{2-\alpha} \\
&\leq K^2 G^\alpha \lambda^{2-\alpha},
\end{aligned}$$

where $j = argmax_{i\in[m]}\mathbb{E}[\|\tilde{\Delta}_{t,i}\|^2]$, and the first inequality is due to the clipping, i.e., $\|\tilde{\Delta}_{t,i}\| \leq \lambda$.

$$\begin{aligned}
\mathbb{E}\|\tilde{\Delta}_t - \mathbb{E}[\tilde{\Delta}_t]\|^2 &= \mathbb{E}\left\|\frac{1}{m}\sum_{i\in[m]}\left(\tilde{\Delta}_{t,i} - \mathbb{E}[\tilde{\Delta}_{t,i}]\right)\right\|^2 \\
&\leq \frac{1}{m^2}\sum_{i\in[m]} \mathbb{E}\|\tilde{\Delta}_{t,i} - \mathbb{E}[\tilde{\Delta}_{t,i}]\|^2 \\
&\leq \frac{1}{m^2}\sum_{i\in[m]} \mathbb{E}\|\tilde{\Delta}_{t,i}\|^2 \\
&\leq \frac{K^2}{m} G^\alpha \lambda^{2-\alpha}.
\end{aligned}$$

$$\|\frac{1}{K}\mathbb{E}[\tilde{\Delta}_t] - \nabla f(\mathbf{x}_t)\| \leq \left\|\nabla f(\mathbf{x}_t) - \frac{1}{K}\mathbb{E}[\Delta_t]\right\| + \frac{1}{K}\left\|\mathbb{E}[\Delta_t] - \mathbb{E}[\tilde{\Delta}_t]\right\|$$

$$\leq \frac{1}{mK}\sum_{i\in[m]}\sum_{k\in[K]} \mathbb{E}\left\|\nabla f_i(\mathbf{x}_t) - \nabla f_i(\mathbf{x}_{t,i}^k)\right\| + \frac{1}{mK}\sum_{i\in[m]}\left\|\mathbb{E}[\Delta_{t,i} - \tilde{\Delta}_{t,i}]\right\|$$

$$
\leq \frac{L\eta_L}{mK} \sum_{i\in[m]} \sum_{k\in[K]} \mathbb{E} \left\| \sum_{j\in[k]} \nabla f_i(\mathbf{x}_{t,i}^j, \xi_{t,i}^j) \right\| + \frac{1}{mK} \sum_{i\in[m]} \mathbb{E}[\|\Delta_{t,i}\| \mathbf{1}_{\{\|\Delta_{t,i}\|\geq\lambda\}}]
$$
$$
\leq L\eta_L KG + KG^\alpha \lambda^{1-\alpha}
$$

where $\mathbf{1}_{\{\cdot\}}$ is the indicator function, the last inequality follows from the fact that $\Delta_{t,i} = \tilde{\Delta}_{t,i}$ if $\|\Delta_{t,i}\| \leq \lambda$ and $\mathbb{E}[\|\Delta_{t,i}\| \mathbf{1}_{\{\|\Delta_{t,i}\|\geq\lambda\}}] \leq \mathbb{E}[\|\Delta_{t,i}\|^\alpha]\lambda^{1-\alpha} \leq K^2G^\alpha\lambda^{1-\alpha}$; the second last inequality is due to L-smoothness, Jenson's inequality (i.e.,$\mathbb{E}[\Delta_{t,i} - \tilde{\Delta}_{t,i}]\| \leq \mathbb{E}\|[\Delta_{t,i} - \tilde{\Delta}_{t,i}]\|$) and the clipping step. Then, we have

$$
\|\frac{1}{K}\mathbb{E}[\tilde{\Delta}_t] - \nabla f(x)\|^2 \leq L^2\eta_L^2K^2G^2 + K^2G^{2\alpha}\lambda^{-2(\alpha-1)} + L\eta_L K^2 G^{1+\alpha}\lambda^{1-\alpha}
$$

□

Theorem 1. (Convergence Rate of FAT-Clipping-PR in the Strongly Convex Case) *Suppose that $f(\cdot)$ is a μ-strongly convex function. Under Assumptions 1–3, if $\eta\eta_L K \geq \frac{2}{\mu T}$, then the output $\bar{\mathbf{x}}_T$ of* FAT-Clipping-PR *being chosen in such a way that $\bar{\mathbf{x}}_T = \mathbf{x}_t$ with probability $\frac{w_t}{\sum_{j\in[T]} w_j}$, where $w_t = (1 - \frac{1}{2}\mu\eta\eta_L K)^{1-t}$, satisfies:*

$$
f(\bar{\mathbf{x}}_T) - f(\mathbf{x}^*) \leq \frac{\mu}{2}\exp\left(-\frac{1}{2}\mu\eta\eta_L KT\right) + \frac{\eta\eta_L K}{2}G^\alpha\lambda^{2-\alpha} + \frac{4}{\mu}\left[2G^{2\alpha}\lambda^{2-2\alpha} + 2L^2\eta_L^2K^2G^\alpha\lambda^{2-\alpha}\right],
$$

where $\mathbf{x}^*$ denotes the global optimal solution. Further, let $\eta\eta_L K = \frac{2c}{\mu}\frac{\ln(T)}{mKT}$, where $c \geq 1$ is a constant satisfying $m^{\frac{2-2\alpha}{\alpha}} K^{\frac{2}{\alpha}} T^{c+\frac{2-2\alpha}{\alpha}} \geq 1$, and let $\eta_L \leq (mKT)^{\frac{1-\alpha}{\alpha}}$. It then follows that

$$
f(\bar{\mathbf{x}}_T) - f(\mathbf{x}^*) = \mathcal{O}((mT)^{\frac{2-2\alpha}{\alpha}} K^{\frac{2}{\alpha}}).
$$

Proof.

$$
\begin{aligned}
&\mathbb{E}[\|\mathbf{x}_{t+1} - x_*\|^2] = \mathbb{E}[\|\mathbf{x}_t - \eta\eta_L\tilde{\Delta}_t - x_*\|^2] \\
&= \|\mathbf{x}_t - x_*\|^2 + \eta^2\eta_L^2\mathbb{E}[\|\tilde{\Delta}_t\|^2] - 2\left\langle \mathbf{x}_t - x_*, \eta\eta_L\left(\mathbb{E}[\tilde{\Delta}_t] - K\nabla f(\mathbf{x}_t) + K\nabla f(\mathbf{x}_t)\right)\right\rangle \\
&\leq (1 - \mu\eta\eta_L K)\|\mathbf{x}_t - x_*\|^2 + \eta^2\eta_L^2\mathbb{E}[\|\tilde{\Delta}_t\|^2] - 2\eta\eta_L K\left\langle \mathbf{x}_t - x_*, \left(\frac{1}{K}\mathbb{E}[\tilde{\Delta}_t] - \nabla f(\mathbf{x}_t)\right)\right\rangle \\
&\quad - 2\eta\eta_L K\left(f(\mathbf{x}_t) - f(\mathbf{x}_*)\right) \\
&\leq (1 - \frac{1}{2}\mu\eta\eta_L K)\|\mathbf{x}_t - x_*\|^2 + \eta^2\eta_L^2\mathbb{E}[\|\tilde{\Delta}_t\|^2] + \frac{8\eta\eta_L K}{\mu}\|\frac{1}{K}\mathbb{E}[\tilde{\Delta}_t] - \nabla f(\mathbf{x}_t)\|^2 \\
&\quad - 2\eta\eta_L K\left(f(\mathbf{x}_t) - f(\mathbf{x}_*)\right).
\end{aligned}
$$

The first inequality follows from the strongly-convex property, i.e., $-\langle \mathbf{x}_t - x_*, \nabla f(\mathbf{x}_t)\rangle \leq -(f(\mathbf{x}_t) - f(\mathbf{x}_*) + \frac{\mu}{2}\|x_t - \mathbf{x}_*\|^2)$, and the last inequality is due to Young's inequality. Then we have

$$
\begin{aligned}
f(\mathbf{x}_t) - f(\mathbf{x}_*) &\leq \frac{1}{2\eta\eta_L K}\left[-\mathbb{E}[\|\mathbf{x}_{t+1} - x_*\|^2] + (1 - \frac{1}{2}\mu\eta\eta_L K)\|\mathbf{x}_t - x_*\|^2\right] \\
&\quad + \frac{\eta\eta_L}{2K}\mathbb{E}[\|\tilde{\Delta}_t\|^2] + \frac{4}{\mu}\|\mathbb{E}[\frac{1}{K}\tilde{\Delta}_t - \nabla f(\mathbf{x}_t)\|^2 \\
&\leq \frac{1}{2\eta\eta_L K}\left[-\mathbb{E}[\|\mathbf{x}_{t+1} - x_*\|^2] + (1 - \frac{1}{2}\mu\eta\eta_L K)\|\mathbf{x}_t - x_*\|^2\right] \\
&\quad + \frac{\eta\eta_L}{2K}K^2G^\alpha\lambda^{2-\alpha} + \frac{4}{\mu}[L^2\eta_L^2K^2G^2 + K^2G^{2\alpha}\lambda^{-2(\alpha-1)} + L\eta_L K^2G^{1+\alpha}\lambda^{1-\alpha}],
\end{aligned}
$$

where the last inequality is due to Lemma 1.

Let $w_t = (1 - \frac{1}{2}\mu\eta\eta_L K)^{1-t}$, $\bar{\mathbf{x}}_T = \mathbf{x}_t$ with probability $\frac{w_t}{\sum_{j\in[T]} w_j}$.

$$
\begin{aligned}
f(\bar{x}_T) - f(\mathbf{x}_*) &\le \frac{1}{\sum_{j\in[T]} w_j} \sum_{t\in[T]} \left(\frac{w_t}{2\eta\eta_L K} \left[-\|\mathbf{x}_{t+1} - x_*\|^2 + (1 - \frac{1}{2}\mu\eta\eta_L K)\|\mathbf{x}_t - x_*\|^2 \right] \right) \\
&\quad + \frac{\eta\eta_L K}{2} G^\alpha \lambda^{2-\alpha} + \frac{4}{\mu}[L^2\eta_L^2 K^2 G^2 + K^2 G^{2\alpha}\lambda^{-2(\alpha-1)} + L\eta_L K^2 G^{1+\alpha}\lambda^{1-\alpha}] \\
&\le \frac{1}{\sum_{j\in[T]} w_j} \sum_{t\in[T]} \left(\frac{1}{2\eta\eta_L K} \left[-w_t\|\mathbf{x}_{t+1} - x_*\|^2 + w_{t-1}\|\mathbf{x}_t - x_*\|^2 \right] \right) \\
&\quad + \frac{\eta\eta_L K}{2} G^\alpha \lambda^{2-\alpha} + \frac{4}{\mu}[L^2\eta_L^2 K^2 G^2 + K^2 G^{2\alpha}\lambda^{-2(\alpha-1)} + L\eta_L K^2 G^{1+\alpha}\lambda^{1-\alpha}] \\
&\le \frac{1}{\sum_{j\in[T]} w_j} \frac{1}{2\eta\eta_L K} \|\mathbf{x}_1 - x_*\|^2 \\
&\quad + \frac{\eta\eta_L K}{2} G^\alpha \lambda^{2-\alpha} + \frac{4}{\mu}[L^2\eta_L^2 K^2 G^2 + K^2 G^{2\alpha}\lambda^{-2(\alpha-1)} + L\eta_L K^2 G^{1+\alpha}\lambda^{1-\alpha}],
\end{aligned}
$$

where the second inequality follows from $w_t \le w_{t-1}$.

$$
\begin{aligned}
2\eta\eta_L K \sum_{t\in[T]} w_t &= 2\eta\eta_L K \left(1 - \frac{1}{2}\mu\eta\eta_L K\right)^{-T} \sum_{t\in[T]} \left(1 - \frac{1}{2}\mu\eta\eta_L K\right)^t \\
&= \frac{4}{\mu} \left(1 - \frac{1}{2}\mu\eta\eta_L K\right)^{-T} \left[1 - \left(1 - \frac{1}{2}\mu\eta\eta_L K\right)^T \right] \\
&\ge \frac{4}{\mu} \left(1 - \frac{1}{2}\mu\eta\eta_L K\right)^{-T} \left[1 - \exp\left(-\frac{1}{2}\mu\eta\eta_L KT\right) \right] \\
&\ge \frac{2}{\mu} \left(1 - \frac{1}{2}\mu\eta\eta_L K\right)^{-T},
\end{aligned}
$$

where the last inequality follows from that $\eta\eta_L K \ge \frac{2}{\mu T}$, the second last inequality is due to $\left(1 - \frac{1}{2}\mu\eta\eta_L K\right)^T \le \exp\left(-\frac{1}{2}\mu\eta\eta_L KT\right)$.

$$
\begin{aligned}
f(\bar{x}_T) - f(\mathbf{x}_*) &\le \frac{\mu}{2} \left(1 - \frac{1}{2}\mu\eta\eta_L K\right)^T + \frac{\eta\eta_L K}{2} G^\alpha \lambda^{2-\alpha} \\
&\quad + \frac{4}{\mu}[L^2\eta_L^2 K^2 G^2 + K^2 G^{2\alpha}\lambda^{-2(\alpha-1)} + L\eta_L K^2 G^{1+\alpha}\lambda^{1-\alpha}] \\
&\le \frac{\mu}{2} \exp\left(-\frac{1}{2}\mu\eta\eta_L KT\right) + \frac{\eta\eta_L K}{2} G^\alpha \lambda^{2-\alpha} \\
&\quad + \frac{4}{\mu}[L^2\eta_L^2 K^2 G^2 + K^2 G^{2\alpha}\lambda^{-2(\alpha-1)} + L\eta_L K^2 G^{1+\alpha}\lambda^{1-\alpha}].
\end{aligned}
$$

Let $\eta\eta_L K = \frac{2c}{\mu}\frac{\ln(T)}{mKT}$ ($c \ge 1$ is a constant and $T^{-c-\frac{2-2\alpha}{\alpha}} \le m^{\frac{2-2\alpha}{\alpha}} K^{\frac{2}{\alpha}}$), $\lambda = (mKT)^{\frac{1}{\alpha}}$, and $\eta_L \le (mKT)^{\frac{1-\alpha}{\alpha}}$,

$$
f(\bar{x}_T) - f(\mathbf{x}_*) \le \frac{1}{T^c} + (mKT)^{\frac{2-2\alpha}{\alpha}} \ln(T) + (mT)^{\frac{2-2\alpha}{\alpha}} K^{\frac{2}{\alpha}} = \mathcal{O}((mT)^{\frac{2-2\alpha}{\alpha}} K^{\frac{2}{\alpha}}).
$$

□

Theorem 2. (Convergence Rate of FAT-Clipping-PR in the Nonconvex Case) *Suppose that $f(\cdot)$ is a nonconvex function. Under Assumptions 1–3, if $\eta\eta_L KL \le 1$, then the sequence of outputs $\{\mathbf{x}_k\}$ generated by* FAT-Clipping-PR *satisfies:*

$$
\min_{t\in[T]} \mathbb{E}\|\nabla f(\mathbf{x}_t)\|^2 \le \frac{2\left(f(\mathbf{x}_1) - f(x_T)\right)}{\eta\eta_L KT} + \left(L^2\eta_L^2 K^2 G^2 + K^2 G^{2\alpha}\lambda^{-2(\alpha-1)} + L\eta_L K^2 G^{1+\alpha}\lambda^{1-\alpha}\right)
$$

$$+ \frac{L\eta\eta_L}{m}\left(KG^\alpha\lambda^{2-\alpha}\right).$$

Further, choosing learning rates and clipping parameter in such a way that $\eta\eta_L = m^{\frac{2\alpha-2}{3\alpha-2}}K^{\frac{-\alpha-2}{3\alpha-2}}T^{\frac{-\alpha}{3\alpha-2}}$, $\eta_L \leq (mT)^{\frac{1-\alpha}{3\alpha-2}}K^{\frac{4-4\alpha}{3\alpha-2}}$, *and* $\lambda = (mK^4T)^{\frac{1}{3\alpha-2}}$, *we have*

$$\min_{t\in[T]}\mathbb{E}\|\nabla f(\mathbf{x}_t)\|^2 = \mathcal{O}((mT)^{\frac{2-2\alpha}{3\alpha-2}}K^{\frac{4-2\alpha}{3\alpha-2}}).$$

Proof. Due to the smoothness in Assumption 1, taking expectation of $f(\mathbf{x}_{t+1})$ over the randomness at communication round t, we have:

$$\begin{aligned}
&\mathbb{E}[f(\mathbf{x}_{t+1})] - f(\mathbf{x}_t) \leq \left\langle \nabla f(\mathbf{x}_t), \mathbb{E}[\mathbf{x}_{t+1} - \mathbf{x}_t]\right\rangle + \frac{L}{2}\mathbb{E}[\|\mathbf{x}_{t+1} - \mathbf{x}_t\|^2] \\
&= -\eta\eta_L\left\langle \nabla f(\mathbf{x}_t), \mathbb{E}[\tilde{\Delta}_t]\right\rangle + \frac{L}{2}\eta^2\eta_L^2\mathbb{E}[\|\tilde{\Delta}_t\|^2] \\
&= -\frac{\eta\eta_L K}{2}\|\nabla f(\mathbf{x}_t)\|^2 - \frac{\eta\eta_L}{2K}\|\mathbb{E}[\tilde{\Delta}_t]\|^2 + \frac{\eta\eta_L K}{2}\|\nabla f(\mathbf{x}_t) - \frac{1}{K}\mathbb{E}[\tilde{\Delta}_t]\|^2 + \frac{L\eta^2\eta_L^2}{2}\mathbb{E}[\|\tilde{\Delta}_t\|^2] \\
&= -\frac{\eta\eta_L K}{2}\|\nabla f(\mathbf{x}_t)\|^2 + \left(-\frac{\eta\eta_L}{2K} + \frac{L\eta^2\eta_L^2}{2}\right)\|\mathbb{E}[\tilde{\Delta}_t]\|^2 + \frac{\eta\eta_L K}{2}\|\nabla f(\mathbf{x}_t) - \frac{1}{K}\mathbb{E}[\tilde{\Delta}_t]\|^2 \\
&\quad + \frac{L\eta^2\eta_L^2}{2}\mathbb{E}[\|\tilde{\Delta}_t - \mathbb{E}[\tilde{\Delta}_t]\|^2] \\
&\leq -\frac{\eta\eta_L K}{2}\|\nabla f(\mathbf{x}_t)\|^2 + \frac{\eta\eta_L K}{2}\underbrace{\|\nabla f(\mathbf{x}_t) - \frac{1}{K}\mathbb{E}[\tilde{\Delta}_t]\|^2}_{A_1} + \frac{L\eta^2\eta_L^2}{2}\underbrace{\mathbb{E}[\|\tilde{\Delta}_t - \mathbb{E}[\tilde{\Delta}_t]\|^2]}_{A_2},
\end{aligned} \tag{5}$$

where the last inequality follows from $\left(-\frac{\eta\eta_L}{2K} + \frac{L\eta^2\eta_L^2}{2}\right) \leq 0$ if $\eta\eta_L KL \leq 1$.

From Lemma 1, we have the bound of A_1 and A_2 in (5). By rearranging and telescoping, we have:

$$\begin{aligned}
\frac{1}{T}\sum_{t\in[T]}\mathbb{E}\|\nabla f(\mathbf{x}_t)\|^2 \leq \frac{2\left(f(\mathbf{x}_1) - f(x_T)\right)}{\eta\eta_L KT} &+ \left(L^2\eta_L^2K^2G^2 + K^2G^{2\alpha}\lambda^{-2(\alpha-1)} + L\eta_L K^2G^{1+\alpha}\lambda^{1-\alpha}\right) \\
&+ \frac{L\eta\eta_L}{m}\left(KG^\alpha\lambda^{2-\alpha}\right).
\end{aligned}$$

Suppose $\eta\eta_L = m^{\frac{2\alpha-2}{3\alpha-2}}K^{\frac{-\alpha-2}{3\alpha-2}}T^{\frac{-\alpha}{3\alpha-2}}$, $\eta_L \leq (mT)^{\frac{1-\alpha}{3\alpha-2}}K^{\frac{4-4\alpha}{3\alpha-2}}$, and $\lambda = (mK^4T)^{\frac{1}{3\alpha-2}}$,

$$\min_{t\in[T]}\mathbb{E}\|\nabla f(\mathbf{x}_t)\|^2 \leq \mathcal{O}((mT)^{\frac{2-2\alpha}{3\alpha-2}}K^{\frac{4-2\alpha}{3\alpha-2}}).$$

□

A.2 Proof of FAT-Clipping-PI

For FAT-Clipping-PI, we have the following notions:

$$\tilde{\nabla} f_i(\mathbf{x}_{t,i}^k, \xi_{t,i}^k) = \min\{1, \frac{\lambda_t}{\|\nabla f_i(\mathbf{x}_{t,i}^k, \xi_{t,i}^k)\|}\}\nabla f_i(\mathbf{x}_{t,i}^k, \xi_{t,i}^k), \tilde{\nabla} f(\mathbf{x}_{t,i}^k) = \mathbb{E}[\tilde{\nabla} f_i(\mathbf{x}_{t,i}^k, \xi_{t,i}^k)];$$

Local steps: $\mathbf{x}_{t,i}^{k+1} = \mathbf{x}_{t,i}^k - \eta_L\tilde{\nabla} f_i(\mathbf{x}_{t,i}^k, \xi_{t,i}^k), k \in [K]$;

$$\Delta_{t,i} = \sum_{k\in[K]}\nabla f_i(\mathbf{x}_{t,i}^k, \xi_{t,i}^k), \tilde{\Delta}_{t,i} = \sum_{k\in[K]}\tilde{\nabla} f_i(\mathbf{x}_{t,i}^k, \xi_{t,i}^k)$$

$$\Delta_t = \frac{1}{m}\sum_{i\in[m]}\Delta_{t,i}, \tilde{\Delta}_t = \frac{1}{m}\sum_{i\in[m]}\tilde{\Delta}_{t,i}$$

$$\mathbf{x}_{t+1} = \mathbf{x}_t - \eta\eta_L\frac{1}{m}\sum_{i\in[m]}\sum_{k\in[K]}\tilde{\nabla} f_i(\mathbf{x}_{t,i}^k, \xi_{t,i}^k) = \mathbf{x}_t - \eta\eta_L\tilde{\Delta}_t.$$

Lemma 2 (Bounded Variance of Stochastic Local Updates for FAT-Clipping-PI)**.** *Assume $f_i(\mathbf{x}, \xi)$ satisfies the Bounded $\alpha-$Moment assumption 3, then we have:*

$$\mathbb{E}[\|\tilde{\Delta}_t\|^2] \leq K^2 G^\alpha \lambda^{2-\alpha},$$
$$\mathbb{E}\|\tilde{\Delta}_t - \mathbb{E}[\tilde{\Delta}_t]\|^2 \leq \frac{K}{m} G^\alpha \lambda^{2-\alpha},$$
$$\|\frac{1}{K}\mathbb{E}[\tilde{\Delta}_t] - \nabla f(x)\|^2 \leq 2G^{2\alpha}\lambda^{-2(\alpha-1)} + 2L^2\eta_L^2 K^2 G^\alpha \lambda^{2-\alpha}.$$

Proof.

$$\begin{aligned}
\mathbb{E}[\|\tilde{\Delta}_t\|^2] &= \frac{1}{m}\sum_{i\in[m]} \mathbb{E}[\|\tilde{\Delta}_{t,i}\|^2] \\
&\leq \frac{1}{m}\sum_{i\in[m]} \mathbb{E}[\|\sum_{j\in[K]} \tilde{\nabla} f(\mathbf{x}_{t,i}^j, \xi_{t,i}^j)\|^2] \\
&\leq \frac{K}{m}\sum_{i\in[m]}\sum_{j\in[K]} \mathbb{E}[\|\tilde{\nabla} f(\mathbf{x}_{t,i}^j, \xi_{t,i}^j)\|^2] \\
&\leq K^2 G^\alpha \lambda^{2-\alpha},
\end{aligned}$$

where the last inequality follows from the fact that $\mathbb{E}\|\tilde{\nabla} f_i(\mathbf{x}_{t,i}^k, \xi_{t,i}^k)\|^2 \leq \mathbb{E}\|\tilde{\nabla} f_i(\mathbf{x}_{t,i}^k, \xi_{t,i}^k)\|^\alpha \lambda^{2-\alpha} \leq G^\alpha \lambda^{2-\alpha}$ (see Lemma 9 in [22]).

$$\begin{aligned}
&\mathbb{E}\|\tilde{\Delta}_t - \mathbb{E}[\tilde{\Delta}_t]\|^2 = \mathbb{E}\left\|\frac{1}{m}\sum_{i\in[m]}\sum_{k\in[K]} \tilde{\nabla} f_i(\mathbf{x}_{t,i}^k, \xi_{t,i}^k) - \frac{1}{m}\sum_{i\in[m]}\sum_{k\in[K]} \tilde{\nabla} f_i(\mathbf{x}_{t,i}^k)\right\|^2 \\
&\leq \frac{1}{m^2}\sum_{i\in[m]}\sum_{k\in[K]} \mathbb{E}\|\tilde{\nabla} f_i(\mathbf{x}_{t,i}^k, \xi_{t,i}^k) - \tilde{\nabla} f_i(\mathbf{x}_{t,i}^k)\|^2 \\
&\leq \frac{1}{m^2}\sum_{i\in[m]}\sum_{k\in[K]} \mathbb{E}\|\tilde{\nabla} f_i(\mathbf{x}_{t,i}^k, \xi_{t,i}^k)\|^2 \\
&\leq \frac{K}{m} G^\alpha \lambda^{2-\alpha},
\end{aligned}$$

where the first inequality follows from the fact that $\{\tilde{\nabla} f_i(\mathbf{x}_{t,i}^k, \xi_{t,i}^k) - \tilde{\nabla} f_i(\mathbf{x}_{t,i}^k)\}$ form a martingale difference sequence (Lemma 4 in [3]), the second inequalities is due to $\mathbb{E}[\|X - \mathbb{E}[X]\|^2] \leq \mathbb{E}[\|X\|^2]$, and the third inequality follows from the fact that $\mathbb{E}\|\tilde{\nabla} f_i(\mathbf{x}_{t,i}^k, \xi_{t,i}^k)\|^2 \leq \mathbb{E}\|\tilde{\nabla} f_i(\mathbf{x}_{t,i}^k, \xi_{t,i}^k)\|^\alpha \lambda^{2-\alpha} \leq G^\alpha \lambda^{2-\alpha}$ (see Lemma 9 in [22]).

$$\begin{aligned}
&\|\frac{1}{K}\mathbb{E}[\tilde{\Delta}_t] - \nabla f(x)\|^2 \leq \left\|\frac{1}{mK}\sum_{i\in[m]}\sum k \in [K] \left(\tilde{\nabla} f_i(\mathbf{x}_{t,i}^k) - f_i(\mathbf{x}_t)\right)\right\|^2 \\
&\leq \frac{1}{mK}\sum_{i\in[m]}\sum_{k\in[K]} \left\|\tilde{\nabla} f_i(\mathbf{x}_{t,i}^k) - f_i(\mathbf{x}_t)\right\|^2 \\
&\leq \frac{1}{mK}\sum_{i\in[m]}\sum_{k\in[K]} \left(2\left\|\tilde{\nabla} f_i(\mathbf{x}_{t,i}^k) - \nabla f_i(\mathbf{x}_{t,i}^k)\right\|^2 + 2\left\|\nabla f_i(\mathbf{x}_{t,i}^k) - f_i(\mathbf{x}_t)\right\|^2\right) \\
&\leq 2G^{2\alpha}\lambda^{-2(\alpha-1)} + 2L^2\frac{1}{mK}\sum_{i\in[m]}\sum_{k\in[K]} \|\mathbf{x}_{t,i}^k - \mathbf{x}_t\|^2 \\
&\leq 2G^{2\alpha}\lambda^{-2(\alpha-1)} + 2L^2\eta_L^2\frac{1}{mK}\sum_{i\in[m]}\sum_{k\in[K]} \left\|\sum_{j\in[K]} \nabla f(x_{t,i}^j, \xi_{t,i}^j)\right\|^2
\end{aligned}$$

$$\leq 2G^{2\alpha}\lambda^{-2(\alpha-1)} + 2L^2\eta_L^2K^2G^\alpha\lambda^{2-\alpha},$$

where the forth inequality is due to $\|\tilde{\nabla} f_i(\mathbf{x}) - \nabla f_i(\mathbf{x})\|^2 \leq G^{2\alpha}\lambda^{-2(\alpha-1)}$ (see Lemma 9 in [22]), and the last inequality follows from the fact that $\mathbb{E}\|\tilde{\nabla} f_i(\mathbf{x}_{t,i}^k, \xi_{t,i}^k)\|^2 \leq G^\alpha\lambda^{2-\alpha}$. □

Theorem 3. (Convergence Rate of FAT-Clipping-PI in the Strongly Convex Case) *Suppose that $f(\cdot)$ is a μ-strongly convex function. Under Assumptions 1–3, if $\eta\eta_L K \geq \frac{2}{\mu T}$, then the output $\bar{\mathbf{x}}_T$ of* FAT-Clipping-PI *being chosen in such a way that $\bar{\mathbf{x}}_T = \mathbf{x}_t$ with probability $\frac{w_t}{\sum_{j\in[T]} w_j}$, where $w_t = (1 - \frac{1}{2}\mu\eta\eta_L K)^{1-t}$, satisfies:*

$$f(\bar{\mathbf{x}}_T) - f(\mathbf{x}^*) \leq \frac{\mu}{2}\exp\left(-\frac{1}{2}\mu\eta\eta_L KT\right) + \frac{\eta\eta_L K}{2}G^\alpha\lambda^{2-\alpha}$$
$$+ \frac{4}{\mu}[2G^{2\alpha}\lambda^{-2(\alpha-1)} + 2L^2\eta_L^2K^2G^\alpha\lambda^{2-\alpha}],$$

where $\mathbf{x}^*$ denotes the global optimal solution. Further, let $\eta\eta_L K = \frac{2c}{\mu}\frac{\ln(T)}{mKT}$, where $c \geq 1$ is a constant satisfying $(mK)^{\frac{2-2\alpha}{\alpha}}T^{c+\frac{2-2\alpha}{\alpha}} \geq 1$, and let $\lambda = (mKT)^{\frac{1}{\alpha}}$, and $\eta_L \leq (mT)^{-\frac{1}{2}}K^{-\frac{3}{2}}$). It then follows that

$$f(\bar{\mathbf{x}}_T) - f(\mathbf{x}^*) = \tilde{\mathcal{O}}((mKT)^{\frac{2-2\alpha}{\alpha}}).$$

Proof. Similarly, we have the following one step iteration:

$$f(\mathbf{x}_t) - f(\mathbf{x}_*) \leq \frac{1}{2\eta\eta_L K}\left[-\mathbb{E}[\|\mathbf{x}_{t+1} - x_*\|^2] + (1 - \frac{1}{2}\mu\eta\eta_L K)\|\mathbf{x}_t - x_*\|^2\right]$$
$$+ \frac{\eta\eta_L}{2K}\mathbb{E}[\|\tilde{\Delta}_t\|^2] + \frac{4}{\mu}\|\mathbb{E}[\frac{1}{K}\tilde{\Delta}_t - \nabla f(\mathbf{x}_t)\|^2$$
$$\leq \frac{1}{2\eta\eta_L K}\left[-\mathbb{E}[\|\mathbf{x}_{t+1} - x_*\|^2] + (1 - \frac{1}{2}\mu\eta\eta_L K)\|\mathbf{x}_t - x_*\|^2\right]$$
$$+ \frac{\eta\eta_L}{2K}K^2G^\alpha\lambda^{2-\alpha} + \frac{4}{\mu}[2G^{2\alpha}\lambda^{-2(\alpha-1)} + 2L^2\eta_L^2K^2G^\alpha\lambda^{2-\alpha}],$$

where the last inequality is due to Lemma 2.

Let $w_t = (1 - \frac{1}{2}\mu\eta\eta_L K)^{1-t}$, $\bar{\mathbf{x}}_T = \mathbf{x}_t$ with probability $\frac{w_t}{\sum_{j\in[T]} w_j}$.

$$f(\bar{x}_T) - f(\mathbf{x}_*) \leq \frac{1}{\sum_{j\in[T]} w_j}\sum_{t\in[T]}\left(\frac{w_t}{2\eta\eta_L K}\left[-\|\mathbf{x}_{t+1} - x_*\|^2 + (1 - \frac{1}{2}\mu\eta\eta_L K)\|\mathbf{x}_t - x_*\|^2\right]\right)$$
$$+ \frac{\eta\eta_L K}{2}G^\alpha\lambda^{2-\alpha} + \frac{4}{\mu}[2G^{2\alpha}\lambda^{-2(\alpha-1)} + 2L^2\eta_L^2K^2G^\alpha\lambda^{2-\alpha}]$$
$$\leq \frac{1}{\sum_{j\in[T]} w_j}\sum_{t\in[T]}\left(\frac{1}{2\eta\eta_L K}\left[-w_t\|\mathbf{x}_{t+1} - x_*\|^2 + w_{t-1}\|\mathbf{x}_t - x_*\|^2\right]\right)$$
$$+ \frac{\eta\eta_L K}{2}G^\alpha\lambda^{2-\alpha} + \frac{4}{\mu}[2G^{2\alpha}\lambda^{-2(\alpha-1)} + 2L^2\eta_L^2K^2G^\alpha\lambda^{2-\alpha}]$$
$$\leq \frac{1}{\sum_{j\in[T]} w_j}\frac{1}{2\eta\eta_L K}\|\mathbf{x}_1 - x_*\|^2$$
$$+ \frac{\eta\eta_L K}{2}G^\alpha\lambda^{2-\alpha} + \frac{4}{\mu}[2G^{2\alpha}\lambda^{-2(\alpha-1)} + 2L^2\eta_L^2K^2G^\alpha\lambda^{2-\alpha}].$$

$$2\eta\eta_L K\sum_{t\in[T]} w_t = 2\eta\eta_L K\left(1 - \frac{1}{2}\mu\eta\eta_L K\right)^{-T}\sum_{t\in[T]}\left(1 - \frac{1}{2}\mu\eta\eta_L K\right)^t$$
$$= \frac{4}{\mu}\left(1 - \frac{1}{2}\mu\eta\eta_L K\right)^{-T}\left[1 - \left(1 - \frac{1}{2}\mu\eta\eta_L K\right)^T\right]$$

$$
\geq \frac{4}{\mu}\left(1-\frac{1}{2}\mu\eta\eta_L K\right)^{-T}\left[1-\exp\left(-\frac{1}{2}\mu\eta\eta_L KT\right)\right]
$$

$$
\geq \frac{2}{\mu}\left(1-\frac{1}{2}\mu\eta\eta_L K\right)^{-T},
$$

where the last inequality follows from that $\eta\eta_L K \geq \frac{2}{\mu T}$, teh second last inequality is due to $\left(1-\frac{1}{2}\mu\eta\eta_L K\right)^T \leq \exp\left(-\frac{1}{2}\mu\eta\eta_L KT\right)$.

$$
\begin{aligned}
f(\bar{x}_T) - f(\mathbf{x}_*) &\leq \frac{\mu}{2}\left(1-\frac{1}{2}\mu\eta\eta_L K\right)^T + \frac{\eta\eta_L K}{2}G^\alpha\lambda^{2-\alpha} + \frac{4}{\mu}[2G^{2\alpha}\lambda^{-2(\alpha-1)} + 2L^2\eta_L^2K^2G^\alpha\lambda^{2-\alpha}] \\
&\leq \frac{\mu}{2}\exp\left(-\frac{1}{2}\mu\eta\eta_L KT\right) + \frac{\eta\eta_L K}{2}G^\alpha\lambda^{2-\alpha} + \frac{4}{\mu}[2G^{2\alpha}\lambda^{-2(\alpha-1)} + 2L^2\eta_L^2K^2G^\alpha\lambda^{2-\alpha}].
\end{aligned}
$$

Let $\eta\eta_L K = \frac{2c}{\mu}\frac{\ln(T)}{mKT}$ ($c \geq 1$ is a constant and $T^{-c-\frac{2-2\alpha}{\alpha}} \leq (mK)^{\frac{2-2\alpha}{\alpha}}$), $\lambda = (mKT)^{\frac{1}{\alpha}}$, and $\eta_L \leq (mT)^{-\frac{1}{2}}K^{-\frac{3}{2}}$,

$$
f(\bar{x}_T) - f(\mathbf{x}_*) \leq \frac{1}{T^c} + (mKT)^{\frac{2-2\alpha}{\alpha}}\ln(T) = \tilde{\mathcal{O}}((mKT)^{\frac{2-2\alpha}{\alpha}}).
$$

□

Theorem 4. (Convergence Rate of FAT-Clipping-PI in the Nonconvex Case) *Suppose that $f(\cdot)$ is a non-convex function. Under Assumptions 1–3, if $\eta\eta_L KL \leq 1$, then the sequence of outputs $\{\mathbf{x}_k\}$ generated by* FAT-Clipping-PI *satisfies:*

$$
\min_{t\in[T]}\mathbb{E}\|\nabla f(\mathbf{x}_t)\|^2 \leq \frac{2\left(f(\mathbf{x}_1)-f(x_T)\right)}{\eta\eta_L KT} + \left(2G^{2\alpha}\lambda^{-2(\alpha-1)} + 2L^2\eta_L^2K^2G^\alpha\lambda^{2-\alpha}\right) + \frac{L\eta\eta_L}{m}\left(G^\alpha\lambda^{2-\alpha}\right).
$$

Further, choosing learning rates and clipping parameter in such a way that $\eta\eta_L = m^{\frac{2\alpha-2}{3\alpha-2}}(KT)^{\frac{-\alpha}{3\alpha-2}}, \eta_L \leq (mKT)^{\frac{-\alpha}{6\alpha-4}}$, and $\lambda = (mKT)^{\frac{1}{3\alpha-2}}$, we have

$$
\min_{t\in[T]}\mathbb{E}\|\nabla f(\mathbf{x}_t)\|^2 \leq \mathcal{O}((mKT)^{\frac{2-2\alpha}{3\alpha-2}}).
$$

Proof. Due to the smoothness in Assumption 1, taking expectation of $f(\mathbf{x}_{t+1})$ over the randomness at communication round t, we have:

$$
\begin{aligned}
&\mathbb{E}[f(\mathbf{x}_{t+1})] - f(\mathbf{x}_t) \leq \left\langle \nabla f(\mathbf{x}_t), \mathbb{E}[\mathbf{x}_{t+1}-\mathbf{x}_t]\right\rangle + \frac{L}{2}\mathbb{E}[\|\mathbf{x}_{t+1}-\mathbf{x}_t\|^2] \\
&= -\eta\eta_L\left\langle \nabla f(\mathbf{x}_t), \mathbb{E}[\tilde{\Delta}_t]\right\rangle + \frac{L}{2}\eta^2\eta_L^2\mathbb{E}[\|\tilde{\Delta}_t\|^2] \\
&= -\frac{\eta\eta_L K}{2}\|\nabla f(\mathbf{x}_t)\|^2 - \frac{\eta\eta_L}{2K}\|\mathbb{E}[\tilde{\Delta}_t]\|^2 + \frac{\eta\eta_L K}{2}\|\nabla f(\mathbf{x}_t) - \frac{1}{K}\mathbb{E}[\tilde{\Delta}_t]\|^2 + \frac{L\eta^2\eta_L^2}{2}\mathbb{E}[\|\tilde{\Delta}_t\|^2] \\
&= -\frac{\eta\eta_L K}{2}\|\nabla f(\mathbf{x}_t)\|^2 + \left(-\frac{\eta\eta_L}{2K} + \frac{L\eta^2\eta_L^2}{2}\right)\|\mathbb{E}[\tilde{\Delta}_t]\|^2 + \frac{\eta\eta_L K}{2}\|\nabla f(\mathbf{x}_t) - \frac{1}{K}\mathbb{E}[\tilde{\Delta}_t]\|^2 + \frac{L\eta^2\eta_L^2}{2}\mathbb{E}[\|\tilde{\Delta}_t - \mathbb{E}[\tilde{\Delta}_t]\|^2] \\
&\leq -\frac{\eta\eta_L K}{2}\|\nabla f(\mathbf{x}_t)\|^2 + \frac{\eta\eta_L K}{2}\underbrace{\|\nabla f(\mathbf{x}_t) - \frac{1}{K}\mathbb{E}[\tilde{\Delta}_t]\|^2}_{A_1} + \frac{L\eta^2\eta_L^2}{2}\underbrace{\mathbb{E}[\|\tilde{\Delta}_t - \mathbb{E}[\tilde{\Delta}_t]\|^2]}_{A_2},
\end{aligned} \tag{6}
$$

where the last inequality follows from $\left(-\frac{\eta\eta_L}{2K} + \frac{L\eta^2\eta_L^2}{2}\right) \leq 0$ if $\eta\eta_L KL \leq 1$.

From Lemma 2, we have the bound of A_1 and A_2 in (6). By rearranging and telescoping, we have:

$$
\frac{1}{T}\sum_{t\in[T]}\mathbb{E}\|\nabla f(\mathbf{x}_t)\|^2 \leq \frac{2\left(f(\mathbf{x}_1)-f(x_T)\right)}{\eta\eta_L KT} + \left(2G^{2\alpha}\lambda^{-2(\alpha-1)} + 2L^2\eta_L^2K^2G^\alpha\lambda^{2-\alpha}\right) + \frac{L\eta\eta_L}{m}\left(G^\alpha\lambda^{2-\alpha}\right).
$$

Suppose $\eta\eta_L = m^{\frac{2\alpha-2}{3\alpha-2}}(KT)^{\frac{-\alpha}{3\alpha-2}}, \eta_L \leq (mKT)^{\frac{-\alpha}{6\alpha-4}}$, and $\lambda = (mKT)^{\frac{1}{3\alpha-2}}$,

$$\min_{t\in[T]} \mathbb{E}\|\nabla f(\mathbf{x}_t)\|^2 \leq \mathcal{O}((mKT)^{\frac{2-2\alpha}{3\alpha-2}}).$$

$\square$

A.3 Proof of FAT-Clipping-PR in Gaussian Noise

In this subsection, we utilize the classic bounded variance and bounded gradient assumption.

Assumption 4. *(Bounded Stochastic Gradient Variance) There exists a constant $\sigma > 0$, such that the variance of each local gradient estimator is bounded by $\mathbb{E}[\|\nabla f_i(\mathbf{x}, \xi) - \nabla f_i(\mathbf{x})\|^2] \leq \sigma^2$, $\forall i \in [m]$.*

Assumption 5. *(Bounded Gradient) There exists a constant $G \geq 0$, such that gradient is bounded by $\|\nabla f_i(\mathbf{x})\|^2 \leq G^2$, $\forall i \in [m]$.*

Lemma 3 (Lemma F.5 [18])**.** *Suppose there exists a constant σ such that the variance of the stochastic gradient of F has bounded variance, i.e., $\mathbb{E}[\|\nabla F(\mathbf{x}, \xi) - \nabla F(\mathbf{x})\|^2] \leq \sigma^2$, and $\|\nabla F(\mathbf{x})\|^2 \leq \frac{\lambda}{2}$, then we have the following inequalities for the clipping $\tilde{\nabla} F(\mathbf{x}_t) = \mathbb{E}[\tilde{\nabla} F(\mathbf{x}, \xi)] = \mathbb{E}[\min\{1, \frac{\lambda}{\|\nabla F(\mathbf{x},\xi)\|}\}\nabla F(\mathbf{x}, \xi)]$:*

$$\begin{aligned}
&\|\mathbb{E}[\tilde{\nabla} F(\mathbf{x}, \xi)] - \nabla F(\mathbf{x})\|^2 \leq \frac{16\sigma^4}{\lambda^2}, \\
&\mathbb{E}\|\tilde{\nabla} F(\mathbf{x}, \xi) - \nabla F(\mathbf{x})\|^2 \leq 18\sigma^2, \\
&\mathbb{E}\|\tilde{\nabla} F(\mathbf{x}, \xi) - \mathbb{E}[\tilde{\nabla} F(\mathbf{x}, \xi)]\|^2 \leq 18\sigma^2.
\end{aligned}$$

We remark that for any stochastic estimator satisfies the above conditions, the above inequalities hold. The proof is the exactly same as that in original proof [18].

Lemma 4 (Bounded Variance of Clipping Stochastic Local Updates in FAT-Clipping-PR)**.** *Assume f_i satisfies the bounded variance assumption, then we have:*

$$\mathbb{E}[\|\Delta_{t,i} - \mathbb{E}[\Delta_{t,i}]\|^2] \leq K\sigma^2.$$

In addition, assume there exists a constant G such that gradient is bounded $\|\nabla f_i(\mathbf{x})\|^2 \leq G^2$, if we set clipping parameter as $\lambda^2 \geq 2K^2G^2$, i.e., $\|\nabla f_i(\mathbf{x})\| \leq \frac{\lambda}{2}$, then we have:

$$\begin{aligned}
\left\|\mathbb{E}[\Delta_{t,i}] - \mathbb{E}[\tilde{\Delta}_{t,i}]\right\|^2 &\leq \frac{16K\sigma^4}{\lambda^2}, \\
\mathbb{E}\|\tilde{\Delta}_t - \mathbb{E}[\tilde{\Delta}_t]\|^2 &\leq \frac{18K}{m}\sigma^4.
\end{aligned}$$

.

Proof.

$$\begin{aligned}
\mathbb{E}[\|\Delta_{t,i} - \mathbb{E}[\Delta_{t,i}]\|^2] &= \mathbb{E}[\|\nabla f(\mathbf{x}_{t,i}^j, \xi_{t,i}^j) - \mathbb{E}[\nabla f(\mathbf{x}_{t,i}^j)]\|^2] \\
&\leq K\sigma^2,
\end{aligned}$$

where $\{\nabla f(\mathbf{x}_{t,i}^j, \xi_{t,i}^j) - \mathbb{E}[\nabla f(\mathbf{x}_{t,i}^j)\}$ forms martingale difference sequence (Lemma 4 in [3]).

Then by applying Lemma 3, we have the bound of $\left\|\mathbb{E}[\Delta_{t,i}] - \mathbb{E}[\tilde{\Delta}_{t,i}]\right\|^2$.

$$\begin{aligned}
\mathbb{E}\|\tilde{\Delta}_t - \mathbb{E}[\tilde{\Delta}_t]\|^2 &= \mathbb{E}\left\|\frac{1}{m}\sum_{i\in[m]} \tilde{\Delta}_{t,i} - \frac{1}{m}\sum_{i\in[m]} \mathbb{E}[\tilde{\Delta}_{t,i}]\right\|^2 \\
&\leq \frac{18K}{m}\sigma^4.
\end{aligned}$$

where the last inequality follows from the fact that $\mathbb{E}[\|\Delta_{t,i} - \mathbb{E}[\Delta_{t,i}]\|^2] \leq K\sigma^2$, $\{\Delta_{t,i} - \mathbb{E}[\Delta_{t,i}]\}$ forms martingale difference sequence and Lemma 3. $\square$

Theorem 5. *Suppose f is non-convex function, under Assumptions 1, 2, 4, and 5, if $\eta\eta_L KL \leq 1$, then the sequence of outputs $\{\mathbf{x}_k\}$ generated by Algorithm* FAT-Clipping-PR *satisfies:*

$$\frac{1}{T}\sum_{t\in[T]}\mathbb{E}\|\nabla f(\mathbf{x}_t)\|^2 \leq \frac{2\left(f(\mathbf{x}_0)-f(\mathbf{x}_T)\right)}{\eta\eta_L KT} + \frac{1}{T}\sum_{t\in[T]}\left(2L^2\eta_L^2K^2(\sigma^2+G^2)+\frac{32\sigma^4}{K^2\lambda_t^2}\right)+\left(\frac{18L\eta\eta_L}{m}\sigma^2\right).$$

Choosing learning rates and clipping parameter as $\eta\eta_L = \frac{m^{1/2}}{(KT)^{1/2}}, \eta_L \leq \frac{1}{(mT)^{1/2}K^{5/2}}$, and $\lambda_t \geq (mT)^{1/4}K^{-3/4}$,

$$\min_{t\in[T]}\mathbb{E}\|\nabla f(\mathbf{x}_t)\|^2 \leq \mathcal{O}((mKT)^{-\frac{1}{2}}).$$

Proof. Due to the smoothness in Assumption 1, taking expectation of $f(\mathbf{x}_{t+1})$ over the randomness at communication round t, we have the same inequality:

$$\mathbb{E}[f(\mathbf{x}_{t+1})]-f(\mathbf{x}_t) \leq -\frac{\eta\eta_L K}{2}\|\nabla f(\mathbf{x}_t)\|^2+\frac{\eta\eta_L K}{2}\underbrace{\|\nabla f(\mathbf{x}_t)-\frac{1}{K}\mathbb{E}[\tilde{\Delta}_t]\|^2}_{A_1}+\frac{L\eta^2\eta_L^2}{2}\underbrace{\mathbb{E}[\|\tilde{\Delta}_t-\mathbb{E}[\tilde{\Delta}_t]\|^2]}_{A_2}, \tag{7}$$

where it requires $\eta\eta_L KL \leq 1$.

Note that the term A_1 in (7) can be bounded as follows:

$$\begin{aligned}
A_1 &= \|\nabla f(\mathbf{x}_t)-\frac{1}{K}\mathbb{E}[\tilde{\Delta}_t]\|^2\\
&= 2\left\|\nabla f(\mathbf{x}_t)-\frac{1}{K}\mathbb{E}[\Delta_t]\right\|^2+\frac{2}{K^2}\left\|\mathbb{E}[\Delta_t]-\mathbb{E}[\tilde{\Delta}_t]\right\|^2\\
&\leq \frac{2}{mK}\sum_{i\in[m]}\sum_{k\in[K]}\left\|\nabla f_i(\mathbf{x}_t)-\nabla f_i(\mathbf{x}_{t,i}^k)\right\|^2+\frac{2}{mK^2}\sum_{i\in[m]}\left\|\mathbb{E}[\Delta_{t,i}]-\mathbb{E}[\tilde{\Delta}_{t,i}]\right\|^2\\
&\leq \frac{2L^2\eta_L^2}{mK}\sum_{i\in[m]}\sum_{k\in[K]}\mathbb{E}\left\|\sum_{j\in[k]}\nabla f_i(\mathbf{x}_{t,i}^j,\xi_{t,i}^j)\right\|^2+\frac{2}{mK^2}\sum_{i\in[m]}\left\|\mathbb{E}[\Delta_{t,i}]-\mathbb{E}[\tilde{\Delta}_{t,i}]\right\|^2\\
&\leq 2L^2\eta_L^2K^2\left(\mathbb{E}\left\|\nabla f_i(\mathbf{x}_{t,i}^k,\xi_{t,i}^k)-\nabla f_i(\mathbf{x}_{t,i}^k)\right\|^2+\left\|\nabla f_i(\mathbf{x}_{t,i}^k)\right\|^2\right)+\frac{32\sigma^4}{K^2\lambda_t^2}\\
&\leq 2L^2\eta_L^2K^2(\sigma^2+G^2)+\frac{32\sigma^4}{K^2\lambda_t^2}
\end{aligned}$$

where the second inequality is due to smoothness assumption 1, the third inequality is due to Lemma 4, and the last inequality follows from bounded variance assumption 4 and bounded gradient assumption 5.

From Lemma 4, the term A_2 in (7) can be bounded as follows:

$$A_2 \leq \frac{18K\sigma^2}{m}.$$

Putting pieces together, we can have the one communication round descent in expectation:

$$\begin{aligned}
\mathbb{E}[f(\mathbf{x}_{t+1})]-f(\mathbf{x}_t) &\leq -\frac{\eta\eta_L K}{2}\|\nabla f(\mathbf{x}_t)\|^2+\frac{\eta\eta_L K}{2}\underbrace{\|\nabla f(\mathbf{x}_t)-\frac{1}{K}\mathbb{E}[\tilde{\Delta}_t]\|^2}_{A_1}+\frac{L\eta^2\eta_L^2}{2}\underbrace{\mathbb{E}[\|\tilde{\Delta}_t-\mathbb{E}[\tilde{\Delta}_t]\|^2]}_{A_2}\\
&\leq -\frac{\eta\eta_L K}{2}\|\nabla f(\mathbf{x}_t)\|^2+\frac{\eta\eta_L K}{2}\left(2L^2\eta_L^2K^2(\sigma^2+G^2)+\frac{32\sigma^4}{K^2\lambda_t^2}\right)+\frac{18LK\eta^2\eta_L^2}{2m}\sigma^2.
\end{aligned}$$

Rearranging and telescoping, we have the final convergence result:

$$\frac{1}{T}\sum_{t\in[T]}\mathbb{E}\|\nabla f(\mathbf{x}_t)\|^2 \leq \frac{2\left(f(\mathbf{x}_0)-f(\mathbf{x}_T)\right)}{\eta\eta_L KT}+\frac{1}{T}\sum_{t\in[T]}\left(2L^2\eta_L^2K^2(\sigma^2+G^2)+\frac{32\sigma^2}{K^2\lambda_t^2}\right)+\left(\frac{18L\eta\eta_L}{m}\sigma^2\right).$$

Suppose $\eta\eta_L = \frac{m^{1/2}}{(KT)^{1/2}}$, $\eta_L \leq \frac{1}{(mT)^{1/2}K^{5/2}}$, and $\lambda_t \geq (mT)^{1/4}K^{-3/4}$,

$$\min_{t\in[T]} \mathbb{E}\|\nabla f(\mathbf{x}_t)\|^2 \leq \mathcal{O}((mKT)^{-\frac{1}{2}}).$$

□

Theorem 6. *Suppose f is μ-strongly convex function, under Assumptions 1–3, if $\eta\eta_L K \geq \frac{2}{\mu T}$, then the outputs $\bar{\mathbf{x}}_T$ in Algorithm 2 (*FAT-Clipping-PR*) by $\bar{\mathbf{x}}_T = \mathbf{x}_t$ with probability $\frac{w_t}{\sum_{j\in[T]} w_j}$ where $w_t = (1-\frac{1}{2}\mu\eta\eta_L K)^{1-t}$ satisfies:*

$$f(\bar{x}_T) - f(\mathbf{x}_*) \leq \frac{\mu}{2}\exp\left(-\frac{1}{2}\mu\eta\eta_L KT\right) + \frac{\eta\eta_L K}{2}G^2 + \frac{4}{\mu}[2L^2\eta_L^2K^2(G^2+\sigma^2) + \frac{32\sigma^4}{\lambda^2}].$$

Suppose $\eta\eta_L K = \frac{2c}{\mu}\frac{\ln(T)}{mKT}$ ($c > 0$ is a constant and $T^{-c+1} \leq (mK)^{-1}$), $\lambda \geq (mKT)^{\frac{1}{2}}$, and $\eta_L \leq (mT)^{-\frac{1}{2}}K^{-\frac{3}{2}}$,

$$f(\bar{x}_T) - f(\mathbf{x}_*) = \tilde{\mathcal{O}}((mKT)^{-1}).$$

Proof.

$$\begin{aligned}
&\|\frac{1}{K}\mathbb{E}[\tilde{\Delta}_t] - \nabla f(x)\|^2 \leq 2\left\|\nabla f(\mathbf{x}_t) - \frac{1}{K}\mathbb{E}[\Delta_t]\right\|^2 + \frac{2}{K}\left\|\mathbb{E}[\Delta_t] - \mathbb{E}[\tilde{\Delta}_t]\right\|^2 \\
&\leq \frac{2}{mK}\sum_{i\in[m]}\sum_{k\in[K]}\mathbb{E}\left\|\nabla f_i(\mathbf{x}_t) - \nabla f_i(\mathbf{x}_{t,i}^k)\right\|^2 + \frac{2}{mK}\sum_{i\in[m]}\left\|\mathbb{E}[\Delta_{t,i}] - \mathbb{E}[\tilde{\Delta}_{t,i}]\right\|^2 \\
&\leq \frac{2L^2\eta_L^2}{mK}\sum_{i\in[m]}\sum_{k\in[K]}\mathbb{E}\left\|\sum_{j\in[k]}\nabla f_i(\mathbf{x}_{t,i}^j, \xi_{t,i}^j)\right\|^2 + \frac{32\sigma^4}{\lambda^2} \\
&\leq 2L^2\eta_L^2K^2(G^2+\sigma^2) + \frac{32\sigma^4}{\lambda^2}.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
f(\mathbf{x}_t) - f(\mathbf{x}_*) &\leq \frac{1}{2\eta\eta_L K}\left[-\mathbb{E}[\|\mathbf{x}_{t+1} - x_*\|^2] + (1-\frac{1}{2}\mu\eta\eta_L K)\|\mathbf{x}_t - x_*\|^2\right] \\
&\quad + \frac{\eta\eta_L}{2K}\mathbb{E}[\|\tilde{\Delta}_t\|^2] + \frac{4}{\mu}\|\mathbb{E}[\frac{1}{K}\tilde{\Delta}_t - \nabla f(\mathbf{x}_t)\|^2 \\
&\leq \frac{1}{2\eta\eta_L K}\left[-\mathbb{E}[\|\mathbf{x}_{t+1} - x_*\|^2] + (1-\frac{1}{2}\mu\eta\eta_L K)\|\mathbf{x}_t - x_*\|^2\right] \\
&\quad + \frac{\eta\eta_L K}{2}G^2 + \frac{4}{\mu}[2L^2\eta_L^2K^2(G^2+\sigma^2) + \frac{32\sigma^4}{\lambda^2}],
\end{aligned}$$

Let $w_t = (1-\frac{1}{2}\mu\eta\eta_L K)^{1-t}$, $\bar{\mathbf{x}}_T = \mathbf{x}_t$ with probability $\frac{w_t}{\sum_{j\in[T]} w_j}$.

$$\begin{aligned}
f(\bar{x}_T) - f(\mathbf{x}_*) &\leq \frac{1}{\sum_{j\in[T]} w_j}\sum_{t\in[T]}\left(\frac{w_t}{2\eta\eta_L K}\left[-\|\mathbf{x}_{t+1} - x_*\|^2 + (1-\frac{1}{2}\mu\eta\eta_L K)\|\mathbf{x}_t - x_*\|^2\right]\right) \\
&\quad + \frac{\eta\eta_L K}{2}G^2 + \frac{4}{\mu}[2L^2\eta_L^2K^2(G^2+\sigma^2) + \frac{32\sigma^4}{\lambda^2}] \\
&\leq \frac{1}{\sum_{j\in[T]} w_j}\frac{1}{2\eta\eta_L K}\|\mathbf{x}_1 - x_*\|^2 + \frac{\eta\eta_L K}{2}G^2 + \frac{4}{\mu}[2L^2\eta_L^2K^2(G^2+\sigma^2) + \frac{32\sigma^4}{\lambda^2}].
\end{aligned}$$

Same as that in heavy-tailed noise case, we have the same bound for $2\eta\eta_L K \sum_{t\in[T]} w_t$:

$$2\eta\eta_L K \sum_{t\in[T]} w_t \geq \frac{2}{\mu}\left(1-\frac{1}{2}\mu\eta\eta_L K\right)^{-T},$$

where it requires $\eta\eta_L K \geq \frac{2}{\mu T}$.

$$\begin{aligned} f(\bar{x}_T)-f(\mathbf{x}_*) &\leq \frac{\mu}{2}\left(1-\frac{1}{2}\mu\eta\eta_L K\right)^{T}+\frac{\eta\eta_L K}{2}G^2+\frac{4}{\mu}[2L^2\eta_L^2K^2(G^2+\sigma^2)+\frac{32\sigma^4}{\lambda^2}] \\ &\leq \frac{\mu}{2}\exp\left(-\frac{1}{2}\mu\eta\eta_L KT\right)+\frac{\eta\eta_L K}{2}G^2+\frac{4}{\mu}[2L^2\eta_L^2K^2(G^2+\sigma^2)+\frac{32\sigma^4}{\lambda^2}]. \end{aligned}$$

Let $\eta\eta_L K = \frac{2c}{\mu}\frac{\ln(T)}{mKT}$ ($c > 0$ is a constant and $T^{-c+1} \leq (mK)^{-1}$), $\lambda \geq (mKT)^{\frac{1}{2}}$, and $\eta_L \leq (mT)^{-\frac{1}{2}}K^{-\frac{3}{2}}$,

$$f(\bar{x}_T)-f(\mathbf{x}_*) = \tilde{\mathcal{O}}((mKT)^{-1}).$$

□

B Experiments in Section 3

In this section, we provide experimental details to demonstrate the fat-tailed noise phenomenon in federated learning. We conduct experiments with CNN on CIFAR-10 dataset as shown in Section 3, and provide additional results of RNN model on Shakespeare dataset. Furthermore, we verify the accuracy of α estimation with logistic regression on MNIST dataset.

B.1 CNN on CIFAR-10 Dataset

B.1.1 Experiment details

We run a convolutional neural network (CNN) model on CIFAR-10 dataset using FedAvg. The CNN architecture is shown in Table 2. To simulate data heterogeneity across clients, we manually distribute the the data to each client in a label-based partition. Specifically, we split the data according to the classes (p) of images that each client has. Then, we randomly distribute these partitioned data to $m = 100$ clients such that each client has only p classes of images in both training and test data, which causes the heterogeneity of data among different clients. For example, for $p = 10$, each client contains training/test data samples with ten classes. Since CIFAR-10 has 10 classes of images, $p = 10$ is the nearly i.i.d case. For the remaining p, each client contains data samples with class p. Therefore, the classes (p) of images in each client's local dataset can be used to represent the non-i.i.d. degree. The smaller the p-value, the more heterogeneous the data between clients.

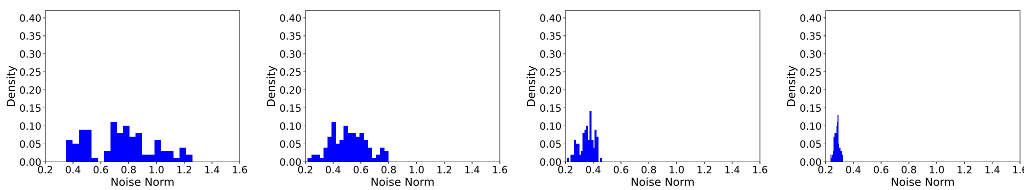
In this experimental setting, we use the global learning rate $\frac{\eta\eta_L}{m} = 1.0$ and the local learning rate $\eta_L = 0.1$. The batch size is set to 500, and the communication round is $T = 4000$. We run this experiment in different cases, including singleSGD and different local epochs $\{1, 2, 5\}$ and non-iid index $p \in \{1, 2, 5, 10\}$. Single SGD means one local update step, which is equivalent to mini-batch SGD.

Table 2: CNN architecture for CIFAR-10.

LAYER TYPE	SIZE
Convolution + ReLu	$3 \times 32 \times 5$
Max Pooling	2×2
Convolution + ReLu	$32 \times 64 \times 5$
Max Pooling	2×2
Fully Connected + ReLU	1600×512
Fully Connected + ReLU	512×128
Fully Connected	128×10

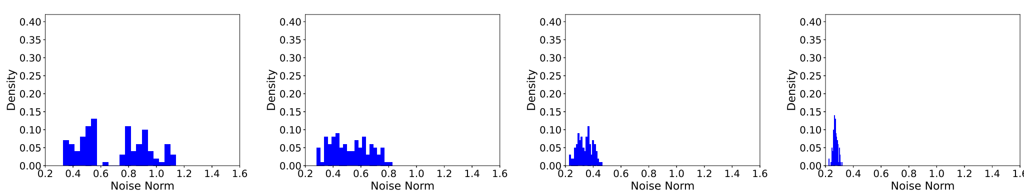
B.1.2 Additional experimental results

We provide additional distributions of the norms of the pseudo-gradient noises in different cases as follows. From Fig. 7- 10, the observation is that the gradient norm statistics are contracted together for more iid cases while dispersed uniformly for more non-iid cases. This is



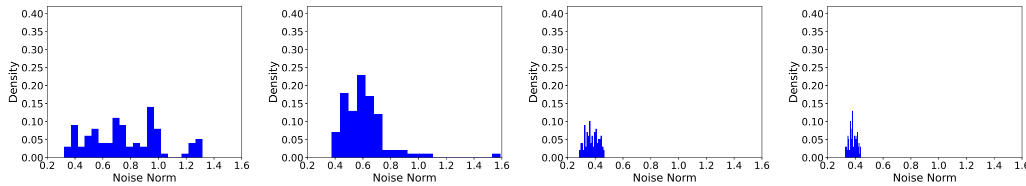
(a) $p = 1$ (b) $p = 2$ (c) $p = 5$ (d) $p = 10$

Figure 7: Distributions of the norms of the pseudo-gradient noises for CIFAR-10 dataset in the case of *Single SGD*.



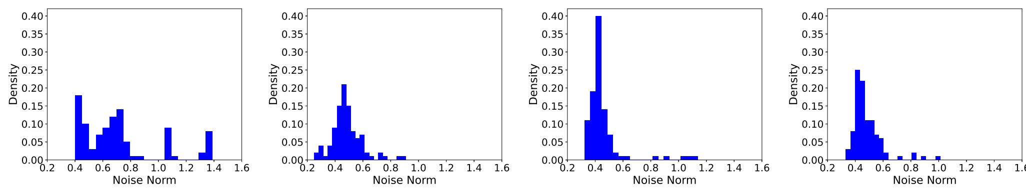
(a) $p = 1$ (b) $p = 2$ (c) $p = 5$ (d) $p = 10$

Figure 8: Distributions of the norms of the pseudo-gradient noises for CIFAR-10 dataset in the case of *Local Epoch=1*.



(a) $p = 1$ (b) $p = 2$ (c) $p = 5$ (d) $p = 10$

Figure 9: Distributions of the norms of the pseudo-gradient noises for CIFAR-10 dataset in the case of *Local Epoch=2*.



(a) $p = 1$ (b) $p = 2$ (c) $p = 5$ (d) $p = 10$

Figure 10: Distributions of the norms of the pseudo-gradient noises for CIFAR-10 dataset in the case of *Local Epoch=5*.

B.2 RNN on Shakespeare Dataset

B.2.1 Experiment details

To provide more evidences of the fat-tailed noise phenomenon, we further run a recurrent neural network (RNN) model on Shakespeare dataset.

Shakespeare dataset is a natural non-iid dataset, and it is built from *The Complete Works of William Shakespeare* [1]. The learning task is to predict next character, and there are 80 classes of characters in total. We use a two-layer LSTM classifier containing 100 hidden units with an 8-dimensional (8D) embedding layer. The model inputs a sequence of 80 characters, embeds each of the characters into a learned 8D space, and then outputs one character per training sample after two LSTM layers and a densely-connected layer. The dataset and model are taken from [45].

There are $m = 143$ clients participating in this experiment. The global learning rate is chosen as 1.0, and the local learning rate is chosen as 0.8. The batch size is set to 10, and the communication round is $T = 150$.

B.2.2 Experimental results

We show the results when local step is set to be one (Single SGD), and multiple local epochs $\{1, 2, 5\}$. In Fig. 11, we observe that the α-value is smaller than 2, and it increases when the number of local epoch increases. This implies that the gradient noise is fat-tailed. Fig. 12 shows that the distributions of the norms of the pseudo-gradient noises are fat-tailed.

B.3 Accuracy of Alpha Estimation (Logistic Regression on MNIST Dataset)

Accurate α-value computation requires the full-gradient calculation, and we have to compute both full-gradient and stochastic gradient in each local step. This is computationally expensive. Instead, we use an estimation to approximate the exact α-value. The full-gradient is replaced by the mean value of the stochastic gradients. We verify the accuracy of this estimation method by running logistic regression on MNIST dataset [46]. The details and the results are described as follows.

B.3.1 Experiment details

MNIST dataset contains ten classes of images, and it is manually partitioned using the same method as to partition CIFAR-10 dataset (see details in Appendix B.1.1). The number of classes (p) that each client has can be used to represent the non-iid level.

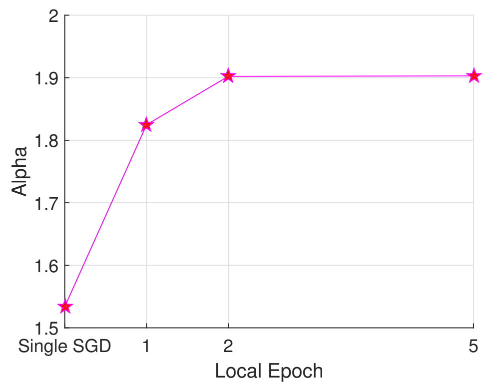
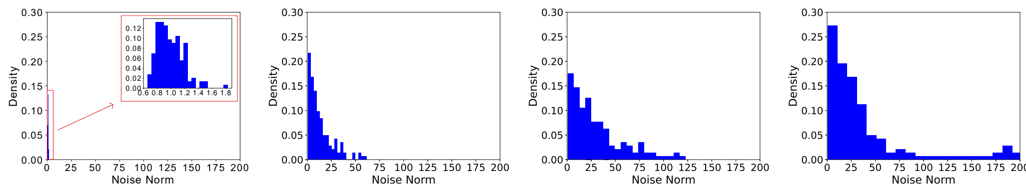


Figure 11: Estimation of α for Shakespeare dataset.



(a) Single SGD (b) Local Epoch=1 (c) Local Epoch=2 (d) Local Epoch=5

Figure 12: Distributions of the norms of the pseudo-gradient noises for Shakespeare dataset.

$m = 100$ clients participate in the experiment. The communication round is $T = 150$. The global learning rate is set to 1.0, and the local learning rate is set to 0.1. The batch size is chosen to be 64.

B.3.2 Experimental results

Table 3 shows the error rate of α-value estimation in different cases, and this implies that the estimation of α-value is within an acceptable margin of error.

C Experiments in Section 5

In this section, we describe the details of the numerical experiments from Section 5 and provide some extra experimental results.

C.1 Experiment details

C.1.1 Strongly Convex Model with Synthetic Data

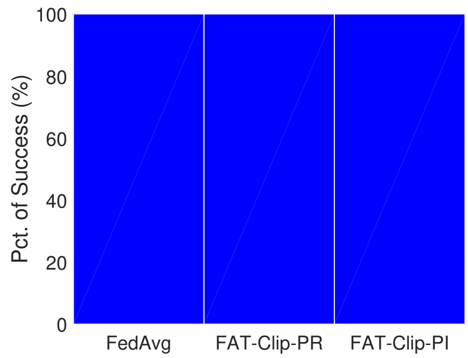
In these experiments, we consider a strongly convex model for Problem (1) as follows:

$$f_i(x) = \mathbb{E}_\xi \left[f_i(x, \xi) \right]$$
$$f_i(x, \xi) = \frac{1}{2} \|x\|^2 + \langle \xi, x \rangle ,$$

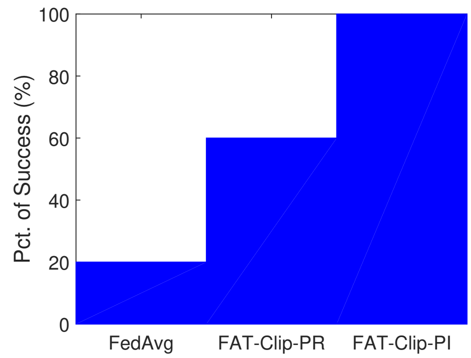
where $x \in \mathbb{R}^{3\times 1}$ and ξ is a random vector. The optimal solution is $f(x^*) = 0$ with $x^* = [0; 0; 0]$.

Table 3: Error rate (%) of α-value estimation.

	NonIID Index (p)			
	1	2	5	10
Single SGD	-2.82	-1.09	-0.12	3.12
Local Epoch=1	1.19	0.37	1.4	2.08
Local Epoch=2	1.8	1.4	1.43	1.74
Local Epoch=5	1.86	0.23	0.56	0.25



(a) non-i.i.d. index $p = 1$



(b) non-i.i.d. index $p = 5$

Figure 13: Percentage of successful training over 5 trials when applying FedAvg, FAT-Clipping-PR and FAT-Clipping-PI to CIFAR-10 dataset in non-i.i.d. cases.

To compare the performance of FedAvg, FAT-Clipping-PI and FAT-Clipping-PR , we consider the noise ξ to be a Cauchy distribution($\alpha < 2$, fat-tailed) with a location parameter of 0 and a scale parameter of 2.1.

To compare the performance of FAT-Clipping-PI and FAT-Clipping-PR under different scenarios, we consider the noise ξ having different tail-indexes ($\alpha = 0.5, 1.0,$ and 1.5) with the same location parameters of 0 and the same scale parameters of 1.

For all the distributions of ξ mentioned above, we use the same experimental setup. There are $m = 5$ clients participating in the training. We choose the starting point $x_0 = [2; 1; 1.5]$. We set the global learning rate $\frac{\eta\eta_L}{m} = 0.1$ and the local learning rate $\eta_L = 0.1$. The local steps we use is $K = 2$, and the communication round is $T = 300$. The clipping parameter in FAT-Clipping-PI we select is $\lambda = 3$, and the clipping parameter in FAT-Clipping-PR is $\lambda = 5$.

C.1.2 CNN (Non-convex Model) on the CIFAR-10

To test the performance of FAT-Clipping-PI and FAT-Clipping-PR for non-convex function, we run a convolutional neural network (CNN) on CIFAR-10 dataset. We compare FAT-Clipping-PI and FAT-Clipping-PR with FedAvg under different data heterogeneity.

In this experimental setting, we randomly select five clients from $m = 10$ clients to participate in each round of the training. The local epoch we use is two. The clipping parameter in FAT-Clipping-PI we select is $\lambda = 50$, and the clipping parameter in FAT-Clipping-PR is $\lambda = 2$. All the remaining settings are the same as described in Appendix B.1.1.

C.2 Additional experimental Results

We provide two additional results when applying FedAvg, FAT-Clipping-PI and FAT-Clipping-PR to the CNN model on CIFAR-10 dataset. In Fig. 13, we show the percentage of successful training over 5 trials in non-i.i.d. cases when the non-i.i.d. index $p = 1$ and $p = 5$. These results further support our finding that FAT-Clipping methods and especially FAT-Clipping-PI reduce catastrophic training failures compared to FedAvg.