A \textbf{MenuGap}(X) = 1 \textit{ when } k = 1

In this brief section we prove that when \( k = 1 \), for any sequence of \( x_i \in \mathbb{R}_{\geq 0}^+ \), \textbf{MenuGap}(X) = 1.

\textbf{Claim 28.} When \( k = 1 \), for any \( X = \{x_i\}_{i=1}^N, x_i \in \mathbb{R}_{\geq 0}^+ \), \textbf{MenuGap}(X) = 1.

\textit{Proof.} Note that when \( k = 1 \), \( ||x_i||_1 = x_i \). Therefore, \textbf{MenuGap}(X, Q) = \sum_i \min_{j<i}(q_i - q_j).

We make the following observation which allows us to look at structured optimal solutions.

\textbf{Observation 29.} Any optimal solution \( Q \) to \textbf{MenuGap}(X) is monotone non-decreasing.

\textit{Proof.} For the sake of contradiction, suppose we are given an optimal solution \( Q \) that is not monotone non-decreasing. Let \( i \) be the smallest index for which \( q_i < q_{i-1} \). Then \( \text{gap}_i^{X,Q} = (q_i - q_{i-1})x_i < 0 \).

Consider instead a solution \( Q' \) where \( q'_j = q_j \) for all \( j \neq i \) and \( q'_i = q_{i-1} \). Now, \( \text{gap}_i^{X,Q'} = 0 \).

Since \( q_{i-1} = q_{i-1} \), \( \text{gap}_j^{X,Q} = \text{gap}_j^{X,Q'} \) for all \( j < i \). Since \( q_{i-1} > q_i \), for any \( j > i \) it holds that \( (q_j - q_{i-1}) < (q_j - q_i) \). Therefore, \( q_i \) is not “setting the gap” for any point after it.

Hence it also holds that \( \text{gap}_j^{X,Q} = \text{gap}_j^{X,Q'} \) for all \( j > i \). Putting everything together we get that \( \text{MenuGap}(X, Q') - \text{MenuGap}(X, Q) = \text{gap}_i^{X,Q'} - \text{gap}_i^{X,Q} > 0 \) contradicting the optimality of \( Q \).

With this observation in hand, since the \( q_i \) are monotone non-decreasing, without loss of generality it holds that \( \text{gap}_i^{X,Q} = \min_{j<i}(q_i - q_j) = q_i - q_{i-1} = q_j \) (for all \( j < i \)). Therefore, we get \( \text{MenuGap}(X, Q) = \sum_i q_i - q_{i-1} = q_N - q_0 \). Since \( q_0 = 0 \) and \( 0 \leq q_N \leq 1 \), we get that \( \text{MenuGap}(X, Q) \leq 1 \).

Finally, note that for any \( X \), we can set \( q_N = 1 \) and \( q_i = 0 \) for all other \( i \), proving that \( \text{MenuGap}(X) \geq 1 \).

\section*{B Omitted Proofs}

\textbf{Proof of Lemma 8.} We prove that for all \( X, C \), \textbf{AlignGap}(X, C) \leq \textbf{MenuGap}(X), which implies the lemma. For a given \( X, C \), define:

- \( \bar{q}_i := c_i \cdot \bar{x}_i \), if \( \text{sgap}_i^{X,C} > 0 \).
- \( \bar{q}_i := \arg \max_{j<i}(c_j \cdot \bar{x}_j) \), if \( \text{sgap}_i^{X,C} \leq 0 \).

Observe first that each \( \bar{q}_i \in [0, 1]^k \), as each \( c_i \bar{x}_i \in [0, 1]^k \) (this follows because each component of \( \bar{x}_i \) is at most \( ||\bar{x}_i||_{\infty} \), and each \( c_i \) is at most \( 1/||\bar{x}_i||_{\infty} \)). Next, observe that if \( \text{sgap}_i^{X,C} \leq 0 \), then \( \text{gap}_i^{X,Q} = 0 \). This is by definition in bullet two above. Finally, observe that if \( \text{sgap}_i^{X,C} > 0 \), then \( \text{gap}_i^{X,Q} \geq \text{gap}_i^{X,C} \). This is because the set of \( \{\bar{q}_j\}_{j<i} \) is a subset of \( \{c_j \bar{x}_j\}_{j<i} \), and because \( \bar{q}_i := c_i \cdot \bar{x}_i \) by bullet one. Therefore, \( \text{gap}_i^{X,Q} \geq \max\{0, \text{sgap}_i^{X,C}\} \) for all \( i \) and the lemma follows.

\textit{Proof of Claim 13.} Take \( M' \) to be exactly the same as \( M \), except having removed all entries with price \( < c \). For every value in the support of \( D \) with \( p^M(\vec{v}) \geq c \) in \( M \), we still have \( p^{M'}(\vec{v}) \geq c \).

This is simply \( \vec{v}' \)'s favorite option in \( M \) is still available in \( M' \), and all options in \( M^* \) were also available in \( M \). For any value with \( p^M(\vec{v}) < c \), we clearly have \( p^{M'}(\vec{v}) \geq 0 \). So for all \( \vec{v} \), we have \( p^{M'}(\vec{v}) \geq p^M(\vec{v}) - c \), and the claim follows by taking an expectation with respect to \( \vec{v} \).

\textit{Proof of Claim 15.} Simply let \( M_1 \) denote the set of menu options from \( M \) whose price lies in \( [c \cdot 2^i, c \cdot 2^{i+1}] \) for an odd integer \( i \), and \( M_2 \) denote the remaining menu options (which lie in \( [c \cdot 2^i, c \cdot 2^{i+1}] \) for an even power of \( i \)). It is easy to see that \( M_1 \) is oddly-priced and \( M_2 \) is evenly-priced. Then for all \( \vec{v} \), we must have \( p^{M_1}(\vec{v}) + p^{M_2}(\vec{v}) \geq p^M(\vec{v}) \). This is because \( \vec{v}' \)'s favorite menu
We first analyze which point sets the gap for
the \( M \) appears in one of the two menus, and is necessarily \( v^* \)'s favorite option on that menu (and they pay non-zero from the other menu). Taking an expectation with respect to \( v \) yields that
\[
\text{Rev}(D, M_1) + \text{Rev}(D, M_2) \geq \text{Rev}(D, M),
\]
completing the proof.

**Proof of Claim 18.** Recall that \((1 + \varepsilon) \cdot ||x||_1 \geq ||x_j||_1\) for all \( v \in B_i \). Therefore, if we set a price of \(||x||_1/(1 + \varepsilon)\) for the grand bundle, every \( v \in B_i \) would choose to purchase the grand bundle. This immediately implies the claim, as: \[
\text{BRev}(D) \geq \frac{||x||_1}{1+\varepsilon} \cdot \Pr_{v \sim D}[||x||_1 \geq \frac{||x||_1}{1+\varepsilon}] \geq \frac{||x||_1}{1+\varepsilon} \cdot \Pr_{v \sim D}[v \in B_i].
\]

**Proof of Claim 19.** Recall that \( \text{gap}^{X,Q}_i \) := \( \min_{j<i} \{ x_i \cdot (\tilde{q}_i - \tilde{q}_j) \} \), and that \( \tilde{q}_j := q^M(x_j) \). For any fixed \( j < i \), recall that because \( M \) was a truthful mechanism, we must have:
\[
\tilde{x}_i \cdot q^M(x_i) - p^M(x_i) \geq \tilde{x}_i \cdot q^M(x_j) - p^M(x_j)
\]
\[
\Rightarrow \tilde{x}_i \cdot (\tilde{q}_i - \tilde{q}_j) \geq p^M(x_i) - p^M(x_j)
\]
\[
\Rightarrow \tilde{x}_i \cdot (\tilde{q}_i - \tilde{q}_j) \geq p^M(x_i)/2.
\]
The first line is simply restating incentive compatibility. The second line is basic algebra, and substituting \( \tilde{q}_j := q^M(x_j) \). The third line invokes the fact that \( p^M(x_i) \geq 2^{\ell(i+1)-\alpha} \), while \( p^M(x_j) \leq 2^{2(\ell-1)\alpha+1} \leq 2^{2(\ell-1)\alpha-1} \).

**Proof of Observation 21.** This follows immediately from weak Lagrangian duality. For a quick refresher on weak Lagrangian duality, observe that for any feasible solution to the LP defining \( \text{AlignGap}^p(X) \) we must have \( \tilde{x}_i \cdot (c_i \tilde{x}_i - c_{i-1} \tilde{x}_{i-1}) - \text{gap}_i \geq 0 \). Therefore, for any feasible solution to the original LP, that solution is also feasible for \( \text{LagRel}_1(X) \), and the objective is only larger. Therefore, the optimal solution to \( \text{LagRel}_1(X) \) must be at least as large as \( \text{AlignGap}^p(X) \).

**Proof of Observation 22.** For all \( i \), \( \max\{0, \text{sgap}_i \} - \text{sgap}_i \leq 0 \). When \( \text{sgap}_i = 0 \), the maximum is achieved (and \( \text{sgap}_i := 0 \) is feasible). Substituting \( \max\{0, \text{sgap}_i \} - \text{sgap}_i = 0 \) for all \( i \) concludes the proof.

**Proof of Proposition 27.** To ease notation throughout the proof, we'll use the notation \( \text{gap}^{X,Q}_{e,j} := \text{gap}^{X,Q}_i \), where \( \tilde{x}_i \) is the \( j \)th point on layer \( \ell \). We will also use the notation \((\ell', j') < (\ell, j)\) if \( \ell' < \ell \), or \( \ell' = \ell \) and \( j' < j \) (that is, if the \( j \)th point in the \( \ell \)th layer comes before the \( j \)th point in the \( \ell' \)th layer). To understand \( \text{gap}^{X,Q}_{e,j} \), we need to understand which point “sets the gap” for \( \tilde{x}_{\ell,j} \), that is, which \((\ell', j') := \arg \min_{(\ell', j') < (\ell, j)} \{ (\tilde{q}_{\ell,j} - \tilde{q}_{\ell', j'}) \cdot \tilde{x}_i \} \).

We first analyze which point sets the gap for \( \tilde{x}_{\ell,j} \) (for even \( \ell \); for odd \( \ell \) the gap is zero and we don’t care which point sets it), and observe that it must either be \( \tilde{q}_{\ell, j-1} \) or \( \tilde{q}_{\ell-2, n_{\ell-2}-1} \) (that is, it must be the previous point in the same layer, or the final point in the previous even layer).

**Claim 30.** For all \( j \), and all even \( \ell \), \( \text{gap}^{X,Q}_{\ell,j} = \tilde{x}_{\ell,j} \cdot \tilde{q}_{\ell,j} - \max \{ \tilde{x}_{\ell,j} \cdot \tilde{q}_{\ell-2, n_{\ell-2}-1}, \tilde{x}_{\ell,j} \cdot \tilde{q}_{\ell, j-1} \} \). \(^{11}\)

**Proof of Claim 30.** First, note that \( \text{gap}^{X,Q}_i := \min_{(\ell', j') < (\ell, j)} \{ \tilde{x}_{\ell,j} \cdot (\tilde{q}_{\ell,j} - \tilde{q}_{\ell', j'}) \} = \tilde{x}_{\ell,j} \cdot \tilde{q}_{\ell,j} - \max_{(\ell', j') < (\ell, j)} \{ \tilde{x}_{\ell,j} \cdot (\tilde{q}_{\ell,j} - \tilde{q}_{\ell', j'}) \} \). To conclude the proof, simply observe that the first component of \( \tilde{q}_{\ell', j'} \) is monotone increasing in \( \ell' \) (for fixed \( j' \)), and the second component is monotone increasing in \( j' \) (for fixed \( \ell' \)). Moreover, the second component of \( \tilde{q}_{\ell', n_{\ell'-1}} \) is 1, and this is the maximum possible. Also, both components of \( \tilde{x}_{\ell,j} \) are non-negative, and therefore we conclude that \( \tilde{x}_{\ell,j} \cdot \tilde{q}_{\ell-2, n_{\ell-2}-1} \geq \tilde{x}_{\ell,j} \cdot \tilde{q}_{\ell', j'} \) whenever \( (\ell', j') \leq (\ell-2, n_{\ell-2}-1) \) (in fact, this extends even to \( (\ell', j') \leq (\ell-1, n_{\ell-1}-1) \) as no new \( \tilde{q} \) are introduced in layer \( \ell \)). Also, \( \tilde{x}_{\ell,j} \cdot \tilde{q}_{\ell,j-1} \leq \tilde{x}_{\ell,j} \cdot \tilde{q}_{\ell', j'} \) whenever \( j' \leq j-1 \).

Now that we know the gap is set either by the last point in the previous layer, or the previous point in the current layer, we can nail down \( \text{gap}^{X,Q}_{e,j} \) exactly.

**Lemma 31.** For all even \( \ell > 2 \), and all \( j \in [0, n_{\ell-1}] \): \( \text{gap}^{X,Q}_{e,j} \geq \frac{\sin(\theta_j)}{\sin(\theta_j+\theta_j)} \).

\(^{11}\)For simplicity of notation, define \( \tilde{q}_{0,j} = \tilde{v} = \tilde{q}_{0,j} \) for all \( j \).
Proof of Lemma 31. To prove the lemma, we simply compute the inner product of $\bar{x}_{\ell,j}$ with the three relevant vectors $\bar{q}_{\ell,j}$, $\bar{q}_{\ell-2,n_{\ell-2}-1}$, $\bar{q}_{\ell-1}$. To this end, recall that:

\[
\begin{align*}
\bar{q}_{\ell,j} &= (z_\ell, 1 - \delta_\ell \cot((j + 1)\theta_\ell)), \\
\bar{q}_{\ell-1} &= (z_{\ell-1}, 1 - \delta_\ell \cot(j\theta_\ell)), \\
\bar{q}_{\ell-2,n_{\ell-2}-1} &= (z_{\ell-2}, 1).
\end{align*}
\]

Therefore, observe that

\[
\begin{align*}
\bar{x}_{\ell,j} \cdot (\bar{q}_{\ell,j} - \bar{q}_{\ell-1}) &= \sin(j\theta_\ell) \cdot \delta_\ell \cdot (\cot(j\theta_\ell) - \cot((j + 1)\theta_\ell)) \\
&= \sin(j\theta_\ell) \cdot \delta_\ell \cdot \left( \frac{\cos(j\theta_\ell)}{\sin(j\theta_\ell)} - \frac{\cos((j + 1)\theta_\ell)}{\sin((j + 1)\theta_\ell)} \right) \\
&= \delta_\ell \cdot \frac{\cos(j\theta_\ell) \sin((j + 1)\theta_\ell) - \sin(j\theta_\ell) \cos((j + 1)\theta_\ell)}{\sin((j + 1)\theta_\ell)} \\
&= \delta_\ell \cdot \frac{\sin(\theta_\ell)}{\sin((j + 1)\theta_\ell)}.
\end{align*}
\]

Similarly,

\[
\begin{align*}
\bar{x}_{\ell,j} \cdot (\bar{q}_{\ell,j} - \bar{q}_{\ell-2,n_{\ell-2}-1}) &= (\delta_\ell + \delta_{\ell-1}) \cdot \cos(j\theta_\ell) - \delta_\ell \cot((j + 1)\theta_\ell) \cdot \sin(j\theta_\ell) \\
&\geq \delta_\ell \cdot \cos(j\theta_\ell) - \delta_\ell \cot((j + 1)\theta_\ell) \cdot \sin(j\theta_\ell) \\
&= \frac{\delta_\ell}{\sin((j + 1)\theta_\ell)} \cdot (\sin((j + 1)\theta_\ell) \cos(j\theta_\ell) - \sin(j\theta_\ell) \cos((j + 1)\theta_\ell)) \\
&= \delta_\ell \cdot \frac{\sin(\theta_\ell)}{\sin((j + 1)\theta_\ell)}.
\end{align*}
\]

This means that no matter which point sets the gap (or if one of the points does not exist), the gap is at least $\delta_\ell \cdot \frac{\sin(\theta_\ell)}{\sin((j + 1)\theta_\ell)}$. \(\Box\)

Finally, we need to sum over each even layer.

**Corollary 32.** For any even $\ell > 2$, $\sum_{j=0}^{n_{\ell-1}-1} \text{gap}^{X,Q}_{\ell,j} \geq \delta_\ell \cdot \ln(n_\ell)/2$.

**Proof of Corollary 32.** Consider the following sequence of calculations:

\[
\begin{align*}
\sum_{j=0}^{n_{\ell-1}-1} \text{gap}^{X,Q}_{\ell,j} &\geq \sum_{j=0}^{n_{\ell-1}-1} \delta_\ell \cdot \frac{\sin(\theta_\ell)}{\sin((j + 1)\theta_\ell)} \\
&\geq \delta_\ell \cdot (\theta_\ell - \theta_\ell^3/6) \cdot \sum_{j=0}^{n_{\ell-1}-1} \frac{1}{(j + 1)\theta_\ell} \\
&\geq \delta_\ell \cdot \ln(n_\ell) \\
&\geq \delta_\ell \cdot \ln(n_\ell) \\
\end{align*}
\]

Above, the first line follows from Lemma 31. The second line uses the fact that $\theta_\ell - \theta_\ell^3/6 \leq \sin(\theta_\ell) \leq \theta_\ell$, because $\theta_\ell \in [0, \pi/2]$. The third line follows as the $n^{th}$ harmonic sum is at least $\ln(n)$. The final line follows as $\theta_\ell^3/6 = \pi^3/(24(n_\ell - 1)^2) \leq 1/2$. \(\Box\)

And finally, we can wrap up the proof of the proposition. Here, we just need to recall that $\delta_\ell := \frac{1}{\alpha n_\ell} = \frac{1}{\alpha \ln^2(\ell)}$. Therefore, we conclude that:

\[
\sum_{\ell \text{ even}} \sum_{j=0}^{n_{\ell-1}-1} \text{gap}^{X,Q}_{\ell,j} \geq \sum_{\ell \text{ even}} \delta_\ell \cdot \ln(n_\ell)/2 = \sum_{\ell \text{ even}} \frac{1}{2\alpha \ell \ln(\ell)} = \infty. \Box
\]
C Proof of Corollary 11

We prove Corollary 11 by making use of Theorem 2 combined with the sequence X from Section 5. The only task is to confirm that ARev(\(D\)) < \(\infty\) for the resulting \(D\), which essentially requires that we execute and analyze the construction fully. Let us quickly review the [HN19] construction, given as input a sequence X:

- Let \(B\) be a very large constant, to be defined later.
- Let \(\tilde{v}_i := B^{2^2i} \cdot \tilde{x}_i/||\tilde{x}_i||_1\) (for all \(i\)).
- Let \(D\) sample \(\tilde{v}_i\) with probability \(1/B^{2^i}\) (for all \(i\)).
- Let \(D\) sample \(\bar{0}\) with probability \(1 - \sum_{i \geq 1} 1/B^{2^i}\).

[HN19] establishes that the above construction yields Theorem 2 (for sufficiently large \(B\), as a function of \(\varepsilon\)). To complete the proof of Corollary 11, we just need to relate ARev(\(D\)) to this construction to AlignGap(X).

**Proposition 33.** The construction above yields a \(D\) satisfying ARev(\(D\)) \(\leq\) AlignGap(X) + 1/B.

**Proof.** Consider any mechanism \(M\). We show that AlignGap(X) \(\geq\) ARev(\(D, M\)) − 1/B. To see this, consider the following choice of \(C\):

- If \(\tilde{v}_i\) is parallel to \(\tilde{q}^M(\tilde{v}_i)\), set \(c_i := ||\tilde{q}^M(\tilde{v}_i)||_2/||\tilde{x}_i||_2\).
- If \(\tilde{v}_i\) is not parallel to \(\tilde{q}^M(\tilde{v}_i)\), set \(c_i := 0\).

We now need to lower bound \(sgap_i^{X,C}\), when \(i\) satisfies the first bullet. Observe that, because \(M\) is truthful, we must have, for all \(j < i\):

\[
\tilde{v}_i \cdot \tilde{q}^M(\tilde{v}_i) - p^M(\tilde{v}_i) \geq \tilde{v}_i \cdot q^M(\tilde{v}_j) - p^M(\tilde{v}_j)
\]

\[
\Rightarrow p^M(\tilde{v}_i) \leq p^M(\tilde{v}_j) + B^{2^j} \cdot \frac{(c_i \tilde{x}_i - c_j \tilde{x}_j)}{||\tilde{x}_i||_1}.
\]

Above, the first line follows from incentive compatibility. The second line follows as \(\tilde{q}^M(\tilde{v}_i) = c_i \tilde{x}_i\) for all \(i\) in the first bulle, and either \(\tilde{q}^M(\tilde{v}_j) = c_j \tilde{x}_j\), or \(c_j = 0\). The final line follows by taking \(j := \arg\min_{j<i} \{|\tilde{v}_i \cdot (c_i \tilde{v}_i - c_j \tilde{v}_j)\}\), and by observing that \(\tilde{v}_j\) cannot possibly pay more than their value for the grand bundle.

We can then conclude that:

\[
ARev(D,M) \leq \sum_i (2B^{2^{i-1}} + B^{2^i} \cdot \text{sgap}_i^{X,C}/||\tilde{x}_i||_1)/B^{2^i}
\]

\[
\leq \sum_i 2/B^{2^{i-1}} + \text{AlignGap}(X)
\]

\[
\leq \text{AlignGap}(X) + 1/B.
\]

\[ \square \]

Because we can take \(B\) as large as we like, we can construct a \(D\) such that ARev(\(D\)) is arbitrarily close to AlignGap(X), while also maintaining that Rev(\(D\)) is arbitrarily close to MenuGap(X). Because Theorem 9 provides a construction \(X\) such that MenuGap(X)/AlignGap(X) = \(\infty\), the [HN19] construction, with sufficiently large \(B\), yields a \(D\) with Rev(\(D\))/ARev(\(D\)) = \(\infty\), completing the proof of Corollary 11.