A Discussions on Tuning β

In this section, we discuss the challenges in tuning β via other approaches. Recall that by the calculation shown after Theorem 1, a β such that $\frac{1}{2}\alpha_T(u^*) \leq \beta \leq \alpha_T(u^*)$ where $u^* \triangleq \operatorname{argmin}_{u\in\bar{\Delta}_N}\sum_{t=1}^T f_t(u)$ ensures a regret bound of $\mathcal{O}(N^2(\ln T)^3)$. We first show the existence of such β when the environment is *oblivious*, that is, $r_1, \ldots, r_T$ are all fixed ahead of time. (However, we emphasize that our adaptive tuning method introduced in Section 3 does not rely on the existence of such β at all and works even against non-oblivious environments.)

When $r_1, r_2, \ldots, r_T$ are fixed and thus u^* is also fixed, one can view $\alpha_T(u^*)$ as a (complicated) function of β. It is not hard to see that this function is continuous: note that x_{t+1} is a continuous function with respect to $\beta, A_t, \eta_t, x_t, \nabla_t$ because x_{t+1} is the minimizer of a strongly convex function parameterized by these quantities. Also, A_t, η_t, ∇_t are continuous functions of $\{x_1, \ldots, x_t\}$.[4] So overall, x_{t+1} is a continuous function of $\{\beta, x_1, \ldots, x_t\}$. By induction, we know that x_t is a continuous function of β for all t. Finally, since $\alpha_T(u^*)$ continuously depends on $\{x_1, \ldots, x_T\}$, it is also a continuous function of β.

Next note that the range of $\alpha_T(u^*)$ is $\left[\frac{1}{16NT}, \frac{1}{2}\right]$ because $8|\nabla_t^\top(u^* - x_t)| \leq 8\|\nabla_t\|_\infty \|u^* - x_t\|_1 \leq 16NT$. Thus by intermediate value theorem, if we vary β from $\frac{1}{32NT}$ to $\frac{1}{2}$, there must exist a β such that $\frac{1}{2}\alpha_T(u^*) \leq \beta \leq \alpha_T(u^*)$, which completes our argument. In fact, by $\alpha_T(u^*)$'s continuity, the set of β's satisfying the inequality will form an interval or a union of intervals.

Given that such β does exist but is unknown, a natural idea is to instantiate M copies of BARRONS's with different β's forming a grid on $\left[\frac{1}{32NT}, \frac{1}{2}\right]$, then use Hedge to learn over these copies, which only introduces an additional $\ln M$ regret since the loss is exp-concave. If any of these β's happens to fall into one of the intervals described above, then the algorithm has overall regret $\mathcal{O}(N^2(\ln T)^3 + \ln M)$.

However, the challenge is to figure out how dense the grid has to be, which depends on the slope (i.e. Lipschitzness) of $\alpha_T(u^*)$ with respect to β. The larger the slope, the denser the grid needs to be. Trivial analysis only shows that the Lipschitzness is exponential in T, which is far from satisfactory. Also note that the running time per round of this algorithm is $(MN^{3.5})$. Therefore even if M is polynomial in T which is good for the regret, it still defeats our purpose of deriving more efficient algorithms.

B Omitted Proofs

We first show that competing with smooth CRP from $\bar{\Delta}_N$ is enough.

Lemma 10. *For any $u' \in \Delta_N$, with $u = \left(1 - \frac{1}{T}\right)u' + \frac{1}{NT} \in \bar{\Delta}_N$ we have*

$$\sum_{t=1}^T f_t(x_t) - \sum_{t=1}^T f_t(u') \leq \sum_{t=1}^T f_t(x_t) - \sum_{t=1}^T f_t(u) + 2.$$

Proof. By convexity of f_t, we have

$$\sum_{t=1}^T f_t(u) - \sum_{t=1}^T f_t(u') \leq \sum_{t=1}^T \nabla f_t(u)^\top(u - u') \leq \sum_{t=1}^T \frac{(u' - u)^\top r_t}{u^\top r_t}$$

$$\leq \sum_{t=1}^T \frac{\left(\frac{u}{1-\frac{1}{T}} - u\right)^\top r_t}{u^\top r_t} = \frac{1}{1 - \frac{1}{T}} \leq 2.$$

$\square$

Next we provide the omitted proofs for several lemmas.

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[4]The fact that η_t is continuous with respect to $\{x_1, \ldots, x_t\}$ depends on our new increasing learning rate scheme and is not true for the scheme used in previous works [2, 21] based on doubling trick.

Proof of Lemma 5. Note that the function $h_t(x) = e^{-2\beta f_t(x)} = \langle x, r_t \rangle^{2\beta}$ is concave since $0 \leq 2\beta \leq 1$. Therefore we have $h_t(u) \leq h_t(x_t) + \langle \nabla h_t(x_t), u - x_t \rangle$. Plugging in $\nabla h_t(x) = -2\beta e^{-2\beta f_t(x)} \nabla f_t(x)$ gives

$$e^{-2\beta f_t(u)} \leq e^{-2\beta f_t(x_t)} \left(1 - 2\beta \langle \nabla_t, u - x_t \rangle \right),$$

or equivalently

$$f_t(u) \geq f_t(x_t) - \frac{1}{2\beta} \ln \left(1 - 2\beta \langle \nabla_t, u - x_t \rangle \right)$$

By the condition on β we also have $\left| 2\beta \langle \nabla_t, u - x_t \rangle \right| \leq \frac{1}{4}$. Using the fact $-\ln(1-z) \geq z + \frac{1}{4}z^2$ for $|z| \leq \frac{1}{4}$ gives:

$$f_t(x_t) - f_t(u) \leq \langle \nabla_t, x_t - u \rangle - \frac{\beta}{2} \langle \nabla_t, x_t - u \rangle^2$$

$$= \langle \nabla_t, x_t - x_{t+1} \rangle + \langle \nabla_t, x_{t+1} - u \rangle - \frac{\beta}{2} \langle \nabla_t, x_t - u \rangle^2$$

$$\leq \langle \nabla_t, x_t - x_{t+1} \rangle + D_{\psi_t}(u, x_t) - D_{\psi_t}(u, x_{t+1}) - \frac{\beta}{2} \langle \nabla_t, x_t - u \rangle^2,$$

where the last step follows standard OMD analysis. More specifically, since x_{t+1} is the minimizer of the function $F_t(x) \triangleq \langle \nabla_t, x \rangle + D_{\psi_t}(x, x_t)$, by the first-order optimality condition, we have $\langle u - x_{t+1}, \nabla F_t(x_{t+1}) \rangle \geq 0$ for all $u \in \bar{\Delta}_N$. Note $\nabla F_t(x_{t+1}) = \nabla_t + \nabla \psi_t(x_{t+1}) - \nabla \psi_t(x_t)$. Rearranging the condition gives $\langle \nabla_t, x_{t+1} - u \rangle \leq \langle \nabla \psi_t(x_{t+1}) - \nabla \psi_t(x_t), u - x_{t+1} \rangle$. Directly using the definition of Bregman divergence, one can verify $\langle \nabla \psi_t(x_{t+1}) - \nabla \psi_t(x_t), u - x_{t+1} \rangle = D_{\psi_t}(u, x_t) - D_{\psi_t}(u, x_{t+1}) - D_{\psi_t}(x_{t+1}, x_t)$, which is further bounded by $D_{\psi_t}(u, x_t) - D_{\psi_t}(u, x_{t+1})$ by the nonnegativity of Bregman divergence. This concludes the proof. $\square$

Proof of Lemma 8. Define $\Psi_t(u) = \sum_{s=1}^{t} f_s(u) + \frac{1}{\gamma} \sum_{i=1}^{N} \ln \frac{1}{u_i}$. We first show that if $\|u_t - u_{t+1}\|_{\nabla^2 \Psi_{t+1}(u_t)} \leq \frac{1}{2}$ holds, then the conclusion follows.

Indeed, note that $\nabla^2 \Psi_{t+1}(u_t) = \sum_{s=1}^{t+1} \frac{r_s r_s^\top}{\langle u_t, r_s \rangle^2} + \frac{1}{\gamma} \left[\frac{1}{u_{t,i}^2} \right]_{\text{diag}} \succeq \frac{1}{\gamma} \left[\frac{1}{u_{t,i}^2} \right]_{\text{diag}}$, where $\left[\frac{1}{u_{t,i}^2} \right]_{\text{diag}}$ represents the N dimensional diagonal matrix whose i-th diagonal element is $\frac{1}{u_{t,i}^2}$. We thus have

$$\|u_t - u_{t+1}\|_{\frac{1}{\gamma}[1/u_{t,i}^2]_{\text{diag}}} \leq \|u_t - u_{t+1}\|_{\nabla^2 \Psi_{t+1}(u_t)} \leq 1/2,$$

which implies $\frac{(u_{t,i} - u_{t+1,i})^2}{\gamma u_{t,i}^2} \leq \frac{1}{4}$, or $1 - \frac{\sqrt{\gamma}}{2} \leq \frac{u_{t+1,i}}{u_{t,i}} \leq 1 + \frac{\sqrt{\gamma}}{2}$ for all $i \in [N]$.

Next, we prove the inequality $\|u_t - u_{t+1}\|_{\nabla^2 \Psi_{t+1}(u_t)} \leq \frac{1}{2}$. Note $u_{t+1} = \operatorname{argmin}_{x \in \bar{\Delta}_N} \Psi_{t+1}(x)$. If we can prove $\Psi_{t+1}(u') > \Psi_{t+1}(u_t)$ for any u' that satisfies $\|u' - u_t\|_{\nabla^2 \Psi_{t+1}(u_t)} = \frac{1}{2}$, then we obtain the desired inequality $\|u_t - u_{t+1}\|_{\nabla^2 \Psi_{t+1}(u_t)} \leq \frac{1}{2}$ by the convexity of Ψ_{t+1}.

By Taylor's expansion, we know there exists some ξ in the line segment joining u' and u_t, such that

$$\Psi_{t+1}(u') = \Psi_{t+1}(u_t) + \nabla \Psi_{t+1}(u_t)^\top (u' - u_t) + \frac{1}{2}(u' - u_t)^\top \nabla^2 \Psi_{t+1}(\xi)(u' - u_t)$$

$$= \Psi_{t+1}(u_t) + \nabla f_{t+1}(u_t)^\top (u' - u_t) + \nabla \Psi_t(u_t)^\top (u' - u_t) + \frac{1}{2} \|u' - u_t\|_{\nabla^2 \Psi_{t+1}(\xi)}^2$$

$$\geq \Psi_{t+1}(u_t) + \nabla f_{t+1}(u_t)^\top (u' - u_t) + \frac{1}{2} \|u' - u_t\|_{\nabla^2 \Psi_{t+1}(\xi)}^2$$

$$\geq \Psi_{t+1}(u_t) - \|\nabla f_{t+1}(u_t)\|_{\nabla^{-2} \Psi_{t+1}(u_t)} \|u' - u_t\|_{\nabla^2 \Psi_{t+1}(u_t)} + \frac{1}{2} \|u' - u_t\|_{\nabla^2 \Psi_{t+1}(\xi)}^2$$

$$= \Psi_{t+1}(u_t) - \frac{1}{2} \|\nabla f_{t+1}(u_t)\|_{\nabla^{-2} \Psi_{t+1}(u_t)} + \frac{1}{2} \|u' - u_t\|_{\nabla^2 \Psi_{t+1}(\xi)}^2 \tag{9}$$

where the first inequality is by the optimality of u_t. As $\nabla^2 \Psi_{t+1}(u_t) \succeq \frac{1}{\gamma} \left[\frac{1}{u_{t,i}^2} \right]_{\text{diag}}$ implies $\nabla^{-2} \Psi_{t+1}(u_t) \preceq \gamma \left[u_{t,i}^2 \right]_{\text{diag}}$, we continue with

$$\|\nabla f_{t+1}(u_t)\|_{\nabla^{-2} \Psi_{t+1}(u_t)}^2 \leq \|\nabla f_{t+1}(u_t)\|_{\gamma \left[u_{t,i}^2 \right]_{\text{diag}}}^2 = \frac{\gamma r_{t+1}^\top \left[u_{t,i}^2 \right]_{\text{diag}} r_{t+1}}{\langle u_t, r_{t+1} \rangle^2} = \frac{\gamma \sum_{i=1}^N u_{t,i}^2 r_{t+1,i}^2}{\langle u_t, r_{t+1} \rangle^2} \leq \gamma.$$
(10)

Note ξ is between u_t and u', so $\|\xi - u_t\|_{\nabla^2 \Psi_{t+1}(u_t)} \leq \frac{1}{2}$ and thus $\frac{\xi_i}{u_{t,i}} \leq 1 + \frac{\sqrt{\gamma}}{2} \leqslant \frac{11}{10}$ according to previous discussions. Therefore, we have

$$\nabla^2 \Psi_{t+1}(\xi) = \sum_{s=1}^{t+1} \frac{r_s r_s^\top}{(r_s^\top \xi)^2} + \frac{1}{\gamma} \left[\frac{1}{\xi_i^2} \right]_{\text{diag}} \succeq \frac{100}{121} \left(\sum_{s=1}^{t+1} \frac{r_s r_s^\top}{(r_s^\top u_t)^2} + \frac{1}{\gamma} \left[\frac{1}{u_{t,i}^2} \right]_{\text{diag}} \right) = \frac{100}{121} \nabla^2 \Psi_{t+1}(u_t).$$
(11)

Now combining inequalities (9), (10) and (11), we arrive at

$$\begin{aligned}
\Psi_{t+1}(u') &\geq \Psi_{t+1}(u_t) - \frac{\sqrt{\gamma}}{2} + \frac{50}{121} \|u' - u_t\|_{\nabla^2 \Psi_{t+1}(u_t)}^2 \\
&= \Psi_{t+1}(u_t) - \frac{\sqrt{\gamma}}{2} + \frac{25}{242} \\
&\geq \Psi_{t+1}(u_t),
\end{aligned}$$

which finishes the proof. $\qquad\square$

Proof of Lemma 9. The proof is similar to the proof of Lemma 8. Denote $F_t(x) = \langle x, \nabla_t \rangle + D_{\psi_t}(x, x_t)$. We again first prove that if $\|x_t - x_{t+1}\|_{\nabla^2 F_t(x_t)} \leq \frac{1}{2}$, then the conclusion follows.

Note $\nabla^2 F_t(x_t) = \beta A_t + \left[\frac{1}{\eta_{t,i} x_{t,i}^2} \right]_{\text{diag}} \succeq \left[\frac{1}{\eta_{t,i} x_{t,i}^2} \right]_{\text{diag}} \succeq \left[\frac{1}{3\eta x_{t,i}^2} \right]_{\text{diag}}$, because $\eta_{t,i} \leq \eta \exp\left(\log_T(\frac{NT}{N}) \right) \leq 3\eta$. Thus we have

$$\|x_t - x_{t+1}\|_{\frac{1}{3\eta}[1/x_{t,i}^2]_{\text{diag}}} \leq \|x_t - x_{t+1}\|_{\nabla^2 F_t(x_t)} \leq 1/2,$$

which implies $\frac{(x_{t,i} - x_{t+1,i})^2}{3\eta x_{t,i}^2} \leq \frac{1}{4}$ and thus $1 - \frac{\sqrt{3\eta}}{2} \leq \frac{x_{t+1,i}}{x_{t,i}} \leq 1 + \frac{\sqrt{3\eta}}{2}$ for all $i \in [N]$.

It remains to prove the inequality $\|x_t - x_{t+1}\|_{\nabla^2 F_t(x_t)} \leq \frac{1}{2}$. Since $x_{t+1} = \operatorname{argmin}_{x \in \bar{\Delta}_N} F_t(x)$, if we can prove $F_t(x') > F_t(x_t)$ for all x' that satisfies $\|x' - x_t\|_{\nabla^2 F_t(x_t)} = \frac{1}{2}$, then we obtain the desired inequality $\|x_t - x_{t+1}\|_{\nabla^2 F_t(x_t)} \leq \frac{1}{2}$ by the convexity of F_t. By Taylor's expansion, there exists some ζ on the line segment joining x' and x_t, such that

$$\begin{aligned}
F_t(x') &= F_t(x_t) + \nabla F_t(x_t)^\top (x' - x_t) + \frac{1}{2}(x' - x_t)^\top \nabla^2 F_t(\zeta)(x' - x_t) \\
&= F_t(x_t) + \nabla_t^\top (x' - x_t) + \frac{1}{2} \|x' - x_t\|_{\nabla^2 F_t(\zeta)}^2 \\
&\geq F_t(x_t) - \|\nabla_t\|_{\nabla^{-2} F_t(x_t)} \|x' - x_t\|_{\nabla^2 F_t(x_t)} + \frac{1}{2} \|x' - x_t\|_{\nabla^2 F_t(\zeta)}^2 \\
&= F_t(x_t) - \frac{1}{2} \|\nabla_t\|_{\nabla^{-2} F_t(x_t)} + \frac{1}{2} \|x' - x_t\|_{\nabla^2 F_t(\zeta)}^2 .
\end{aligned}$$
(12)

As $\nabla^2 F_t(x_t) = \beta A_t + \left[\frac{1}{\eta_{t,i} x_{t,i}^2} \right]_{\text{diag}} \succeq \frac{1}{3\eta} \left[\frac{1}{x_{t,i}^2} \right]_{\text{diag}}$, we have $\nabla^{-2} F_t(x_t) \preceq 3\eta \left[x_{t,i}^2 \right]_{\text{diag}}$. Therefore

$$\|\nabla_t\|_{\nabla^{-2} F_t(x_t)}^2 \leq \|\nabla_t\|_{3\eta[x_{t,i}^2]_{\text{diag}}}^2 = \frac{3\eta r_t^\top \left[x_{t,i}^2 \right]_{\text{diag}} r_t}{\langle x_t, r_t \rangle^2} = \frac{3\eta \sum_{i=1}^N x_{t,i}^2 r_{t,i}^2}{\langle x_t, r_t \rangle^2} \leq 3\eta.$$
(13)

Since ζ is between x_t and x', we have $\|\zeta - x_t\|_{\nabla^2 F_t(x_t)} < \frac{1}{2}$ and thus $\frac{\zeta_i}{x_{t,i}} \leq 1 + \frac{\sqrt{3\eta}}{2} \leq \frac{21}{20}$ according to previous discussions and the fact $\eta \leqslant \frac{1}{300}$. Therefore, we have

$$\nabla^2 F_t(\zeta) = \sum_{s=1}^{t} \frac{r_s r_s^\top}{(r_s^\top \zeta)^2} + \left[\frac{1}{\eta_{t,i} \zeta_i^2} \right]_{\text{diag}} \succeq \frac{400}{441} \left(\sum_{s=1}^{t} \frac{r_s r_s^\top}{(r_s^\top x_t)^2} + \left[\frac{1}{\eta_{t,i} x_{t,i}^2} \right]_{\text{diag}} \right) = \frac{400}{441} \nabla^2 F_t(x_t).$$

(14)

Now combining inequalities (12), (13) and (14), we get

$$F_t(x') \geq F_t(x_t) - \frac{\sqrt{3\eta}}{2} + \frac{200}{441} \|x' - x_t\|_{\nabla^2 F_t(x_t)}^2$$
$$= F_t(x_t) - \frac{\sqrt{3\eta}}{2} + \frac{50}{441}$$
$$\geq F_t(u_t),$$

which finishes the proof. □