A New Alternating Direction Method for Linear Programming: Supplementary Material

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Abstract

It is well known that, for a linear program (LP) with constraint matrix $\mathbf{A} \in \mathbb{R}^{m\times n}$, the Alternating Direction Method of Multiplier converges globally and linearly at a rate $O((\|\mathbf{A}\|_F^2 + mn)\log(1/\epsilon))$. However, such a rate is related to the problem dimension and the algorithm exhibits a slow and fluctuating "tail convergence" in practice. In this paper, we propose a new variable splitting method of LP and prove that our method has a convergence rate of $O(\|\mathbf{A}\|^2 \log(1/\epsilon))$. The proof is based on simultaneously estimating the distance from a pair of primal dual iterates to the optimal primal and dual solution set by certain residuals. In practice, we result in a new first-order LP solver that can exploit both the sparsity and the specific structure of matrix $\mathbf{A}$ and a significant speedup for important problems such as basis pursuit, inverse covariance matrix estimation, L1 SVM and nonnegative matrix factorization problem compared with current fastest LP solvers.

1 Introduction

We are interested in applying the Alternating Direction Method of Multiplier (ADMM) to solve a linear program (LP) of the form

$$\min_{\mathbf{x}\in\mathbb{R}^n} \quad \mathbf{c}^T\mathbf{x} \quad s.t. \quad \mathbf{A}\mathbf{x} = \mathbf{b}, x_i \geq 0, i \in [n_b]. \tag{1}$$

where $\mathbf{c} \in \mathbb{R}^n$, $\mathbf{A} \in \mathbb{R}^{m\times n}$ is the constraint matrix, $\mathbf{b} \in \mathbb{R}^m$ and $[n_b] = \{1, 2, \ldots, n_b\}$. This problem plays a major role in numerical optimization, and has been used in a large variety of application areas. For example, several important machine learning problems including the nonnegative matrix factorization (NMF) [1], l_1-regularized SVM [2], sparse inverse covariance matrix estimation (SICE) [3], the basis pursuit (BP) [4], and the MAP inference [5] problem can be cast into an LP setting.

The complexity of the traditional LP solver is still at least quadratic in the problem dimension, i.e., the Interior Point method (IPM) with a weighted path finding strategy. However, many recent problems in machine learning have extremely large-scale targeting data but exhibit a sparse structure, i.e., $nnz(\mathbf{A}) \ll mn$, where $nnz(\mathbf{A})$ is the number of non-zero elements in the constraint matrix $\mathbf{A}$. This characteristic severely limits the ability of the IPM or Simplex technique to solve these problems. On the other hand, first-order methods have received extensive attention recently due to their ability to deal with large data sets. These methods require a matrix vector multiplication $\mathbf{A}\mathbf{x}$ in each iteration with complexity linear in $nnz(\mathbf{A})$. However, the key challenge in designing a first-order algorithm is that LPs are usually non-smooth and non-strongly convex optimization problems (may not have a unique solution). Utilizing the standard primal and dual stochastic sub-gradient descent method will result in an extremely slow convergence rate, i.e., $O(1/\epsilon^2)$ [6].

The ADMM was first developed in 1975 [7], and since then there have been several LP solvers based on this technique. Compared with the traditional Augmented Lagrangian Method (ALM), this

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method splits the variable into several blocks, and optimizes the augmented Lagrangian (AL) function in a Gauss-Seidel fashion, which often results in relatively easier subproblems to solve. However, this method suffers from a slow convergence when the number of blocks increases. Moreover, the challenge of applying the ADMM to the LP is that the LP problem does not exhibit an explicit separable structure among variables, which are difficult to split in the traditional sense. The notable work [8] first applies the ADMM to solve the LP by augmenting the original $n-$dimensional variables into $nm-$dimensions, and the resultant Augmented Lagrangian function is separable among n blocks of variables. They prove that this method converges globally and linearly. However, the rate of this method is dependent on the problem dimension m, n, and converges quite slowly when m, n are large. Thus, they leave an open question on whether other efficient splitting methods exist, resulting in convergence analysis in the space with lower dimension m or n.

In this paper, we propose a new splitting method for LP, which splits the equality and inequality constraints into two blocks. The resultant subproblems in each iteration are a linear system with a positive definite matrix, and n one-dimensional truncation operations. We prove our new method converges globally and linearly at a faster rate compared with the method in [8]. Specifically, the main contributions of this paper can be summarized as follows: (i) We show that the existing ADMM in [8] exhibits a slow and fluctuating "tail convergence", and provide a theoretical understanding of why this phenomenon occurs. (ii) We propose a new ADMM method for LP and provide a new analysis of the linear convergence rate of this new method, which only involves $O(m+n)-$dimensional iterates. This result answers the open question proposed in [8]. (iii) We show that when the matrix $\mathbf{A}$ possesses some specific structure, the resultant subproblem can be solved in closed form. For the general constraint matrix $\mathbf{A}$, we design an efficiently implemented Accelerated Coordinate Descent Method (ACDM) to solve the subproblem in $O(\log(1/\epsilon)nnz(\mathbf{A}))$ time. (iv) Practically, we show that our proposed algorithm significantly speeds up solving the basis pursuit, l_1-regularized SVM, sparse inverse covariance matrix estimation, and the nonnegative matrix factorization problem compared with existing splitting method [8] and the current fastest first-order LP solver in [9].

2 Preliminaries

In this section, we first review several definitions that will be used in the sequel. We also include several LP-based machine problems that can be cast into the LP setting. Finally we illustrate some observations from the existing method.

2.1 Notation

The proximal operator with parameter ρ of a closed and convex function $f : \mathbb{R}^n \to \mathbb{R}$ is defined as

$$\mathbf{prox}_{\rho f}(\mathbf{x}) = \arg\min_{\mathbf{u}} f(\mathbf{u}) + \frac{1}{2\rho}\|\mathbf{u}-\mathbf{x}\|^2. \tag{2}$$

A twice differentiable function $f : \mathbb{R}^n \to \mathbb{R}$ has strong convexity parameter ρ if and only if its Hessian satisfies $\nabla^2 f(\mathbf{x}) \succeq \rho\mathbf{I}, \forall\mathbf{x}$. We use $\|\cdot\|$ to denote standard l_2 norm for vector or spectral norm for matrix, $\|\cdot\|_1$ to denote the l_1 norm and $\|\cdot\|_F$ to denote the Frobenius norm. A twice differentiable function $f : \mathbb{R}^n \to \mathbb{R}$ has the component-wise Lipschitz continuous gradient with constant L_i if and only if $\|\nabla_i f(\mathbf{x}) - \nabla_i f(\mathbf{y})\| \leq L_i\|\mathbf{x}-\mathbf{y}\|, \forall\mathbf{x},\mathbf{y}$. For example, for the quadratic function $F(\mathbf{x}) = \frac{1}{2}\|\mathbf{A}\mathbf{x}-\mathbf{b}\|^2$, the gradient $\nabla F(\mathbf{x}) = \mathbf{A}^T(\mathbf{A}\mathbf{x}-\mathbf{b})$ and the Hessian $\nabla^2 F(\mathbf{x}) = \mathbf{A}^T\mathbf{A}$. Hence the parameter ρ and L_i satisfies (choose $\mathbf{y} = \mathbf{x}+t\mathbf{e}_i$, where $t \in \mathbb{R}$, $\mathbf{e}_i \in \mathbb{R}^n$ is the unit vector), $\mathbf{x}\mathbf{A}^T\mathbf{A}\mathbf{x} \geq \rho\|\mathbf{x}\|^2$ and $t\mathbf{A}_i^T\mathbf{A}\mathbf{e}_i \leq L_i|t|, \forall\mathbf{x}, t$. Thus, the ρ is the smallest eigenvalue of $\mathbf{A}^T\mathbf{A}$ and $L_i = \|\mathbf{A}_i\|^2$, where $\mathbf{A}_i$ is ith column of the matrix $\mathbf{A}$. The projection operator of point $\mathbf{x}$ into convex set $\mathcal{S}$ is defined as $[\mathbf{x}]_{\mathcal{S}} = \arg\min_{\mathbf{u}\in\mathcal{S}}\|\mathbf{x}-\mathbf{u}\|$. If $\mathcal{S}$ is the non-negative cone, let $[\mathbf{x}]_+ \triangleq [\mathbf{x}]_{\mathcal{S}}$. Let $V_i = [0,\infty)$ for $i \in [n_b]$ and $V_i = \mathbb{R}$ for $i \in [n]\backslash[n_b]$.

2.2 Applications

Basis pursuit problem: The problem of basis pursuit [4] is a fundamental decoding model in the compressive sensing. It aims at recovering the original signal from the compressed one with preserving the sparsity.

$$\min_{\mathbf{x}\in R^n} \|\mathbf{x}\|_1$$

$$s.t. \quad \mathbf{A}\mathbf{x} = \mathbf{b}, \tag{3}$$

where $\mathbf{A} \in \mathbb{R}^{m\times n}$ is the sensing matrix and $\mathbf{b}$ is the compressed measurement. In practice, the dimension $m \ll n$ and matrix $\mathbf{A}$ are formed by randomly taking a subset of rows from orthonormal transform matrices, such as DCT (discrete cosine transform), DFT (discrete Fourier transform) or DWHT (discrete Walsh-Hadamard transform) matrices.

l_1-**regularized SVM:** The problem of l_1-regularized support vector machine [2] aims at finding a classifier in the 2-class SVM,

$$\min_{\boldsymbol{\beta}} \sum_{i=1}^{n} \Big[1 - y_i(\boldsymbol{\beta}^T \mathbf{x}_i)\Big]_+ + \lambda\|\boldsymbol{\beta}\|_1, \tag{4}$$

where $(\mathbf{x}_i, y_i)$ are the ith training data and label, $\boldsymbol{\beta} \in \mathbb{R}^p$ is the linear classifier, and λ is the tuning parameter. It can be generalized to the scenario of multi-classes by replacing the hinge loss term in the objective function of maximizing the distances among different class, i.e., $\boldsymbol{\beta}_1^T\mathbf{x}_i - \boldsymbol{\beta}_2^T\mathbf{x}_i$.

Sparse Inverse Covariance Matrix Estimation: This problem aims to find a sparse matrix to approximate the inverse of the covariance matrix $\mathbf{S} \in \mathbb{R}^{p\times p}$. One popular approach [3] is to solve p independent problems of the following form

$$\min_{\beta\in\mathbb{R}^{p-1}} \|\boldsymbol{\beta}\|_1 \quad \text{s.t.} \quad \|\mathbf{S}_{-i,j} - \mathbf{S}_{-i,-i}\boldsymbol{\beta}\|_\infty \leq \delta, \tag{5}$$

where $\mathbf{S}_{-i,j}$ is the jth column of $\mathbf{S}$ with its ith entry removed, $\mathbf{S}_{-i,-i}$ is the submatrix of $\mathbf{S}$ with its ith row and column removed, and δ is predefined approximation threshold. In practice, the covariance matrix $\mathbf{S}$ is dense, but it can be decomposed into the product of two sparse matrices, i.e., $\mathbf{S} = \mathbf{X}\mathbf{X}^T$, which can be exploited by introducing the auxiliary variables in LP.

Nonnegative Matrix Factorization: Given n nonnegative $m-$dimensional vectors collected in matrix $\mathbf{M} \in \mathbb{R}_+^{m\times n}$, the NMF determines two nonnegative matrix $\mathbf{W} \in \mathbb{R}_+^{m\times r}$ and $\mathbf{H} \in \mathbb{R}_+^{r\times n}$ such that $\mathbf{M} \approx \mathbf{W}\mathbf{H}$. It is a powerful technique in dimensionality reduction and can be solved in polynomial time when the matrix $\mathbf{M}$ satisfies a separability condition. One of most popular approaches [1, 10] to solve this problem is

$$\begin{aligned} &\min_{\mathbf{X}\in R_+^{n\times n}} \mathbf{p}^T \mathrm{diag}(\mathbf{X}) \\ &s.t. \quad \|\tilde{\mathbf{M}} - \tilde{\mathbf{M}}\mathbf{X}\|_1 \leq \lambda\epsilon, X_{ii} \leq 1, X_{ij} \leq X_{ii}, \forall i, j, \end{aligned}$$

where $\mathbf{p}$ is any $n-$dimensional vector with distinct entries, $\tilde{\mathbf{M}}$ is a normalized noisy matrix, λ is predefined approximation threshold.

2.3 Tail Convergence of the Existing ADMM Method

The existing ADMM in [8] solves the LP (1) by following procedure: in each iteration k, go through the following two steps:

1. Primal update: $x_i^{k+1} = \left[x_i^k + \frac{1}{\|\mathbf{A}_i\|^2}\left(\frac{\mathbf{A}_i^T(\mathbf{b}-\mathbf{A}\mathbf{x}^k)}{q} - \frac{c_i - \mathbf{A}_i^T\mathbf{z}^k}{\lambda}\right)\right]_{V_i}, i = 1, \ldots, n.$
2. Dual update: $\mathbf{z}^{k+1} = \mathbf{z}^k - \frac{\lambda}{q}(\mathbf{A}\mathbf{x}^k - \mathbf{b})$.

In the Fig. 1, we plot the solving accuracy versus the number of iterations for solving three kinds of problems: l_1-regularized SVM, sparse inverse covariance matrix estimation and nonnegative matrix factorization problem. Here the solving accuracy is defined as the relative accuracy $(\mathbf{c}^T\mathbf{x}^k - \mathbf{c}^T\mathbf{x}^*)/(\mathbf{c}^T\mathbf{x}^*)$, where $\mathbf{x}^*$ is obtained approximately by running our method with a specific topping criterion. The detailed information of date set and LP is listed in TABLE 1.

We can observe that it converges fast in the initial phase, but exhibits a slow and fluctuating convergence when the iterates approach the optimal set. This method originates from a specific splitting method in the standard $2-$block ADMM [11]. To provide some understanding of this phenomenon, we show that this method can be actually recovered by an inexact Uzawa method [12]. The Augmented Lagrangian function of the problem (1) is denoted by $L(\mathbf{x}, \mathbf{z}) = \mathbf{c}^T\mathbf{x} + \frac{\rho}{2}\|\mathbf{A}\mathbf{x} - \mathbf{b} - \mathbf{z}/\rho\|^2$.

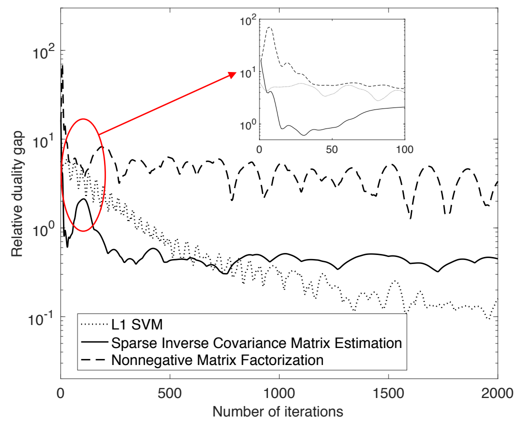

Figure 1: The relative duality gap versus the number of iterations. Here one iteration represents one dual updating step in existing ADMM.

Table 1: Data Statistics for Testing Existing ADMM (w8a is used for L1 SVM, rcv1 is used for SICE and a1a is used for NMF problem)

Data Set	#Samples	#Features	m	n	$nnz(\mathbf{A})$
w8a	49749	300	50095	30097	1161572
rcv1	15564	47236	256417	161947	7339063
a1a	1605	123	3150615	2958015	42006060

In each iteration k, the inexact Uzawa method first minimizes a local second-order approximation of the quadratic term in $L(\mathbf{x}, \mathbf{z}^k)$ with respect to primal variables $\mathbf{x}$, specifically,

$$\mathbf{x}^{k+1} = \arg\min_{x_i \in V_i} \mathbf{c}^T\mathbf{x} + \langle \rho\mathbf{A}^T(\mathbf{A}\mathbf{x}^k - \mathbf{b} - \mathbf{z}^k/\rho), \mathbf{x} - \mathbf{x}^k\rangle + \frac{1}{2}\|\mathbf{x} - \mathbf{x}^k\|_{\mathbf{D}}, \tag{6}$$

then update the dual variables by $\mathbf{z}^{k+1} = \mathbf{z}^k - \rho(\mathbf{A}\mathbf{x}^{k+1} - \mathbf{b})$. Let the proximity parameter $\rho = \lambda/q$ and matrix $\mathbf{D}$ equal to the diagonal matrix diag$\{\dots, 1/q\|\mathbf{A}_i\|^2, \dots\}$, then we can recover the above algorithm by the first-order optimality condition of (6). This equivalence allows us to illustrate the main reason for the slow and fluctuating "tail convergence" comes from the *inefficiency of such a local approximation of the Augmented Lagrangian function when the iterates approach the optimal set*.

One straightforward idea to resolve this issue is to minimize the Augmented Lagrangian function exactly instead of its local approximation, which leads to the classic ALM. There exists a line of works focusing on analyzing the convergence of applying ALM to LP [9, 13, 14]. This method will produce a sequence of constrained quadratic programs (QP) that are difficult to solve. The work [9] prove that the proximal Coordinate Descent method can solve each QPs at a linear rate even when matrix $\mathbf{A}$ is not full column rank. However, there exists several drawbacks in this approach: (i) the practical solving time of each subproblem is quite long when $\mathbf{A}$ is rank-deficient; (ii) the theoretical performance and complexity of using recent accelerated techniques in proximal optimization [15] with the ALM is unknown; (iii) it cannot exploit the specific structure of matrix $\mathbf{A}$ when solving each constrained QP. Therefore, it motivates us to investigate the new and efficient variable splitting method for such a problem.

3 New Splitting Method in ADMM

We first separate the equality and inequality constraints of the above LP (1) by adding another group of variables $\mathbf{y} \in \mathbb{R}^n$.

$$\begin{aligned} \min \quad & \mathbf{c}^T\mathbf{x} \\ s.t. \quad & \mathbf{A}\mathbf{x} = \mathbf{b}, \mathbf{x} = \mathbf{y}, \\ & y_i \geq 0, i \in [n_b]. \end{aligned} \tag{7}$$

Algorithm 1 Alternating Direction Method of Multiplier with Inexact Subproblem Solver

Initialize $\mathbf{z}^0 \in \mathbb{R}^{m+n}$, choose parameter $\rho > 0$.
repeat
1. Primal update: find $\mathbf{x}^{k+1}$ such that $F_k(\mathbf{x}^{k+1}) - \min_{\mathbf{x}\in\mathbb{R}^n} F_k(\mathbf{x}) \le \epsilon_k$.
2. Primal update: for each i, let $y_i^{k+1} = \left[x_i^{k+1} + z_{y,i}^k/\rho\right]_{V_i}$.
3. Dual update: $\mathbf{z}_x^{k+1} = \mathbf{z}_x^k + \rho(\mathbf{A}\mathbf{x}^{k+1} - \mathbf{b})$, $\mathbf{z}_y^{k+1} = \mathbf{z}_y^k + \rho(\mathbf{x}^{k+1} - \mathbf{y}^{k+1})$.

until $\|\mathbf{A}\mathbf{x}^{k+1} - \mathbf{b}\|_\infty \le \epsilon$ and $\|\mathbf{x}^{k+1} - \mathbf{y}^{k+1}\|_\infty \le \epsilon$

The dual of problem (7) takes the following form.

$$\begin{aligned} \min \quad & \mathbf{b}^T\mathbf{z}_x \\ s.t. \quad & -\mathbf{A}^T\mathbf{z}_x - \mathbf{z}_y = \mathbf{c}, \\ & z_{y,i} \le 0, i \in [n_b], z_{y,i} = 0, i \in [n]\backslash[n_b]. \end{aligned} \tag{8}$$

Let $\mathbf{z}_x, \mathbf{z}_y$ be the Lagrange multipliers for constraints $\mathbf{A}\mathbf{x} = \mathbf{b}$, $\mathbf{x} = \mathbf{y}$,respectively. Define the indicator function $g(\mathbf{y})$ of the non-negative cone: $g(\mathbf{y}) = 0$ if $y_i \ge 0, \forall i \in [n_b]$; otherwise $g(\mathbf{y}) = +\infty$. Then the augmented Lagrangian function of the primal problem (7) is defined as

$$L(\mathbf{x},\mathbf{y},\mathbf{z}) = \mathbf{c}^T\mathbf{x} + g(\mathbf{y}) + \mathbf{z}^T(\mathbf{A}_1\mathbf{x} + \mathbf{A}_2\mathbf{y} - \overline{\mathbf{b}}) + \frac{\rho}{2}\|\mathbf{A}_1\mathbf{x} + \mathbf{A}_2\mathbf{y} - \overline{\mathbf{b}}\|^2, \tag{9}$$

where $\mathbf{z} = [\mathbf{z}_x; \mathbf{z}_y]$. The matrix $\mathbf{A}_1$, $\mathbf{A}_2$ and vector $\overline{\mathbf{b}}$ are denoted by

$$\mathbf{A}_1 = \begin{bmatrix}\mathbf{A}\\ \mathbf{I}\end{bmatrix}, \mathbf{A}_2 = \begin{bmatrix}\mathbf{0}\\ -\mathbf{I}\end{bmatrix}, \text{ and } \overline{\mathbf{b}} = \begin{bmatrix}\mathbf{b}\\ \mathbf{0}\end{bmatrix}. \tag{10}$$

In each iteration k, the standard ADMM go through following three steps:

1. Primal update: $\mathbf{x}^{k+1} = \arg\min_{\mathbf{x}\in\mathbb{R}^n} L(\mathbf{x}, \mathbf{y}^k, \mathbf{z}^k)$.
2. Primal update: $\mathbf{y}^{k+1} = \arg\min_{\mathbf{y}\in\mathbb{R}^n} L(\mathbf{x}^{k+1}, \mathbf{y}, \mathbf{z}^k)$.
3. Dual update: $\mathbf{z}^{k+1} = \mathbf{z}^k + \rho(\mathbf{A}_1\mathbf{x}^{k+1} + \mathbf{A}_2\mathbf{y}^{k+1} - \overline{\mathbf{b}})$.

The first step is an unconstrained quadratic program, which can be simplified as

$$\mathbf{x}^{k+1} = \arg\min_{\mathbf{x}} F_k(\mathbf{x}) \triangleq \mathbf{c}^T\mathbf{x} + (\mathbf{z}^k)^T\mathbf{A}_1\mathbf{x} + \frac{\rho}{2}\|\mathbf{A}_1\mathbf{x} + \mathbf{A}_2\mathbf{y}^k - \overline{\mathbf{b}}\|^2. \tag{11}$$

The gradient of the function $F_k(\mathbf{x})$ can be expressed as

$$\nabla F_k(\mathbf{x}) = \rho(\mathbf{A}^T\mathbf{A} + \mathbf{I})\mathbf{x} + \mathbf{A}_1^T[\mathbf{z}^k + \rho(\mathbf{A}_2\mathbf{y}^k - \overline{\mathbf{b}})] + \mathbf{c}, \tag{12}$$

and the Hessian of function $F_k(\mathbf{x})$ is

$$\nabla^2 F_k(\mathbf{x}) = \rho(\mathbf{A}^T\mathbf{A} + \mathbf{I}). \tag{13}$$

Further, based on the first-order optimality condition, the first step is equivalent to solve a linear system, which requires inverting the Hessian matrix (13). In practice, the complexity is quite high to be exactly solved unless the Hessian exhibits some specific structures. Thus, we relax the first step into the inexact minimization: find $\mathbf{x}^{k+1}$ such that

$$F_k(\mathbf{x}^{k+1}) - \min_{\mathbf{x}\in\mathbb{R}^n} F_k(\mathbf{x}) \le \epsilon_k, \tag{14}$$

where ϵ_k is the given accuracy. Transforming the indicator function $g(\mathbf{y})$ back to the constraints, the second step can be separated into n one$-$dimensional optimization problems: for each i,

$$y_i^{k+1} = \arg\min_{y_i\in V_i} -z_{y,i}^k y_i + \frac{\rho}{2}(y_i - x_i^{k+1})^2 = \left[x_i^{k+1} + z_{y,i}^k/\rho\right]_{V_i}.$$

The resultant algorithm is sketched in Algorithm 1. In some applications such as BP, L1 SVM and SICE, the objective function contains the l_1 norm of the variables, and the optimization problem has the form of

$$\min \quad \mathbf{c}^T\mathbf{x} + \|\mathbf{x}\|_1 \tag{15}$$

$$s.t. \quad \mathbf{A}\mathbf{x} = \mathbf{b}. \tag{16}$$

Transforming to the canonical form (1) will add additional n variables and $2n$ constraints. One important feature in our method is that we can split the objective function by adding group of variable $\mathbf{y}$.

$$\min \quad \mathbf{c}^T\mathbf{x} + \|\mathbf{y}\|_1 \tag{17}$$

$$s.t. \quad \mathbf{A}\mathbf{x} = \mathbf{b}, \mathbf{x} = \mathbf{y}. \tag{18}$$

Similarly, applying the ADMM to problem (17), we can obtain the following two subproblems: the first step is an unconstrained QP,

$$\mathbf{x}^{k+1} = \arg\min_{\mathbf{x}} \mathbf{c}^T\mathbf{x} + (\mathbf{z}^k)^T\mathbf{A}_1\mathbf{x} + \frac{\rho}{2}\|\mathbf{A}_1\mathbf{x} + \mathbf{A}_2\mathbf{y}^k - \overline{\mathbf{b}}\|^2.$$

The second step is n one$-$dimensional shrinkage operations: for each i,

$$y_i^{k+1} = \begin{cases} x_i^{k+1} + (z_i^k - 1)/\rho, & \text{if } x_i^{k+1} + z_i^k/\rho \geq 1/\rho \\ x_i^{k+1} + (z_i^k + 1)/\rho, & \text{if } x_i^{k+1} + z_i^k/\rho \leq -1/\rho \\ 0, & \text{otherwise} \end{cases}.$$

4 Convergence Analysis of New ADMM

In this section, we prove that the Algorithm 1 converges at a global and linear rate, and provide a roadmap of the main technical development. We can first write the primal problem (7) as the following standard 2$-$block form.

$$\min_{\mathbf{x},\mathbf{y}} f(\mathbf{x}) + g(\mathbf{y}) \quad s.t. \quad \mathbf{A}_1\mathbf{x} + \mathbf{A}_2\mathbf{y} = \overline{\mathbf{b}}, \tag{19}$$

where $f(\mathbf{x}) = \mathbf{c}^T\mathbf{x}$ and $g(\mathbf{y})$ is the indicator function as defined before. Most works in the literature prove that the 2-block ADMM converges globally and linearly via assuming that one of the functions f and g is strongly convex [16, 17, 18]. Unfortunately, both the linear function f and the indicator function g in the LP do not satisfy this property, which poses a significant challenge on the current analytical framework. There exists several recent works trying to address this problem in some sense. In work [19], they have demonstrated that when the dual step size ρ is sufficiently small (impractical), the ADMM converges globally linearly, while no implicit rate is given. The work [14] shows that the ADMM is locally linearly converged when applying to LP. They utilize a unique combination of iterates and conduct a spectral analysis. However, they still leave an open question whether ADMM converges globally and linearly when applying to the LP in the above form.

In the sequel, we will answer this question positively and provide an accurate analysis of such a splitting method. The main technical development is based on a geometric argument: we first prove that the set formed by optimal primal and dual solutions of LP (7) is a $(3n + m)-$dimensional polyhedron $\mathcal{S}^*$; then we utilize certain global error bound to simultaneously estimate the distance from iterates $\mathbf{x}^{k+1}, \mathbf{y}^k, \mathbf{z}^k$ to $\mathcal{S}^*$. All detailed proofs are given in the Appendix.

Lemma 1. (Convergence of 2-block ADMM [11]) *Let $\mathbf{p}^k = \mathbf{z}^k - \rho\mathbf{A}_2\mathbf{y}^k$, we have*

$$\|\mathbf{p}^{k+1} - [\mathbf{p}^{k+1}]_{G^*}\|^2 \leq \|\mathbf{p}^k - [\mathbf{p}^k]_{G^*}\|^2 - \|\mathbf{p}^{k+1} - \mathbf{p}^k\|^2,$$

where $G^* \triangleq \{\mathbf{p}^* \in \mathbb{R}^{m+n} | T(\mathbf{p}^*) = \mathbf{p}^*\}$, and the definition of operator T is given in (55) in Appendix. Moreover, if the LP (7) has a pair of optimal primal and dual solution, the iterates $\mathbf{x}^k$,$\mathbf{y}^k$ and $\mathbf{z}^k$ converges to an optimal solution; Otherwise, at least one of the iterates is unbounded.

Lemma 1 is tailored from applying the classic Douglas-Rachford splitting method to the LP. This result guarantees that the sequence $\mathbf{p}^k$ produced by ADMM globally converges under a mild assumption. However, to establish the linear convergence rate, the key lies in estimating the other side inequality,

$$\|\mathbf{p}^k - [\mathbf{p}^k]_{G^*}\| \leq \gamma\|\mathbf{p}^{k+1} - \mathbf{p}^k\|, \gamma > 0. \tag{20}$$

Then one can combine these two results together to prove that sequence $\mathbf{p}^k$ converges globally and linearly with $\|\mathbf{p}^{k+1} - [\mathbf{p}^{k+1}]_{G^*}\|^2 \leq (1 - 1/\gamma^2) \cdot \|\mathbf{p}^k - [\mathbf{p}^k]_{G^*}\|^2$, which further can be used to show the $R-$linear convergence of iterates $\mathbf{x}^k, \mathbf{y}^k$ and $\mathbf{z}^k$. To estimate the constant γ, we first describe the geometry formed by the optimal primal solutions $\mathbf{x}^*, \mathbf{y}^*$ and dual solutions $\mathbf{z}^*$ of the LP (7).

Lemma 2. (Geometry of the optimal solution set of LP) *The variables* $(\mathbf{x}^*, \mathbf{y}^*)$ *are the optimal primal solutions and* $\mathbf{z}^*$ *are optimal dual solutions of LP (7) if and only if (i)* $\mathbf{A}\mathbf{x}^* = \mathbf{b}$*,* $\mathbf{x}^* = \mathbf{y}^*$*; (ii)* $-\mathbf{A}^T\mathbf{z}^*_{\mathbf{x}} - \mathbf{z}^*_y = \mathbf{c}$*; (iii)* $y^*_i \geq 0, z^*_{y,i} \leq 0, i \in [n_b]$*;* $z^*_{y,i} = 0, i \in [n]\backslash[n_b]$*; (iv)* $\mathbf{c}^T\mathbf{x}^* + \mathbf{b}^T\mathbf{z}^*_x = 0$*.*

In Lemma 2, one interesting element is to utilize the strong duality condition (iv) to eliminate the complementary slackness in the standard KKT condition. Then, the set of optimal primal and dual solutions is described only by affine constraints, which further implies that the optimal solution set is an $(m + 3n)-$dimensional polyhedron. We use $\mathcal{S}^*$ to denote such a polyhedron.

Lemma 3. (Hoffman bound [20, 21]) *Consider a polyhedron set* $\mathcal{S} = \{\mathbf{x} \in \mathbb{R}^d | \mathbf{E}\mathbf{x} = \mathbf{t}, \mathbf{C}\mathbf{x} \leq \mathbf{d}\}$*. For any point* $\mathbf{x} \in \mathbb{R}^d$*, we have*

$$\|\mathbf{x} - [\mathbf{x}]_{\mathcal{S}}\| \leq \theta_S \left\| \begin{bmatrix} \mathbf{E}\mathbf{x} - \mathbf{t} \\ [\mathbf{C}\mathbf{x} - \mathbf{d}]_+ \end{bmatrix} \right\|, \tag{21}$$

where θ_S *is the Hoffman constant that depends on the structure of polyhedron* $\mathcal{S}$*.*

According to the result in Lemma 2, it seems that we can use the Hoffman bound to estimate the distance between the current iterates $(\mathbf{x}^k, \mathbf{y}^k, \mathbf{z}^k)$ and the solution set $\mathcal{S}^*$ via the their primal and dual residual. However, to obtain the form of inequality (20), we need to bound such a residual in terms of $\|\mathbf{p}^k - \mathbf{p}^{k+1}\|$. Indeed, we have these results.

Lemma 4. (Estimation of residual) *The sequence* $(\mathbf{x}^{k+1}, \mathbf{y}^k, \mathbf{z}^k)$ *produced by Algorithm 1 satisfies*

$$\begin{cases} \mathbf{A}_1\mathbf{x}^{k+1} + \mathbf{A}_2\mathbf{y}^k - \overline{\mathbf{b}} = (\mathbf{p}^{k+1} - \mathbf{p}^k)/\rho, \\ \mathbf{c} + \mathbf{A}_1^T\mathbf{z}^k = \mathbf{A}_1^T(\mathbf{p}^k - \mathbf{p}^{k+1}), \\ \mathbf{c}^T\mathbf{x}^{k+1} + \mathbf{b}^T\mathbf{z}^k_x = (\mathbf{A}_1\mathbf{x}^{k+1} - \mathbf{z}^k/\rho)^T(\mathbf{p}^k - \mathbf{p}^{k+1}), \\ y^k_i \geq 0, z^k_{y,i} \leq 0, i \in [n_b]; z^k_{y,i} = 0, i \in [n]\backslash[n_b]. \end{cases}$$

One observation from Lemma 4 is that Algorithm 1 automatically preserves the boundness and the complementary slackness of both primal and dual iterates. Instead, in the previous algorithm in [8], the complementary slackness is not preserved during the iteration. Combining the results in Lemma 2, Lemma 3 and Lemma 4, we are readily to estimate the constant γ.

Lemma 5. (Estimation of linear rate) *The sequence* $\mathbf{p}^k = \mathbf{z}^k - \rho\mathbf{A}_2\mathbf{y}_k$ *produced by Algorithm 1 satisfies* $\|\mathbf{p}^k - [\mathbf{p}^k]_{G^*}\| \leq \gamma\|\mathbf{p}^{k+1} - \mathbf{p}^k\|$*, where the rate* γ *is given by*

$$\gamma = (1 + \rho)\left[\frac{R_z + 1}{\rho} + (R_x + 1)\|\mathbf{A}_1^T\|\right]\theta_{S^*}. \tag{22}$$

$R_x = \sup_k \|\mathbf{x}^k\| < +\infty$*,* $R_z = \sup_k \|\mathbf{z}^k\| < +\infty$ *are the maximum radius of iterates* $\mathbf{x}^k$ *and* $\mathbf{z}^k$*.*

Then we can establish the global and linear convergence of Algorithm 1.

Theorem 1. (Linear convergence of Algorithm 1) *Denote* $\mathbf{z}^k$ *as the primal iterates produced by Algorithm 1. To guarantee that there exists an optimal dual solution* $\mathbf{z}^*$ *such that* $\|\mathbf{z}^k - \mathbf{z}^*\| \leq \epsilon$*, it suffices to run Algorithm 1 for number of iterations* $K = 2\gamma^2 \log(2D_0/\epsilon)$ *with the solving accuracy* ϵ_k *satisfying* $\epsilon_k \leq \epsilon^2/8K^2$*, where* $D_0 = \|\mathbf{p}^0 - [\mathbf{p}^0]_{G^*}\|$*.*

The proof of Theorem 1 consists of two steps: first, we establish the global and linear convergence rate of Algorithm 1 when $\epsilon_k = 0, \forall k$ (exact subproblem solver); then we relax this condition and prove that when ϵ_k is less than a specified threshold, the algorithm still shares a convergence rate of the same order. The results of primal iterates $\mathbf{x}^k$ and $\mathbf{y}^k$ are similar.

5 Efficient Subproblem Solver

In this section, we will show that, due to our specific splitting method, each subproblem in line 1 of Algorithm 1 can be either solved in closed-form expression or efficiently solved by the Accelerated Coordinate Descent Method.

Algorithm 2 Efficiently Subproblem Solver

Initialize $\mathbf{u}_0, \mathbf{v}_0$, $\overline{\mathbf{u}}_0 = \mathbf{A}\mathbf{u}_0$, $\overline{\mathbf{v}}_0 = \mathbf{A}\mathbf{v}_0$, parameter τ, η, S, matrix $\mathbf{M}$ by (26) and distribution $p = [\dots, \sqrt{1 + \|\mathbf{A}_i\|^2/S}, \dots]$ and let $\mathbf{d}^k = \mathbf{A}_1^T[\mathbf{z}^k + \rho(\mathbf{A}_2\mathbf{y}^k - \overline{\mathbf{b}})] + \mathbf{c}$.
repeat
  $[\mathbf{u}_t, \mathbf{v}_t]^T = \mathbf{M}_{t-1} \cdot [\overline{\mathbf{u}}, \overline{\mathbf{v}}]^T$ and $[\overline{\mathbf{u}}_t, \overline{\mathbf{v}}_t]^T = \mathbf{M}_{t-1} \cdot [\overline{\mathbf{u}}, \overline{\mathbf{v}}]^T$.
  Sample i from $[n]$ based on probability distribution p.
  $\nabla_i F_k(\mathbf{u}_t) = \rho(\mathbf{A}_i)^T\overline{\mathbf{u}}_t + \rho u_{t,i} + d_i^k$, and calculate $\mathbf{s}_t^i$ by (26).
  $\mathbf{M}_t = \mathbf{M} \cdot \mathbf{M}_{t-1}$. Update $\begin{bmatrix}\mathbf{u}^T \\ \mathbf{v}^T\end{bmatrix} = \begin{bmatrix}\mathbf{u}^T \\ \mathbf{v}^T\end{bmatrix} - \mathbf{M}_t^{-1}\mathbf{s}_t^i$, $\begin{bmatrix}\overline{\mathbf{u}}^T \\ \overline{\mathbf{v}}^T\end{bmatrix} = \begin{bmatrix}\overline{\mathbf{u}}^T \\ \overline{\mathbf{v}}^T\end{bmatrix} - \mathbf{M}_t^{-1}\mathbf{s}_t^i\mathbf{A}^T$,
until Converge
Output $\mathbf{x}^{k+1} = (\mathbf{u}_T - \tau\mathbf{v}_T)/(1-\tau)$.

5.1 Well-structured Constraint Matrix

Let the gradient (12) vanishes, then the primal iterates $\mathbf{x}^{k+1}$ can be exactly determined by

$$\mathbf{x}^{k+1} = \rho^{-1}(\mathbf{I} + \mathbf{A}^T\mathbf{A})^{-1}\mathbf{d}^k, \text{ with } \mathbf{d}^k = -\mathbf{A}_1^T[\mathbf{z}^k + \rho(\mathbf{A}_2\mathbf{y}^k - \overline{\mathbf{b}})] - \mathbf{c}, \tag{23}$$

which requires inverting an $n \times n$ positive definite matrix $\mathbf{I} + \mathbf{A}^T\mathbf{A}$, or equivalently, inverting an $m \times m$ positive definite matrix $\mathbf{I} + \mathbf{A}\mathbf{A}^T$ via the following Sherman–Morrison–Woodbury identity,

$$(\mathbf{I} + \mathbf{A}^T\mathbf{A})^{-1} = \mathbf{I} - \mathbf{A}^T(\mathbf{I} + \mathbf{A}\mathbf{A}^T)^{-1}\mathbf{A}. \tag{24}$$

One basic fact is that we only need to invert such a matrix once and then use this cached factorization in subsequent iterations. Therefore, there are several cases for which the above factorization can be efficiently calculated: (i) Factorization has a closed-form expression. For example, in the LP-based MAP inference [5], the matrix $\mathbf{I} + \mathbf{A}^T\mathbf{A}$ is block diagonal, and each block has been shown to possess a closed-form factorization. Another important application is that, in the basis pursuit problem, the encoding matrices such as DFT (discrete Fourier transform) and DWHT (discrete Walsh-Hadamard transform) matrices have orthonormal rows and satisfy $\mathbf{A}\mathbf{A}^T = \mathbf{I}$. Based on (23), each $\mathbf{x}^{k+1} = \rho^{-1}(\mathbf{I} - \frac{1}{2}\mathbf{A}^T\mathbf{A})\mathbf{d}^k$ and can be calculated in $O(n\log(n))$ time by certain fast transforms. (ii) Factorization has a low-complexity: the dimension m (or n) is small, i.e., $m = 10^4$. Such a factorization can be calculated in $O(m^3)$ and the complexity of each iteration is only $O(nnz(\mathbf{A}) + m^2)$.

Remark 1. *In the traditional Augmented Lagrangian method, the resultant subproblem is a constrained and non-strongly convex QP (Hessian is not invertible), which does not allow the above close-form expression. Besides, in the ALCD [9], the coordinate descent (CD) step only picks one column in each iteration and cannot exploit the nice structure of matrix* $\mathbf{A}$. *One idea is to modify the CD step in [9] to the proximal gradient descent. However, it will greatly increase the computation time due to the large number of inner gradient descent steps.*

5.2 General Constraint Matrix

However, in other applications, the constraint matrix $\mathbf{A}$ only exhibits the sparsity, which is difficult to invert. To resolve this issue, we resort to the current fastest accelerated coordinate descent method [22]. This method has an order improvement up to $O(\sqrt{n})$ of iteration complexity compared with previous accelerated coordinate descent methods [23]. However, the naive evaluation of partial derivative of function $F_k(\mathbf{x})$ in ACDM takes $O(nnz(\mathbf{A}))$ time; second, the time cost of full vector operation in each iteration of ACDM is $O(n)$. We will show that these difficulties can be tackled by a carefully designed implementation technique[1] and the main procedure is listed in Algorithm 2.

The main three steps of the ACDM is, in each iteration t,

1. $\mathbf{u}_t = \tau\mathbf{v}_t + (1-\tau)\mathbf{x}_t^{k+1}$.
2. Sample coordinate i with probability proportional to $\sqrt{L_i}$ and update $\mathbf{x}_{t+1}^{k+1} = \mathbf{u}_t - \frac{1}{L_i}\nabla_i F_k(u_t)\mathbf{e}_i$, where L_i is the Lipschitz constant of $\nabla_i F_k(\cdot)$.

[1]This technique is motivated by [23].

3. $\mathbf{v}_{t+1} = \frac{1}{1+\eta\rho}\Big(\mathbf{v}_t + \eta\rho\mathbf{u}_t - \frac{\eta}{p_i}\nabla_i F_k(\mathbf{u}_t)\mathbf{e}_i\Big)$.

Here we drop the superscript $k+1$ of intermediate variables $\mathbf{u}, \mathbf{v}$ in each inner iteration. In the above procedure, $\eta = \frac{1}{\tau S^2}, \tau = \frac{2}{1+\sqrt{4S^2/\rho+1}}$, $S = \sum_{i=1}^n \sqrt{\|\mathbf{A}_i\|^2+1}$. This choice of parameter is based on letting $\beta = 0$ in the original ACDM [22] and the estimation of component-wise Lipschitz constant $L_i = \|\mathbf{A}_i\|^2 + 1$ and strongly convexity parameter $\sigma \geq \rho$ of problem (11). The above procedure can be further written as

$$
\begin{aligned}
\mathbf{u}_{t+1} &= \tau\mathbf{v}_{t+1} + (1-\tau)\mathbf{x}_{t+1}^{k+1} \\
&= \frac{1+\eta\rho-\tau}{1+\eta\rho}\mathbf{u}^t + \frac{\tau}{1+\eta\rho}\mathbf{v}^t - \left[\frac{\eta\tau}{p_i(1+\eta\rho)} + \frac{1-\tau}{L_i}\right]\nabla_i F_k(\mathbf{u}_t)\mathbf{e}_i.
\end{aligned}
$$

Combining this formula with the third step of ACDM, we can write each iteration as following matrix iteration:

$$
\begin{bmatrix}\mathbf{u}_{t+1}^T \\ \mathbf{v}_{t+1}^T\end{bmatrix} = \mathbf{M}\begin{bmatrix}\mathbf{u}_t^T \\ \mathbf{v}_t^T\end{bmatrix} - \mathbf{s}_t^i. \tag{25}
$$

The matrix $\mathbf{M}$ and vector $\mathbf{s}_t^i$ are defined as

$$
\mathbf{M} = \begin{bmatrix}1-\alpha_v & \alpha_v \\ \beta_u & 1-\beta_u\end{bmatrix} \text{ with } \begin{bmatrix}\alpha_v \\ \beta_u\end{bmatrix} = \begin{bmatrix}\frac{\tau}{1+\eta\rho} \\ \frac{\eta\rho}{1+\eta\rho}\end{bmatrix} \text{ and } \mathbf{s}_t^i = \begin{bmatrix}\Big(\frac{\eta\tau}{p_i(1+\eta\rho)} + \frac{1-\tau}{L_i}\Big)\nabla_i F_k(\mathbf{u}_t)\mathbf{e}_i^T \\ \frac{\eta}{p_i(1+\eta\rho)}\nabla_i F_k(\mathbf{u}_t)\mathbf{e}_i^T\end{bmatrix}, \tag{26}
$$

Therefore, we can implement ACDM in each iteration as

$$
\begin{bmatrix}\mathbf{u}_{t+1}^T \\ \mathbf{v}_{t+1}^T\end{bmatrix} = \mathbf{M}_{t+1}\begin{bmatrix}\mathbf{u}^T \\ \mathbf{v}^T\end{bmatrix}, \tag{27}
$$

and update $\mathbf{u}$, $\mathbf{v}$ and matrix $\mathbf{M}_t$ by

$$
\mathbf{M}_{t+1} = \mathbf{M}\cdot\mathbf{M}_t \text{ and } \begin{bmatrix}\mathbf{u}^T \\ \mathbf{v}^T\end{bmatrix} := \begin{bmatrix}\mathbf{u}^T \\ \mathbf{v}^T\end{bmatrix} - \mathbf{M}_{t+1}^{-1}\mathbf{s}_t^i. \tag{28}
$$

The rest is to calculate the partial derivative

$$
\nabla_i F_k(\mathbf{u}_t) = \rho(\mathbf{A}_i)^T\mathbf{A}\mathbf{u}_t + \rho u_{t,i} + \mathbf{A}_{1,i}^T[\mathbf{z}^k + \rho(\mathbf{A}_2\mathbf{y}^k - \overline{\mathbf{b}})] + c_i,
$$

where $\mathbf{A}_{1,i}$ is ith column of matrix $\mathbf{A}_1$. Utilizing auxillary variable $\overline{\mathbf{u}}_t$ and $\overline{\mathbf{v}}_t$ to represent $\mathbf{A}\mathbf{u}_t$ and $\mathbf{A}\mathbf{v}_t$ and multiplying (25) by matrix $\mathbf{A}$, we have

$$
\begin{bmatrix}(\mathbf{A}\mathbf{u}_{t+1})^T \\ (\mathbf{A}\mathbf{v}_{t+1})^T\end{bmatrix} = \mathbf{M}\begin{bmatrix}(\mathbf{A}\mathbf{u}_t)^T \\ (\mathbf{A}\mathbf{v}_t)^T\end{bmatrix} + \mathbf{s}_t^i\mathbf{A}^T \tag{29}
$$

$$
\iff \begin{bmatrix}\overline{\mathbf{u}}_{t+1}^T \\ \overline{\mathbf{v}}_{t+1}^T\end{bmatrix} = \mathbf{M}\begin{bmatrix}\overline{\mathbf{u}}_t^T \\ \overline{\mathbf{v}}_t^T\end{bmatrix} + \mathbf{s}_t^i\mathbf{A}^T. \tag{30}
$$

Therefore, to implement ACDM in each iteration, we can just maintain vectors $\overline{\mathbf{u}}$ and $\overline{\mathbf{v}}$ such that

$$
\begin{bmatrix}\overline{\mathbf{u}}_{t+1}^T \\ \overline{\mathbf{v}}_{t+1}^T\end{bmatrix} = \mathbf{M}\begin{bmatrix}\overline{\mathbf{u}}^T \\ \overline{\mathbf{v}}^T.\end{bmatrix} \tag{31}
$$

With this representation, each update step can be implemented by

$$
\mathbf{M}_{t+1} = \mathbf{M}\cdot\mathbf{M}_t \text{ and } \begin{bmatrix}\overline{\mathbf{u}}^T \\ \overline{\mathbf{v}}^T\end{bmatrix} := \begin{bmatrix}\overline{\mathbf{u}}^T \\ \overline{\mathbf{v}}^T\end{bmatrix} - \mathbf{M}_{t+1}^{-1}\mathbf{s}_t^i\mathbf{A}^T. \tag{32}
$$

Lemma 6. (Inner complexity) *In each iteration of Algorithm 2, if the current picked coordinate is i, the update can be finished in $O(nnz(\mathbf{A}_i))$ time, moreover, to guarantee that $F_k(\mathbf{x}^{k+1}) - \min_{\mathbf{x}} F_k(\mathbf{x}) \leq \epsilon_k$ with probability $1-p$, it suffices to run Algorithm 2 for number of iterations*

$$
T_k \geq O(1)\cdot\sum_{i=1}^n \|\mathbf{A}_i\|\log\left(\frac{D_0^k}{\epsilon_k p}\right), \quad D_0^k = \|\overline{F}_k(\mathbf{u}_0) - \min_{\mathbf{x}}\overline{F}_k(\mathbf{x})\|. \tag{33}
$$

Table 2: Data Statistics for Experiments in Basis Pursuit Problem

Data Set	Signal dimension	Measurements dimension
bp1	8192	1024
bp2	16384	2048
bp3	32768	4096

Table 3: Data Statistics for Experiments in L1 SVM, SICE and NMF Problem

Data Set	#Samples	#Features	nnz
news20	15935	62061	1272569
real-sim	72309	20958	3709083
arcene	900	10000	540941
colon	62	2000	124000
sonar	208	60	12479
w2a	3470	300	40373

The above iteration complexity is obtained by choosing parameter $\beta = 0$ in [22] and utilizing the Theorem 1 in [24] to transform the convergence in expectation to the form of probability.

Theorem 2. (Overall complexity) *Denote $\mathbf{z}^k$ as the dual iterates produced by Algorithm 1. To guarantee that there exists an optimal solution $\mathbf{z}^*$ such that $\|\mathbf{z}^k - \mathbf{z}^*\| \leq \epsilon$ with probability $1-p$, it suffices to run Algorithm 1 for*

$$k \geq 2\gamma^2 \log(2D_0/\epsilon) \tag{34}$$

outer iterations and solve each sub-problem (11) for the number of inner iterations

$$T \geq O(1) \cdot \sum_{i=1}^{n} \|\mathbf{A}_i\| \log\left(\frac{\rho(D_0^k)^{\frac{1}{3}}\gamma^2}{\epsilon^{\frac{2}{3}}p^{\frac{1}{3}}} \log\left(\frac{2D_0}{\epsilon}\right)\right). \tag{35}$$

The results for the primal iterates $\mathbf{x}^k$ and $\mathbf{y}^k$ are similar. In the existing ADMM [8], each primal and dual update only requires $O(nnz(\mathbf{A}))$ time to solve. The complexity of this method is

$$O(a_m\mu^2(a_mR_x + d_mR_z)^2(\sqrt{mn} + \|\mathbf{A}\|_F)^2 nnz(\mathbf{A})\log(1/\epsilon)),$$

where $a_m = \max_i \|\mathbf{A}_i\|$, d_m is the largest number of non-zero elements of each row of matrix $\mathbf{A}$, and μ is the Hoffman constant depends on the optimal solution set of LP. Based on Theorem 2, an estimation of the worst-case complexity of Algorithm 1 is

$$O(a_m\theta_{S^*}^2(R_x\|\mathbf{A}\| + R_z)^2 nnz(\mathbf{A})\log^2(1/\epsilon)).$$

Remark that our method has a weak dependence on the problem dimension compared with the existing ADMM. Since the Frobenius norm of a matrix satisfies $\|\mathbf{A}\|_2 \leq \|\mathbf{A}\|_F$, our method is faster than the one in [8].

6 Numerical Results

In this section, we examine the performance of our algorithm and compare it with the state-of-art of algorithms developed for solving the LP. The first is the existing ADMM in [8]. The second is the ALCD method in [9], which is reported to be the current fastest first-order LP solver. They have shown that this algorithm can significantly speed up solving several important machine learning problems compared with the Simplex and IPM. We name our Algorithm 1 as LPADMM. In the experiments, we require that the accuracy of subproblem solver $\epsilon_k = 10^{-3}$ and the stopping criteria is that both primal residual $\|\mathbf{A}_1\mathbf{x}^k + \mathbf{A}_2\mathbf{y}^k - \overline{\mathbf{b}}\|_\infty$ and dual residual $\|\mathbf{A}_1^T\mathbf{z}^k + \mathbf{c}\|_\infty$ is less than 10^{-3}. All the LP instances are generated from the basis pursuit, L1 SVM, SICE and NMF problems. The data source and statistics are listed in TABLE 2 and TABLE 3.

For the basis pursuit problem, we adopt the following popular signal generation model [4]. In particular, the target signal $\mathbf{x} \in \mathbb{R}^n$ is set to

$$x_i = \mathbf{1}\{i \in \Lambda\}\Theta_i^{(1)}10^{2\Theta_i^{(2)}}, \tag{36}$$

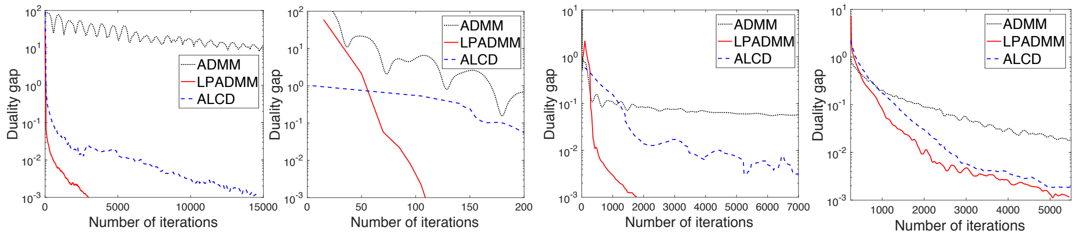

Figure 2: The duality gap versus the number of iterations. From left to right figures are the BP, NMF, the L1 SVM and and the SICE problem.

Table 4: Timing Results for BP, SICE, NMF and L1 SVM Problem (in sec. long means > 60 hours)

Data	m	n	nnz($\mathbf{A}$)	LPADMM		ALCD		ADMM	
				Time	Iterations	Time	Iterations	Time	Iterations
bp1	17408	16384	8421376	**22**	**3155**	864	14534	long	long
bp2	34816	32768	33619968	**79**	**4657**	2846	19036	long	long
bp3	69632	65536	134348800	**217**	**6287**	12862	24760	long	long
arcene	50095	30097	1151775	**801**	**15198**	1978	176060	21329	2035415
real-sim	176986	135072	7609186	**955**	**4274**	1906	18262	19697	249363
sonar	80912	68224	2756832	**258**	**5446**	659	13789	3828	151972
colon	217580	161040	8439626	**395**	**216**	455	1288	7423	83680
w2a	12048256	12146960	167299110	**19630**	**2525**	45388	8492	long	long
news20	2785205	2498375	53625267	**7765**	**2205**	9173	6174	long	long

where Λ is constructed by selecting randomly s indices of set $[1, 2, \ldots, n]$, $\Theta_i^{(1)}, i \in \Lambda$ is the Bernoulli random with values $\{+1, -1\}$ in equal probability, $\Theta_i^{(2)}, i \in \Lambda$ is a uniformly distributed random variable in $[0, 1]$. The sparsity level is set to 0.20. The dynamic range of signal is 40dB. The measurement matrix $\mathbf{A} \in \mathbb{R}^{m \times n}$ is generated by randomly selecting rows from the classic Walsh-Hadamard transform matrix of order 2^j. For the L1 SVM problem, we set the penalty parameter $\eta = 1$. For the SICE problem, we adopt the technique introduced in [3] and provide the result on only one of p independent problems. For NMF problem, we set the approximation tolerance to be 0.01 times number of samples.

We first compare the convergence rate of different algorithms in solving the above problems. We use the bp1 for BP problem, data set colon cancer for NMF problem, news20 for L1 SVM problem and real-sim for SICE problem. We set proximity parameter $\rho = 1$. We adopt the relative duality gap as the comparison metric, which is defined as $\|\mathbf{c}^T\mathbf{x}^k + \mathbf{b}^T\mathbf{z}_x^k\|/\|\mathbf{c}^T\mathbf{x}^*\|$, where $\mathbf{x}^*$ is obtained approximately by running our method with a strict stopping condition. In our simulation, one iteration represents n coordinate descent steps for ALCD and LPADMM, and one dual updating step for ADMM. As can be seen in the Fig. 2, our new method exhibits a global and linear convergence rate and matches our theoretical performance bound. Besides, it converges faster than both the ALCD and existing ADMM method, especially in solving the BP and NMF problem.

We next examine the performance of our algorithm from the perspective of time efficiency (both clocking time and number of iterations). We adopt the dynamic step size rule for ALCD to optimize its performance. Note that, exchanging the role of the primal and dual problem in (7), we can obtain the dual version of both ADMM and ACLD, which can be used to tackle the primal or dual sparse problem. We run both methods and adopt the minimum time. The stopping criterion requires that the primal and dual residual and the relative duality gap is less than 10^{-3}. The data set bp1,bp2,bp3 is used for basis pursuit problem, news20 is used for L1 SVM problem; arcene, real-sim are used for SICE problem; sonar, colon and w2a are used for NMF problem. Among all experiments, we can observe that our proposed algorithm requires approximately $10\% - 40\%$ iterations and $10\% - 85\%$ time of the ALCD method, and become particularly advantageous for basis pursuit problem ($50\times$ speed up) or ill posed problems such as SICE and NMF problem. The main reason is that, due to our splitting method, each subproblem is a well-conditioned linear system that can be efficiently solved. In particular, for the basis pursuit problem, the primal iterates $\mathbf{x}^k$ is updated by closed-form expression (23), which can be calculated in $O(n\log(n))$ time by Fast Walsh–Hadamard transform.

Under the same stopping criterion, we vary the Augmented Lagrangian parameter ρ and see how it influences the required number of iterations to obtain a given accuracy solution. We also run 100 times with the same ρ for each data set to avoid random noise. We observe that, when the parameter ρ increases 100 times (from 1 to 100), the number of iterations decreases or increases roughly 20% compared with the existing results in TABLE 1 (section of numerical results). Moreover, even when the ρ is drastically increased from 1 to 100, the largest number of iterations and clocking time produced by our algorithm are still much less than the smallest one produced by other algorithms.

7 Conclusions

In this paper, we proposed a new variable splitting method to solve the linear programming problem. The theoretical contribution of this work is that we prove that $2-$block ADMM converges globally and linearly when applying to the linear program. The obtained convergence rate has a weak dependence of the problem dimension and is less than the best known result. Compared with the existing LP solvers, our algorithms not only provides a flexibility to exploit the specific structure of constraint matrix $\mathbf{A}$, but also can be naturally combined with the existing acceleration techniques to significantly speed up solving the large-scale machine learning problems. The future work focuses on generalizing our theoretical framework and exhibiting the global linear convergence rate when applying ADMM to solve a convex quadratic program.

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A Proof of Lemma 1

The proof of Lemma 1 is tailored from [11]. Several intermediate results in this proof will be useful in the derivation of other lemmas. Thus, we provide this modification for completeness. We first write the three steps of ADMM as,

$$\mathbf{x}^{k+1} = \arg\min_{\mathbf{x}} f(\mathbf{x}) + (\mathbf{z}^k)^T \mathbf{A}_1 \mathbf{x} + \frac{\rho}{2}\|\mathbf{A}_1\mathbf{x} + \mathbf{A}_2\mathbf{y}^k - \overline{\mathbf{b}}\|^2, \tag{37}$$

$$\mathbf{y}^{k+1} = \arg\min_{\mathbf{y}} g(\mathbf{y}) + (\mathbf{z}^k)^T \mathbf{A}_2 \mathbf{y} + \frac{\rho}{2}\|\mathbf{A}_1\mathbf{x}^{k+1} + \mathbf{A}_2\mathbf{y} - \overline{\mathbf{b}}\|^2, \tag{38}$$

$$\mathbf{z}^{k+1} = \mathbf{z}^k + \rho(\mathbf{A}_1\mathbf{x}^{k+1} + \mathbf{A}_2\mathbf{y}^{k+1} - \overline{\mathbf{b}}). \tag{39}$$

We will prove (37)-(39) are equivalent to the following proximal operations.

$$\mathbf{w}^{k+1} = \mathbf{prox}_{\rho\overline{f}}(\mathbf{z}^k + \rho\mathbf{A}_2\mathbf{y}^k), \tag{40}$$

$$\mathbf{z}^{k+1} = \mathbf{prox}_{\rho\overline{g}}(\mathbf{w}^{k+1} - \rho\mathbf{A}_2\mathbf{y}^k), \tag{41}$$

$$\rho\mathbf{A}_2\mathbf{y}^{k+1} = \rho\mathbf{A}_2\mathbf{y}^k + \mathbf{z}^{k+1} - \mathbf{w}^{k+1}, \tag{42}$$

where $\mathbf{w}^{k+1} = \mathbf{z}^k + \rho(\mathbf{A}_1\mathbf{x}^{k+1} + \mathbf{A}_2\mathbf{y}^k - \overline{\mathbf{b}})$ are the dual variables in the optimization problem (11). Functions f and g is defined as

$$\overline{f}(\mathbf{u}) \triangleq \overline{\mathbf{b}}^T\mathbf{u} + f^*(-\mathbf{A}_1^T\mathbf{x}) \quad \text{and} \quad \overline{g}(\mathbf{u}) \triangleq g^*(-\mathbf{A}_2^T\mathbf{x}), \tag{43}$$

where f^* and g^* is the convex conjugate of function f and g, defined as $f^*(\mathbf{y}) = \sup_{\mathbf{x}} \mathbf{y}^T\mathbf{x} - f(\mathbf{x})$ and $g^*(\mathbf{y}) = \sup_{\mathbf{x}} \mathbf{y}^T\mathbf{x} - g(\mathbf{x})$.

Claim 1: equivalence of first step (37) $\iff$ (40)

$$\begin{aligned}
&\mathbf{x}^{k+1} = \arg\min_{\mathbf{x}} f(\mathbf{x}) + (\mathbf{z}^k)^T \mathbf{A}_1 \mathbf{x} + \frac{\rho}{2}\|\mathbf{A}_1\mathbf{x} + \mathbf{A}_2\mathbf{y}^k - \overline{\mathbf{b}}\|^2 \\
\overset{(a)}{\iff} &\mathbf{0} \in \partial f(\mathbf{x}^{k+1}) + \mathbf{A}_1^T \mathbf{z}^k + \rho \mathbf{A}_1^T(\mathbf{A}_1\mathbf{x}^{k+1} + \mathbf{A}_2\mathbf{y}^k - \overline{\mathbf{b}}) \\
\overset{(b)}{\iff} & -\mathbf{A}_1^T\mathbf{w}^{k+1} \in \partial f(\mathbf{x}^{k+1}) \\
\overset{(c)}{\iff} &\mathbf{x}^{k+1} \in \partial f^*(-\mathbf{A}_1^T\mathbf{w}^{k+1}) \\
\overset{(d)}{\iff} & -\mathbf{A}_1\mathbf{x}^{k+1} \in -\mathbf{A}_1\partial f^*(-\mathbf{A}_1^T\mathbf{w}^{k+1}) \\
\overset{(e)}{\iff} &\mathbf{0} \in \overline{\mathbf{b}} - \mathbf{A}_1\partial f^*(-\mathbf{A}_1^T\mathbf{w}^{k+1}) + \frac{1}{\rho}(\mathbf{w}^{k+1} - \mathbf{z}^k - \rho\mathbf{A}_2\mathbf{y}^k) \\
\overset{(f)}{\iff} &\mathbf{w}^{k+1} = \mathbf{prox}_{\rho\overline{f}}(\mathbf{z}^k + \rho\mathbf{A}_2\mathbf{y}^k).
\end{aligned} \tag{44}$$

The above, (a) is based on the first-order optimality condition of unconstrained optimization, (b) utilizes the definition of $\mathbf{w}^{k+1}$, (c) is based on the fact that function f is closed and convex, (d) utilizes the fact that matrix $\mathbf{A}_1$ has full column rank, since $\mathbf{A}_1 = [\mathbf{A}; \mathbf{I}]$ contains the identity matrix in its column space. (e) utilizes the definition of $\mathbf{w}^{k+1}$ again. (f) is based on the first-order optimality condition of proximal operator.

Claim 2: equivalence of second step (38) $\iff$ (41)

$$\begin{aligned}
&\mathbf{y}^{k+1} = \arg\min_{\mathbf{y}} g(\mathbf{y}) + (\mathbf{z}^k)^T \mathbf{A}_2 \mathbf{y} + \frac{\rho}{2}\|\mathbf{A}_1\mathbf{x}^{k+1} + \mathbf{A}_2\mathbf{y} - \overline{\mathbf{b}}\|^2 \\
\overset{(a)}{\iff} &\mathbf{0} \in \partial g(\mathbf{y}^{k+1}) + \mathbf{A}_2^T \mathbf{z}^k + \rho \mathbf{A}_2^T(\mathbf{A}_1\mathbf{x}^{k+1} + \mathbf{A}_2\mathbf{y}^{k+1} - \overline{\mathbf{b}}) \\
\overset{(b)}{\iff} & -\mathbf{A}_2^T\mathbf{z}^{k+1} \in \partial g(\mathbf{y}^{k+1}) \\
\overset{(c)}{\iff} &\mathbf{y}^{k+1} \in \partial g^*(-\mathbf{A}_2^T\mathbf{z}^{k+1}) \\
\overset{(d)}{\iff} & -\mathbf{A}_2\mathbf{y}^{k+1} \in -\mathbf{A}_2\partial g^*(-\mathbf{A}_2^T\mathbf{z}^{k+1}) \\
\overset{(e)}{\iff} &\mathbf{0} \in -\mathbf{A}_2\partial g^*(-\mathbf{A}_2^T\mathbf{z}^{k+1}) + \frac{1}{\rho}(\mathbf{z}^{k+1} - \mathbf{w}^{k+1} + \rho\mathbf{A}_2\mathbf{y}^k) \\
\overset{(f)}{\iff} &\mathbf{z}^{k+1} = \mathbf{prox}_{\rho\overline{g}}(\mathbf{w}^{k+1} - \rho\mathbf{A}_2\mathbf{y}^k).
\end{aligned} \tag{45}$$

The above, (a) is based on the first-order optimality condition of unconstrained optimization, (b) utilizes the definition of dual iterates $\mathbf{z}^{k+1}$, (c) is based on the fact that function g is closed and convex. (d) utilizes the fact that matrix $\mathbf{A}_2$ has full column rank, since $\mathbf{A}_1 = [\mathbf{0}; -\mathbf{I}]$ contains the identity matrix in its column space. (e) utilizes the definition of $\mathbf{z}^{k+1}$ again. (f) is based on the first-order optimality condition of proximal operator.

Claim 3: equivalence between (39) $\iff$ (42)

$$\begin{aligned}
&\rho\mathbf{A}_2\mathbf{y}^k + \mathbf{z}^{k+1} - \mathbf{w}^{k+1} \\
&\quad\overset{(a)}{=} \rho\mathbf{A}_2\mathbf{y}^k + [\mathbf{z}^k + \rho(\mathbf{A}_1\mathbf{x}^{k+1} + \mathbf{A}_2\mathbf{y}^{k+1} - \overline{\mathbf{b}})] - [\mathbf{z}^k + \rho(\mathbf{A}_1\mathbf{x}^{k+1} + \mathbf{A}_2\mathbf{y}^k - \overline{\mathbf{b}})] \\
&\quad= \rho\mathbf{A}_2\mathbf{y}^{k+1}.
\end{aligned}$$

The above, (a) is based on the definition of $\mathbf{w}^{k+1}$ and $\mathbf{z}^{k+1}$.

Combining claim 1-3, we arrive at the desired equivalence between ADMM and the proximal operations. Based on the definition of iterates $\mathbf{p}^k = \mathbf{z}^k - \rho\mathbf{A}_2\mathbf{y}^k$, the proximal operations (40)-(42) are equivalent to

$$\mathbf{w}^{k+1} = \mathbf{prox}_{\rho\overline{f}}(2\mathbf{z}^k - \mathbf{p}^k), \tag{46}$$

$$\mathbf{z}^{k+1} = \mathbf{prox}_{\rho\overline{g}}(\mathbf{w}^{k+1} - \mathbf{z}^k + \mathbf{p}^k), \tag{47}$$

$$\mathbf{p}^{k+1} = \mathbf{p}^k + \mathbf{w}^{k+1} - \mathbf{z}^k. \tag{48}$$

It can be further simplified as

$$\mathbf{p}^{k+1} = \mathbf{p}^k + \mathbf{prox}_{\rho\overline{f}}(2\mathbf{z}^k - \mathbf{p}^k) - \mathbf{z}^k, \tag{49}$$

$$\mathbf{z}^{k+1} = \mathbf{prox}_{\rho\overline{g}}(\mathbf{p}^{k+1}). \tag{50}$$

Start at $\mathbf{p}^0$ and renumber the iterates, we have

$$\mathbf{z}^{k+1} = \mathbf{prox}_{\rho\overline{g}}(\mathbf{p}^k), \tag{51}$$

$$\mathbf{p}^{k+1} = \mathbf{p}^k + \mathbf{prox}_{\rho\overline{f}}(2\mathbf{z}^{k+1} - \mathbf{p}^k) - \mathbf{z}^{k+1}, \tag{52}$$

which is the classic Douglas-Rachford splitting method [11]. We can further write it as following iteration of proximal operator.

$$\mathbf{p}^{k+1} = T(\mathbf{p}^k), \tag{53}$$

where the operator T is defined as

$$T(\mathbf{x}) = \mathbf{x} + \mathbf{prox}_{\rho\overline{f}}(2\mathbf{prox}_{\rho\overline{g}}(\mathbf{x}) - \mathbf{x}) - \mathbf{prox}_{\rho\overline{g}}(\mathbf{x}). \tag{54}$$

Claim 4: Let $G(\mathbf{x}) = \mathbf{x} - T(\mathbf{x})$, then G is firmly non-expansive.

Based on the fact that proximal operator $\mathbf{prox}_{\rho\overline{f}}(\cdot)$ and $\mathbf{prox}_{\rho\overline{g}}$ are firmly non-expansive, we have $\forall \mathbf{p}, \mathbf{p}' \in \mathbb{R}^{m+n}$,

$$\langle \mathbf{prox}_{\rho\overline{g}}(\mathbf{p}) - \mathbf{prox}_{\rho\overline{g}}(\mathbf{p}'), \mathbf{p} - \mathbf{p}' \rangle \geq \|\mathbf{prox}_{\rho\overline{g}}(\mathbf{p}) - \mathbf{prox}_{\rho\overline{g}}(\mathbf{p}')\|^2,$$

$$\langle \mathbf{prox}_{\rho\overline{f}}(2\mathbf{prox}_{\rho\overline{g}}(\mathbf{p}) - \mathbf{p}) - \mathbf{prox}_{\rho\overline{f}}(2\mathbf{prox}_{\rho\overline{g}}(\mathbf{p}') - \mathbf{p}'), (2\mathbf{prox}_{\rho\overline{g}}(\mathbf{p}) - \mathbf{p}) - (2\mathbf{prox}_{\rho\overline{g}}(\mathbf{p}') - \mathbf{p}') \rangle$$
$$\geq \|\mathbf{prox}_{\rho\overline{f}}(2\mathbf{prox}_{\rho\overline{g}}(\mathbf{p}) - \mathbf{p}) - \mathbf{prox}_{\rho\overline{f}}(2\mathbf{prox}_{\rho\overline{g}}(\mathbf{p}') - \mathbf{p}')\|^2.$$

Summing the above two inequalities together and rearranging the terms in both sides, we have

$$\langle G(\mathbf{p}) - G(\mathbf{p}'), \mathbf{p} - \mathbf{p}' \rangle \geq \|G(\mathbf{p}) - G(\mathbf{p}')\|^2, \tag{55}$$

which implies the desired result.

Let $\mathbf{p}^*$ be any point satisfying $T(\mathbf{p}^*) = \mathbf{p}^*$, we have

$$\begin{aligned}
\|\mathbf{p}^{k+1} - \mathbf{p}^*\|^2 &= \|\mathbf{p}^{k+1} - \mathbf{p}^k + \mathbf{p}^k - \mathbf{p}^*\|^2 \\
&= \|\mathbf{p}^k - \mathbf{p}^*\|^2 + \|\mathbf{p}^{k+1} - \mathbf{p}^k\|^2 + 2\langle \mathbf{p}^{k+1} - \mathbf{p}^k, \mathbf{p}^k - \mathbf{p}^* \rangle \\
&\overset{(a)}{=} \|\mathbf{p}^k - \mathbf{p}^*\|^2 + \|G(\mathbf{p}^k)\|^2 - 2\langle G(\mathbf{p}^k) - G(\mathbf{p}^*), \mathbf{p}^k - \mathbf{p}^* \rangle \\
&\overset{(b)}{\leq} \|\mathbf{p}^k - \mathbf{p}^*\|^2 + \|G(\mathbf{p}^k)\|^2 - 2\|G(\mathbf{p}^k) - G(\mathbf{p}^*)\|^2 \\
&= \|\mathbf{p}^k - \mathbf{p}^*\|^2 - \|\mathbf{p}^k - \mathbf{p}^{k+1}\|^2.
\end{aligned} \tag{56}$$

The above, (a) is based on the definition of $G(\cdot)$ and $\mathbf{p}^*$ is the zero point of operator $G(\cdot)$, (b) is based on the Claim 4 that the operator $G(\cdot)$ is non-firmly expansive. Since $\mathbf{p}^*$ is any zero point of operator $G(\cdot)$, let $\mathbf{p}^* = [\mathbf{p}^k]_{G^*}$, where $G^* = \{\mathbf{p}^* | T(\mathbf{p}^*) = \mathbf{p}^*\}$, we have

$$\|\mathbf{p}^{k+1} - [\mathbf{p}^{k+1}]_{G^*}\|^2 \leq \|\mathbf{p}^{k+1} - [\mathbf{p}^k]_{G^*}\|^2 \leq \|\mathbf{p}^k - [\mathbf{p}^k]_{G^*}\|^2 - \|\mathbf{p}^k - \mathbf{p}^{k+1}\|^2. \tag{57}$$

The convergence and boundedness of $\mathbf{x}^k, \mathbf{y}^k, \mathbf{z}^k$ can be obtained from the above inequality. More details can be seen in Theorem 2 in [8]. Thus, the lemma follows.

B Proof of Lemma 2

Let's first prove the "if" direction. Suppose that $(\mathbf{x}^*, \mathbf{y}^*, \mathbf{z}^*)$ satisfy the condition (i)-(iv). The condition (i)-(iii) implies that the $(\mathbf{x}^*, \mathbf{y}^*, \mathbf{z}^*)$ satisfies the primal feasibility: $A\mathbf{x}^* = \mathbf{b}$, $\mathbf{x}^* = \mathbf{y}^*$, $y_i^* \geq 0, i \in [n_b]$; dual feasibility: $-\mathbf{A}^T\mathbf{z}_x^* - \mathbf{z}_y^* = \mathbf{c}$, $z_{y,i}^* \leq 0, i \in [n_b], z_{y,i}^* = 0, i \in [n] \backslash [n_b]$. Based on the weak duality theorem, for any other primal feasible variables $\mathbf{x}, \mathbf{y}$ and dual feasible variables $\mathbf{z}_x, \mathbf{z}_y$,

$$\mathbf{c}^T\mathbf{x} \geq -\mathbf{b}^T\mathbf{z}_{\mathbf{x}}^* \quad \text{and} \quad \mathbf{c}^T\mathbf{x}^* \geq -\mathbf{b}^T\mathbf{z}_x.$$

Then, combining this result with the condition of (iv), we have

$$\mathbf{c}^T\mathbf{x} \geq \mathbf{c}^T\mathbf{x}^* \quad \text{and} \quad \mathbf{b}^T\mathbf{z}_x \geq \mathbf{b}^T\mathbf{z}_{\mathbf{x}}^*, \tag{58}$$

which implies the $(\mathbf{x}^*, \mathbf{y}^*, \mathbf{z}^*)$ are optimal primal and dual solutions of the LP (7).

We then prove the "only if" direction. Suppose that the $(\mathbf{x}^*, \mathbf{y}^*, \mathbf{z}^*)$ are optimal primal and dual solutions of the LP (7). By the strong duality theorem of LP, we arrive the last condition (iv). Based on the KKT condition, the primal feasibility implies that $\mathbf{A}\mathbf{x}^* = \mathbf{b}, \mathbf{x}^* = \mathbf{y}^*$ and $y_i^* \geq 0, i \in [n_b]$; the constraints on the dual variable implies that $z_{y,i}^* \leq 0, i \in [n_b]$; the Lagrangian of LP: $\mathbf{c}^T\mathbf{x} + g(\mathbf{y}) + \mathbf{z}_x^T(\mathbf{A}\mathbf{x} - \mathbf{b}) + \mathbf{z}_y^T(\mathbf{x} - \mathbf{y})$ with respect to $\mathbf{x}, \mathbf{y}$ vanishes implies that $-\mathbf{A}^T\mathbf{z}_x^* - \mathbf{z}_y^* = \mathbf{c}$ and $z_{y,i}^* = 0, i \in [n]\backslash[n_b]$.

C Proof of Lemma 4

We first show that $\mathbf{p}^{k+1} - \mathbf{p}^k = \rho(\mathbf{A}_1\mathbf{x}^{k+1} + \mathbf{A}_2\mathbf{y}^k - \overline{\mathbf{b}})$. Based on the definition of $\mathbf{p}_k = \mathbf{z}^k - \rho\mathbf{A}_2\mathbf{y}^k$, we have

$$\begin{aligned}\mathbf{p}^{k+1} - \mathbf{p}^k &= (\mathbf{z}^{k+1} - \rho\mathbf{A}_2\mathbf{y}^{k+1}) - (\mathbf{z}^k - \rho\mathbf{A}_2\mathbf{y}^k) \\ &\overset{(a)}{=} \rho(\mathbf{A}_1\mathbf{x}^{k+1} + \mathbf{A}_2\mathbf{y}^k - \overline{\mathbf{b}}).\end{aligned} \tag{59}$$

The above, (a) is based on the dual updating step of ADMM.

Second, we show $\mathbf{c} + \mathbf{A}_1^T\mathbf{z}^k = \mathbf{A}_1^T(\mathbf{p}^k - \mathbf{p}^{k+1})$. Based on the first-order optimality condition of the first step of ADMM (44) and the fact that $\partial f(\mathbf{x}^{k+1}) = \{\mathbf{c}\}$, we have

$$\mathbf{c} + \mathbf{A}_1^T\mathbf{z}^k + \rho\mathbf{A}_1^T(\mathbf{A}_1\mathbf{x}^{k+1} + \mathbf{A}_2\mathbf{y}^k - \overline{\mathbf{b}}) = \mathbf{0}. \tag{60}$$

Utilizing the result of (59), we can obtain

$$\mathbf{c} + \mathbf{A}_1^T\mathbf{z}^k + \mathbf{A}_1^T(\mathbf{p}^{k+1} - \mathbf{p}^k) = \mathbf{0}. \tag{61}$$

Third, we show that $\mathbf{c}^T\mathbf{x}^{k+1} + \mathbf{b}^T\mathbf{z}_x^k = \left\langle \mathbf{A}_1\mathbf{x}^{k+1} - \mathbf{z}^k/\rho, \mathbf{p}^k - \mathbf{p}^{k+1} \right\rangle$. Based on the result of (59) and (61), we have

$$\mathbf{A}_1\mathbf{x}^{k+1} + \mathbf{A}_2\mathbf{y}^k - \overline{\mathbf{b}} = (\mathbf{p}^{k+1} - \mathbf{p}^k)/\rho, \tag{62}$$

$$\mathbf{c} + \mathbf{A}_1^T\mathbf{z}^k = \mathbf{A}_1^T(\mathbf{p}^k - \mathbf{p}^{k+1}). \tag{63}$$

Further, based on the second and third steps of ADMM, for those bounded variables, we have ,

$$y_i^{k+1} = \left[x_i^{k+1} + z_{y,i}^k/\rho\right]_+ = \max\left\{0, x_i^{k+1} + z_{y,i}^k/\rho\right\} \geq 0, i \in [n_b], \tag{64}$$

$$z_{y,i}^{k+1} = z_{y,i}^k + \rho(x_i^{k+1} - y_i^{k+1}) = \min\left\{z_{y,i}^k + \rho x_i^{k+1}, 0\right\} \leq 0, i \in [n_b]. \tag{65}$$

For the free variables, we have

$$y_i^{k+1} = x_i^{k+1} + z_{y,i}^k/\rho \quad \text{and} \quad z_{y,i}^{k+1} = z_{y,i}^k + \rho(x_i^{k+1} - y_i^{k+1}) = 0, i \in [n]\backslash[n_b]. \tag{66}$$

Thus, we can obtain,

$$y_i^{k+1} \cdot z_{y,i}^{k+1} = \max\left\{0, x_i^{k+1} + z_{y,i}^k/\rho\right\} \cdot \min\left\{z_{y,i}^k + \rho x_i^{k+1}, 0\right\} = 0, \quad i \in [n_b], \tag{67}$$

$$y_i^{k+1} \cdot z_{y,i}^{k+1} = (x_i^{k+1} + z_{y,i}^k/\rho) \cdot 0 = 0, \quad i \in [n]\backslash[n_b]. \tag{68}$$

The equations (62)-(68) imply that the pair of iterates $(\mathbf{x}^{k+1}, \mathbf{y}^k)$ and z^k satisfies the complementary slackness of the following perturbed primal and dual LPs. Thus the iterates $(\mathbf{x}^{k+1}, \mathbf{y}^k)$ and $\mathbf{z}^k$

Approximate primal LP

$$\begin{aligned}\min \quad & \langle \mathbf{c} - \mathbf{A}_1^T(\mathbf{p}^k - \mathbf{p}^{k+1}), \mathbf{x} \rangle \\ s.t. \quad & \mathbf{A}_1\mathbf{x} + \mathbf{A}_2\mathbf{y} = \overline{\mathbf{b}} - (\mathbf{p}^{k+1} - \mathbf{p}^k)/\rho \\ & y_i \geq 0, i \in [n_b].\end{aligned}$$

Approximate dual LP

$$\begin{aligned}\min \quad & \left\langle \overline{\mathbf{b}} - (\mathbf{p}^{k+1} - \mathbf{p}^k)/\rho, \mathbf{z} \right\rangle \\ s.t. \quad & -\mathbf{A}_1^T\mathbf{z} = \mathbf{c} - \mathbf{A}_1^T(\mathbf{p}^k - \mathbf{p}^{k+1}), \\ & z_{y,i} \leq 0, i \in [n_b], z_{y,i} = 0, i \in [n]\backslash[n_b].\end{aligned}$$

constitute the optimal primal and dual solutions of the above LPs. According to the strong duality, we have

$$\left\langle \mathbf{c} - \mathbf{A}_1^T(\mathbf{p}^k - \mathbf{p}^{k+1}), \mathbf{x}^{k+1} \right\rangle + \left\langle \overline{\mathbf{b}} - (\mathbf{p}^{k+1} - \mathbf{p}^k)/\rho, \mathbf{z}^k \right\rangle = 0. \tag{69}$$

Rearranging the terms in the above equation, we have

$$\mathbf{c}^T\mathbf{x}^{k+1} + \mathbf{b}^T\mathbf{z}_{\mathbf{x}}^k = \left\langle \mathbf{A}_1\mathbf{x}^{k+1} - \mathbf{z}^k/\rho, \mathbf{p}^k - \mathbf{p}^{k+1} \right\rangle. \tag{70}$$

Thus, the lemma follows.

D Proof of Lemma 5

Let $\mathcal{S}^*$ denote the solution set described by Lemma 2. Since $\mathcal{S}^*$ is a non-empty polyhedron, we can utilize the Hoffman bound in Lemma 3 to bound the distance between the primal, dual iterates and the optimal solution set $\mathcal{S}^*$.

$$\begin{aligned}
\left\| \begin{bmatrix} \mathbf{x}^{k+1} \\ \mathbf{y}^k \\ \mathbf{z}^k \end{bmatrix} - \left[\begin{bmatrix} \mathbf{x}^{k+1} \\ \mathbf{y}^k \\ \mathbf{z}^k \end{bmatrix}\right]_{\mathcal{S}^*} \right\| &\le \theta_{S^*} \left\| \begin{bmatrix} \mathbf{A}_1\mathbf{x}^{k+1} + \mathbf{A}_2\mathbf{y}^{k+1} - \mathbf{b} \\ -\mathbf{A}_1^T\mathbf{z}^k - \mathbf{c} \\ [-\mathbf{y}^k]_+ \\ [\mathbf{z}_y^k]_+ \\ \mathbf{c}^T\mathbf{x}^{k+1} + \mathbf{b}^T\mathbf{z}_x^k \end{bmatrix} \right\| \\
&\overset{(a)}{=} \theta_{S^*} \left\| \begin{bmatrix} \mathbf{A}_1\mathbf{x}^{k+1} + \mathbf{A}_2\mathbf{y}^{k+1} - \mathbf{b} \\ -\mathbf{A}_1^T\mathbf{z}^k - \mathbf{c} \\ \mathbf{c}^T\mathbf{x}^{k+1} + \mathbf{b}^T\mathbf{z}_x^k \end{bmatrix} \right\| \\
&\overset{(b)}{=} \theta_{S^*} \left\| \begin{bmatrix} (\mathbf{p}^{k+1} - \mathbf{p}^k)/\rho \\ \mathbf{A}_1^T(\mathbf{p}^{k+1} - \mathbf{p}^k) \\ \langle \mathbf{A}_1\mathbf{x}^{k+1} - \mathbf{z}_x^k/\rho, \mathbf{p}^k - \mathbf{p}^{k+1} \rangle \end{bmatrix} \right\| \\
&\overset{(c)}{\le} \theta_{S^*} \left(\|\mathbf{p}^{k+1} - \mathbf{p}^k\|/\rho + \|\mathbf{A}_1^T(\mathbf{p}^{k+1} - \mathbf{p}^k)\| \right) + \\
&\quad\ \theta_{S^*} \left\| \langle \mathbf{A}_1\mathbf{x}^{k+1} - \mathbf{z}_x^k/\rho, \mathbf{p}^k - \mathbf{p}^{k+1} \rangle \right\| \\
&\overset{(d)}{\le} \theta_{S^*} \left[(1 + R_z)/\rho + (R_x + 1)\|\mathbf{A}_1^T\| \right] \|\mathbf{p}^{k+1} - \mathbf{p}^k\|. \qquad (71)
\end{aligned}$$

The above, (a) is based on (64) and (65) such that $[-\mathbf{y}^k]_+ = 0$ and $[\mathbf{z}_y^k]_+ = 0$ (Note that the projection operator $[\cdot]_+$ is elementary wise and we omit the constraints that $z_{y,i} = 0, i \in [n]\backslash[n_b]$ since it is always satisfied (Lemma 4), (b) is based on the estimation of residuals in Lemma 4, (c) utilizes the triangle inequality, (d) utilizes the following spectrum inequality

$$\|\mathbf{A}_1^T x\| \le \|\mathbf{A}_1^T\| \|\mathbf{x}\|, \qquad (72)$$

where $\|\mathbf{A}_1^T\|$ is the spectral norm of matrix $\mathbf{A}_1^T$, defined as $\|\mathbf{A}_1^T\|^2 = \rho_{max}(\mathbf{A}_1\mathbf{A}_1^T)$ (the maximum eigenvalue of matrix $\mathbf{A}_1\mathbf{A}_1^T$). Besides,

$$\begin{aligned}
\left\| \left\langle \mathbf{A}_1\mathbf{x}^{k+1} - \frac{1}{\rho}\mathbf{z}_x^k, \mathbf{p}^k - \mathbf{p}^{k+1} \right\rangle \right\| &= \left\| \langle \mathbf{x}^{k+1}, \mathbf{A}_1^T(\mathbf{p}^k - \mathbf{p}^{k+1}) \rangle - \langle \mathbf{z}_x^k/\rho, \mathbf{p}^k - \mathbf{p}^{k+1} \rangle \right\| \\
&\overset{(e)}{\le} \left\| \langle \mathbf{x}^{k+1}, \mathbf{A}_1^T(\mathbf{p}^k - \mathbf{p}^{k+1}) \rangle \right\| + \left\| \langle \mathbf{z}_x^k/\rho, \mathbf{p}^k - \mathbf{p}^{k+1} \rangle \right\| \\
&\overset{(f)}{\le} R_x\|\mathbf{A}_1^T\|\|\mathbf{p}^k - \mathbf{p}^{k+1}\| + \frac{R_z}{\rho}\|\mathbf{p}^k - \mathbf{p}^{k+1}\|. \qquad (73)
\end{aligned}$$

The above, (e) is based on the triangle inequality, (f) utilizes Cauchy-Schwarz inequality and spectrum inequality. Here R_x and R_z is defined as

$$R_x = \sup_k \|\mathbf{x}^k\| \quad \text{and} \quad R_z = \sup_k \|\mathbf{z}_x^k\|. \qquad (74)$$

Based on the above results, we have the following two inequalities,

$$\|\mathbf{y}^k - [\mathbf{y}^k]_{\mathcal{S}^*}\| \le \gamma'\|\mathbf{p}^{k+1} - \mathbf{p}^k\| \text{ and } \|\mathbf{z}^k - [\mathbf{z}^k]_{\mathcal{S}^*}\| \le \gamma'\|\mathbf{p}^{k+1} - \mathbf{p}^k\|, \qquad (75)$$

where $[\mathbf{x}^{k+1}]_{\mathcal{S}^*}$, $[\mathbf{y}^k]_{\mathcal{S}^*}$ and $[\mathbf{z}^k]_{\mathcal{S}^*}$ are the sub-vector of the $\left[\begin{bmatrix} \mathbf{x}^{k+1} \\ \mathbf{y}^k \\ \mathbf{z}^k \end{bmatrix}\right]_{\mathcal{S}^*}$ with corresponding coordinates of $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$, and the estimation $\gamma' = \left[(R_z + 1)/\rho + (R_x + 1)\|\mathbf{A}_1^T\| \right] \theta_{S^*}$.

Claim: $\mathbf{p}^* = [\mathbf{z}^k]_{\mathcal{S}^*} - \rho\mathbf{A}_2[\mathbf{y}^k]_{\mathcal{S}^*}$ belongs to the optimal solution set G^*, that is $G(\mathbf{p}^*) = 0$ (defined in Lemma 1).

Since $([\mathbf{x}^{k+1}]_{\mathcal{S}^*}, [\mathbf{y}^k]_{\mathcal{S}^*}, [\mathbf{z}^k]_{\mathcal{S}^*})$ are the optimal primal and dual solutions of LPs (7) and (8), they satisfy the following conditions.

1. Primal feasibility: $\mathbf{A}_1[\mathbf{x}^{k+1}]_{\mathcal{S}^*} + \mathbf{A}_2[\mathbf{y}^k]_{\mathcal{S}^*} = \overline{\mathbf{b}}$;
2. Dual feasibility: $\mathbf{c} + A_1^T[\mathbf{z}^k]_{\mathcal{S}^*} = \mathbf{0}$;
3. Complementary slackness: $\langle -\mathbf{A}_2^T[\mathbf{z}^k]_{\mathcal{S}^*}, [\mathbf{y}^k]_{\mathcal{S}^*}\rangle = \mathbf{0}$.

Then, we have

$$\begin{aligned}
G(\mathbf{p}^*) &= \mathbf{prox}_{\rho\overline{g}}(\mathbf{p}^*) - \mathbf{prox}_{\rho\overline{f}}(2\mathbf{prox}_{\rho\overline{g}}(\mathbf{p}^*) - \mathbf{p}^*) \\
&\overset{(a)}{=} [\mathbf{z}^k]_{\mathcal{S}^*} - \mathbf{prox}_{\rho\overline{f}}(2[\mathbf{z}^k]_{\mathcal{S}^*} - \mathbf{p}^*) \\
&\overset{(b)}{=} [\mathbf{z}^k]_{\mathcal{S}^*} - \mathbf{prox}_{\rho\overline{f}}([\mathbf{z}^k]_{\mathcal{S}^*} + \rho\mathbf{A}_2[\mathbf{y}^k]_{\mathcal{S}^*}) \\
&\overset{(c)}{=} \mathbf{0}.
\end{aligned}$$

The above, (a) is based on the following argument

$$\begin{aligned}
[\mathbf{z}^k]_{\mathcal{S}^*} = \mathbf{prox}_{\rho\overline{g}}(\mathbf{p}^*) &\overset{(d)}{\Leftarrow} \mathbf{0} \in -\mathbf{A}_2\partial g^*(-\mathbf{A}_2^T[\mathbf{z}^k]_{\mathcal{S}^*}) + ([\mathbf{z}^k]_{\mathcal{S}^*} - \mathbf{p}^*)/\rho \\
&\overset{(e)}{\Leftarrow} \mathbf{A}_2[\mathbf{y}^k]_{\mathcal{S}^*} \in \mathbf{A}_2\partial g^*(-\mathbf{A}_2^T[\mathbf{z}^k]_{\mathcal{S}^*}) \\
&\overset{(f)}{\Leftarrow} -\mathbf{A}_2^T[\mathbf{z}^k]_{\mathcal{S}^*} \in \partial g([\mathbf{y}^k]_{\mathcal{S}^*}) \\
&\overset{(g)}{\Leftarrow} \langle -\mathbf{A}_2^T[\mathbf{z}^k]_{\mathcal{S}^*}, [\mathbf{y}^k]_{\mathcal{S}^*}\rangle \geq \langle -\mathbf{A}_2^T[\mathbf{z}^k]_{\mathcal{S}^*}, \mathbf{y}\rangle, \forall y_i \geq 0, i \in [n_b].
\end{aligned}$$

The above, (d) is based on the first-order optimality condition of the proximal operator, (e) utilizes the definition of $\mathbf{p}^*$, (f) is based on full column rank property of matrix $\mathbf{A}_2$, (g) is based on the definition of the subgradients of the indicator function $g(\mathbf{y})$. The last inequality always holds because the left-hand-side is equal to 0 by complementary slackness (condition 3), and the right-hand-side is negative by definition of $\mathcal{S}^*$.

The step (b) utilizes the definition of $\mathbf{p}^*$. The step (c) is based on the following argument

$$\begin{aligned}
[\mathbf{z}^k]_{\mathcal{S}^*} = \mathbf{prox}_{\rho\overline{f}}([\mathbf{z}^k]_{\mathcal{S}^*} + \rho\mathbf{A}_2[\mathbf{y}^k]_{\mathcal{S}^*}) &\overset{(h)}{\Leftarrow} \mathbf{0} \in \overline{\mathbf{b}} - \mathbf{A}_1\partial f^*(-\mathbf{A}_1^T[\mathbf{z}^k]_{\mathcal{S}^*}) + \\
&\qquad ([\mathbf{z}^k]_{\mathcal{S}^*} - [\mathbf{z}^k]_{\mathcal{S}^*} - \rho\mathbf{A}_2[\mathbf{y}^k]_{\mathcal{S}^*})/\rho \\
&\overset{(i)}{\Leftarrow} -\mathbf{A}_1[\mathbf{x}^{k+1}]_{\mathcal{S}^*} \in -\mathbf{A}_1\partial f^*(-\mathbf{A}_1^T[\mathbf{z}^k]_{\mathcal{S}^*}) \\
&\overset{(j)}{\Leftarrow} -\mathbf{A}_1^T[\mathbf{z}^k]_{\mathcal{S}^*} \in \partial f([\mathbf{x}^{k+1}]_{\mathcal{S}^*}).
\end{aligned}$$

The above, (h) is based on the first-order optimality condition of the proximal operator, (i) utilizes the primal feasibility condition $\mathbf{A}_1[\mathbf{x}^{k+1}]_{\mathcal{S}^*} + \mathbf{A}_2[\mathbf{y}^k]_{\mathcal{S}^*} = \overline{\mathbf{b}}$, (j) is based on the similar argument of (f). The last equality always holds by dual feasibility condition $\mathbf{c} + \mathbf{A}_1^T[\mathbf{z}^k]_{\mathcal{S}^*} = \mathbf{0}$ and fact that $\partial f(\mathbf{x}) = \{\mathbf{c}\}$. Thus the claim follows.

Then, we have

$$\begin{aligned}
\|\mathbf{p}^k - [\mathbf{p}^k]_{G^*}\| &\overset{(a)}{\leq} \|\mathbf{p}^k - \mathbf{p}^*\| \\
&= \|\mathbf{z}^k - [\mathbf{z}^k]_{\mathcal{S}^*} - \rho\mathbf{A}_2(\mathbf{y}^k - [\mathbf{y}^k]_{\mathcal{S}^*})\| \\
&\overset{(b)}{\leq} \|\mathbf{z}^k - [\mathbf{z}^k]_{\mathcal{S}^*}\| + \rho\|\mathbf{y}^k - [\mathbf{y}^k]_{\mathcal{S}^*}\| \\
&\overset{(c)}{\leq} (1+\rho)\gamma'\|\mathbf{p}^k - \mathbf{p}^{k+1}\|. \qquad (76)
\end{aligned}$$

The above, (a) is based on definition of projection operator $[\cdot]_{G^*}$ and the claim that $\mathbf{p}^*$ belongs to G^*, (b) utilizes the triangle inequality and the definition of matrix $\mathbf{A}_2$, (c) utilizes results of (75). Therefore, the lemma follows.

E Proof of Theorem 1

We first prove the following convergence result each subproblem is exactly solved ($\epsilon_k = 0$).

Lemma 7. (Linear convergence of Algorithm 1 with exact subproblem solver) *Denote $\mathbf{z}^k$ as the dual iterates produced by Algorithm 1. In each iteration k, if the accuracy $\epsilon_k = 0$ and $k \geq 2\gamma^2 \log(D_0/\epsilon)$, then there exists an optimal dual solution $\mathbf{z}^*$ such that $\|\mathbf{z}^{k+1} - \mathbf{z}^*\| \leq \epsilon$, where $D_0 = \|\mathbf{p}^0 - [\mathbf{p}^0]_{G^*}\|$.*

Proof. We first show the accuracy of $\|\mathbf{z}^k - \mathbf{z}^*\|$. Combining the results of Lemma 1 and Lemma 5, we have

$$\|\mathbf{p}^{k+1} - [\mathbf{p}^{k+1}]_{G^*}\|^2 \leq \left(1 - \frac{1}{\gamma^2}\right) \|\mathbf{p}^k - [\mathbf{p}^k]_{G^*}\|^2. \tag{77}$$

Further,

$$\|\mathbf{p}^{k+1} - [\mathbf{p}^{k+1}]_{G^*}\| \leq \sqrt{1 - \frac{1}{\gamma^2}} \cdot \|\mathbf{p}^k - [\mathbf{p}^k]_{G^*}\|. \tag{78}$$

Then, telescoping (78), we have

$$\|\mathbf{p}^k - [\mathbf{p}^k]_{G^*}\| \leq \left(1 - \frac{1}{\gamma^2}\right)^{\frac{k}{2}} \cdot \|\mathbf{p}^0 - [\mathbf{p}^0]_{G^*}\|. \tag{79}$$

Thus, let the number of iterations k satisfies

$$k \geq 2\gamma^2 \log\left(\frac{D_0}{\epsilon}\right), \tag{80}$$

where the constant D_0 is the distance between the initial point and optimal solution set, defined as $D_0 = \|\mathbf{p}^0 - [\mathbf{p}^0]_{G^*}\|$. Then we have

$$\begin{aligned}
\|\mathbf{p}^k - [\mathbf{p}^k]_{G^*}\| &\leq \left(1 - \frac{1}{\gamma^2}\right)^{\gamma^2 \log\left(\frac{D_0}{\epsilon}\right)} \cdot \|\mathbf{p}^0 - [\mathbf{p}^0]_{G^*}\| \\
&= \exp\left\{\gamma^2 \log\left(\frac{\epsilon}{D_0}\right) \log\left(\frac{\gamma^2}{\gamma^2 - 1}\right) + \log(D_0)\right\} \\
&\overset{(a)}{\leq} \exp\left\{\log\left(\frac{\epsilon}{D_0}\right) + \log(D_0)\right\} = \epsilon,
\end{aligned} \tag{81}$$

The above, (a) is based on the inequality: $\gamma^2 \log\left(\frac{\gamma^2}{\gamma^2-1}\right) \geq 1$,when $\gamma > 1$. Here the fact that $\gamma > 1$ derives from the result of Lemma 1.

We then show that the distance between the dual iterates $\mathbf{z}^k$ and the optimal solution set is also bounded by ϵ, when k satisfies the condition (80). Based on the first-order optimality condition, we have

$$\begin{aligned}
\mathbf{z}^k = \mathbf{prox}_{\rho\overline{g}}(\mathbf{p}^k) &\overset{(a)}{\Longleftrightarrow} -\mathbf{A}_2\mathbf{y}^k \in -\mathbf{A}_2 \partial g^*(-\mathbf{A}_2\mathbf{z}^k) \\
&\overset{(b)}{\Longleftrightarrow} \mathbf{y}^k \in \partial g^*(-\mathbf{A}_2\mathbf{z}^k).
\end{aligned} \tag{82}$$

The above, (a) is based on the definition of $\mathbf{p}^k = \mathbf{z}^k - \rho\mathbf{A}_2\mathbf{y}^k$, (b) utilizes the fact that matrix $\mathbf{A}_2$ has full column rank. The last equality is indeed the second step of the ADMM, as indicated in (45). Based on the definition of optimal solution set G^*, there exists an optimal primal and dual solution of LP $\mathbf{y}^*$ and $\mathbf{z}^*$ such that $[\mathbf{p}^k]_{G^*} = \mathbf{z}^* - \rho\mathbf{A}_2\mathbf{y}^*$. Similarly, we have that $\mathbf{z}^* = \mathbf{prox}_{\rho\overline{g}}([\mathbf{p}^k]_{G^*})$. According to the non-expansiveness of the proximal operator, we have there exists optimal multiplier $\mathbf{z}^*$ of ADMM such that

$$\|\mathbf{z}^k - \mathbf{z}^*\| = \|\mathbf{prox}_{\rho\overline{g}}(\mathbf{p}^k) - \mathbf{prox}_{\rho\overline{g}}([\mathbf{p}^k]_{G^*})\| \leq \|\mathbf{p}^k - [\mathbf{p}^k]_{G^*}\|. \tag{83}$$

Hence, if $k \geq 2\gamma^2 \log\left(\frac{D_0}{\epsilon}\right)$, then $\|\mathbf{z}^k - \mathbf{z}^*\| \leq \epsilon$.

Secondly, we show the accuracy of $\|\mathbf{x}^k - \mathbf{x}^*\|$ and $\|\mathbf{y}^k - \mathbf{y}^*\|$. From the result of (75), we have

$$\|\mathbf{x}^{k+1} - [\mathbf{x}^{k+1}]_{\mathcal{S}^*}\| \leq \gamma' \|\mathbf{p}^{k+1} - \mathbf{p}^k\| \text{ and } \|\mathbf{y}^k - [\mathbf{y}^k]_{\mathcal{S}^*}\| \leq \gamma' \|\mathbf{p}^{k+1} - \mathbf{p}^k\|. \tag{84}$$

According to Lemma 1, we have

$$\|\mathbf{p}^{k+1} - \mathbf{p}^k\| \le \|\mathbf{p}^k - [\mathbf{p}^k]_{G^*}\|.$$

Thus, to guarantee that both $\|\mathbf{x}^{k+1} - [\mathbf{x}^{k+1}]_{\mathcal{S}^*}\| \le \epsilon$ and $\|\mathbf{y}^k - [\mathbf{y}^k]_{\mathcal{S}^*}\| \le \epsilon$, we requires

$$k \ge 2\gamma^2 \log\left(\frac{D_0\gamma}{(1+\rho)\epsilon}\right).$$

The accuracy of the duality gap is related to the accuracy of both the primal and dual iterates by

$$\begin{aligned}\|\mathbf{c}^T\mathbf{x}^{k+1} + \mathbf{b}^T z_x^k\| &\overset{(a)}{=} \|\mathbf{c}^T\mathbf{x}^{k+1} - \mathbf{c}^T\mathbf{x}^* - \mathbf{b}^T\mathbf{z}_x^* + \mathbf{b}^T\mathbf{z}_x^k\| \le \|\mathbf{c}^T\mathbf{x}^{k+1} - \mathbf{c}^T\mathbf{x}^*\| + \|\mathbf{b}^T\mathbf{z}^* - \mathbf{b}^T\mathbf{z}_x^k\| \\ &\overset{(c)}{\le} \|\mathbf{c}\| \cdot \|\mathbf{x}^{k+1} - \mathbf{x}^*\| + \|\mathbf{b}\| \cdot \|\mathbf{z}_x^k - z_{\mathbf{x}}^*\|.\end{aligned}$$

The above, (a) is based on the strong duality theorem that $\mathbf{c}^T\mathbf{x}^* + \mathbf{b}^T\mathbf{z}_x^* = 0$, (c) follows from the Cauchy-Schwarz inequality. Thus, the lemma follows. □

We next generalize the convergence result to the inexact subproblem solver ($\epsilon_k > 0$). Let the iterates under the inexact update be denoted by $\overline{\mathbf{x}}^k, \overline{\mathbf{y}}^k, \overline{\mathbf{z}}^k, \overline{\mathbf{p}}^k = \overline{\mathbf{z}}^k - \rho\mathbf{A}_2\overline{\mathbf{y}}^k$ and corresponding AL function

$$\overline{F}_k(\mathbf{x}) = \mathbf{c}^T\mathbf{x} + (\overline{\mathbf{z}}^k)^T\mathbf{A}_1\mathbf{x} + \frac{\rho}{2}\|\mathbf{A}_1\mathbf{x} + \mathbf{A}_2\overline{\mathbf{y}}^k - \overline{\mathbf{b}}\|^2. \tag{85}$$

We first construct the relation between the primal accuracy ϵ_k and the accuracy of the dual iterates in the subproblem (11) by a standard primal dual argument; then we connect the accuracy of such dual iterates with the $\overline{\mathbf{p}}^k$.

Lemma 8. (Relation between primal and dual accuracy) *Let* $\overline{\mathbf{w}}^{k+1} = \overline{\mathbf{z}}^k + \rho(\mathbf{A}_1\overline{\mathbf{x}}^{k+1} + A_2\overline{\mathbf{y}}^k - \overline{\mathbf{b}})$. *If the* $\overline{F}_k(\overline{\mathbf{x}}^{k+1}) - \min_x \overline{F}_k(\mathbf{x}) \le \epsilon_k$*, then the dual iterates satisfy.*

$$\|\overline{\mathbf{w}}^{k+1} - \mathbf{prox}_{\rho\overline{f}}(\overline{\mathbf{z}}^k + \rho\mathbf{A}_2\overline{\mathbf{y}}^k)\| \le \sqrt{2\rho\epsilon_k}.$$

Proof. The quadratic function with inexact update in the step 1 of Algorithm 1 is

$$\overline{F}_k(\mathbf{x}) \triangleq f(\mathbf{x}) + (\overline{\mathbf{z}}^k)^T(\mathbf{A}_1\mathbf{x} - \overline{\mathbf{b}}) + \frac{\rho}{2}\|\mathbf{A}_1\mathbf{x} + \mathbf{A}_2\overline{\mathbf{y}}^k - \overline{\mathbf{b}}\|^2. \tag{86}$$

The proof of this lemma is based on the following two claims.

Claim 1: The following two problems are primal and dual optimization problems.

$$\text{primal: } \min_{\mathbf{x}} \overline{F}_k(\mathbf{x}) - \frac{\rho}{2}\|\mathbf{A}_2\overline{\mathbf{y}}^k\|^2 \iff \text{dual: } \max_{\mathbf{w}} -\overline{\mathbf{b}}^T\mathbf{w} - f^*(-\mathbf{A}_1^T\mathbf{w}) - \frac{1}{2\rho}\|\mathbf{w} - \overline{\mathbf{z}}^k - \rho\mathbf{A}_2\overline{\mathbf{y}}^k\|^2, \tag{87}$$

Let $\mathbf{A}_1\mathbf{x} + \mathbf{A}_2\overline{\mathbf{y}}^k - \overline{\mathbf{b}} = \overline{\mathbf{t}}$, then the primal problem is equivalent to

$$\begin{aligned}\min_{x,\overline{\mathbf{t}}} \quad & f(\mathbf{x}) + (\overline{\mathbf{z}}^k)^T(\overline{\mathbf{t}} - \mathbf{A}_2\overline{\mathbf{y}}^k) + \frac{\rho}{2}\|\overline{\mathbf{t}}\|^2 - \frac{\rho}{2}\|\mathbf{A}_2\overline{\mathbf{y}}^k\|^2 \\ s.t. \quad & \mathbf{A}_1\mathbf{x} + \mathbf{A}_2\overline{\mathbf{y}}^k - \overline{\mathbf{b}} = \overline{\mathbf{t}}.\end{aligned} \tag{88}$$

Then, the dual optimization problem can be written as minimizing the Lagrangian function w.r.t x and $\overline{\mathbf{t}}$.

$$\begin{aligned}&\min_{x,\overline{\mathbf{t}}} f(\mathbf{x}) + (\overline{\mathbf{z}}^k)^T(\overline{\mathbf{t}} - \mathbf{A}_2\overline{\mathbf{y}}^k) + \frac{\rho}{2}\|\overline{\mathbf{t}}\|^2 + w^T(\mathbf{A}_1\mathbf{x} + \mathbf{A}_2\overline{\mathbf{y}}^k - \overline{\mathbf{b}} - \overline{\mathbf{t}}) - \frac{\rho}{2}\|\mathbf{A}_2\overline{\mathbf{y}}^k\|^2 \\ &\overset{(a)}{=} \min_x \left[f(\mathbf{x}) + \langle\mathbf{A}_1^T\mathbf{w}, \mathbf{x}\rangle\right] + \min_{\overline{\mathbf{t}}}\left[(\overline{\mathbf{z}}^k)^T(\overline{\mathbf{t}} - \mathbf{A}_2\overline{\mathbf{y}}^k) + \frac{\rho}{2}\|\overline{\mathbf{t}}\|^2 + w^T(\mathbf{A}_2\overline{\mathbf{y}}^k - \overline{\mathbf{b}} - \overline{\mathbf{t}}) - \frac{\rho}{2}\|\mathbf{A}_2\overline{\mathbf{y}}^k\|^2\right] \\ &\overset{(b)}{=} -\overline{\mathbf{b}}^T\mathbf{w} - f^*(-\mathbf{A}_1^T\mathbf{w}) + \min_{\overline{\mathbf{t}}}\left[(\overline{\mathbf{z}}^k)^T(\overline{\mathbf{t}} - \mathbf{A}_2\overline{\mathbf{y}}^k) + \frac{\rho}{2}\|\overline{\mathbf{t}}\|^2 + w^T(\mathbf{A}_2\overline{\mathbf{y}}^k - \overline{\mathbf{t}}) - \frac{\rho}{2}\|\mathbf{A}_2\overline{\mathbf{y}}^k\|^2\right] \\ &\overset{(c)}{=} -\overline{\mathbf{b}}^T\mathbf{w} - f^*(-\mathbf{A}_1^T\mathbf{w}) - \frac{1}{2\rho}\|\mathbf{w} - \overline{\mathbf{z}}^k - \rho\mathbf{A}_2\overline{\mathbf{y}}^k\|^2.\end{aligned} \tag{89}$$

The above, (a) is based on the separability between variable x and $\overline{\mathbf{t}}$ in the above optimization problem, (b) utilizes the definition of the convex conjugate function, (c) is obtained by setting $\overline{\mathbf{t}} = (\mathbf{w} - \overline{\mathbf{z}}^k)/\rho$.

Claim 2: The following two problems are primal and dual optimization problems.

$$\text{primal:} \min_{\mathbf{x}} \tilde{F}_k(\mathbf{x}) \triangleq f(\mathbf{x}) + \mathbf{s}^T(\mathbf{A}_1\mathbf{x} - \overline{\mathbf{b}}) + \frac{\rho}{2}\|\mathbf{A}_1\mathbf{x} + \mathbf{A}_2\overline{\mathbf{y}}^k - \overline{\mathbf{b}}\|^2 - \frac{\rho}{2}\|\mathbf{A}_2\overline{\mathbf{y}}^k\|^2$$
$$\Longleftrightarrow \quad \text{dual:} \max_{w} -\overline{\mathbf{b}}^T\mathbf{w} - f^*(-\mathbf{A}_1^T\mathbf{w}) - \frac{1}{2\rho}\|\mathbf{w} - \mathbf{s} - \rho\mathbf{A}_2\overline{\mathbf{y}}^k\|^2 \tag{90}$$

Let $\mathbf{A}_1\mathbf{x} + \mathbf{A}_2\overline{\mathbf{y}}^k - \overline{\mathbf{b}} = \overline{\mathbf{t}}$, then the primal problem is equivalent to

$$\begin{aligned} \min_{x,\overline{\mathbf{t}}} \quad & f(\mathbf{x}) + s^T(\overline{\mathbf{t}} - \mathbf{A}_2\overline{\mathbf{y}}^k) + \frac{\rho}{2}\|\overline{\mathbf{t}}\|^2 - \frac{\rho}{2}\|\mathbf{A}_2\overline{\mathbf{y}}^k\|^2 \\ s.t. \quad & \mathbf{A}_1\mathbf{x} + \mathbf{A}_2\overline{\mathbf{y}}^k - \overline{\mathbf{b}} = \overline{\mathbf{t}}. \end{aligned} \tag{91}$$

Then, the dual optimization problem can be written as minimizing the Lagrangian function w.r.t x and $\overline{\mathbf{t}}$.

$$\begin{aligned} &\min_{x,\overline{\mathbf{t}}} f(\mathbf{x}) + \mathbf{s}^T(\overline{\mathbf{t}} - \mathbf{A}_2\overline{\mathbf{y}}^k) + \frac{\rho}{2}\|\overline{\mathbf{t}}\|^2 + \mathbf{w}^T(\mathbf{A}_1\mathbf{x} + \mathbf{A}_2\overline{\mathbf{y}}^k - \overline{\mathbf{b}} - \overline{\mathbf{t}}) - \frac{\rho}{2}\|\mathbf{A}_2\overline{\mathbf{y}}^k\|^2 \\ &\overset{(a)}{=} \min_{x}\left[f(\mathbf{x}) + \langle \mathbf{A}_1^T\mathbf{w}, \mathbf{x}\rangle\right] + \min_{\overline{\mathbf{t}}}\left[\mathbf{s}^T(\overline{\mathbf{t}} - \mathbf{A}_2\overline{\mathbf{y}}^k) + \frac{\rho}{2}\|\overline{\mathbf{t}}\|^2 + \mathbf{w}^T(\mathbf{A}_2\overline{\mathbf{y}}^k - \overline{\mathbf{b}} - \overline{\mathbf{t}})\right] - \frac{\rho}{2}\|\mathbf{A}_2\overline{\mathbf{y}}^k\|^2 \\ &\overset{(b)}{=} -\overline{\mathbf{b}}^T\mathbf{w} - f^*(-\mathbf{A}_1^T\mathbf{w}) + \min_{\overline{\mathbf{t}}}\left[\mathbf{s}^T(\overline{\mathbf{t}} - \mathbf{A}_2\overline{\mathbf{y}}^k) + \frac{\rho}{2}\|\overline{\mathbf{t}}\|^2 + \mathbf{w}^T(\mathbf{A}_2\overline{\mathbf{y}}^k - \overline{\mathbf{t}})\right] - \frac{\rho}{2}\|\mathbf{A}_2\overline{\mathbf{y}}^k\|^2 \\ &\overset{(c)}{=} -\overline{\mathbf{b}}^T\mathbf{w} - f^*(-\mathbf{A}_1^T\mathbf{w}) - \frac{1}{2\rho}\|\mathbf{w} - \mathbf{s} - \rho\mathbf{A}_2\overline{\mathbf{y}}^k\|^2. \end{aligned} \tag{92}$$

The above, (a) is based on the separability between variable x and $\overline{\mathbf{t}}$ in the above optimization problem, (b) utilizes the definition of the convex conjugate function, (c) utilizes the first-order optimality condition of the second optimization problem such that $\overline{\mathbf{t}} = (\mathbf{w} - \mathbf{s})/\rho$. Define the following iterates,

$$\tilde{\mathbf{x}}^{k+1} = \arg\min_{x} \overline{F}_k(\mathbf{x}) - \frac{\rho}{2}\|\mathbf{A}_2\mathbf{y}^k\|^2 \text{ and } \tilde{\mathbf{w}}^{k+1} = \mathbf{prox}_{\rho\overline{f}}(\overline{\mathbf{z}}^k + \rho\mathbf{A}_2\overline{\mathbf{y}}^k). \tag{93}$$

Then the pair of sequences $\tilde{\mathbf{x}}^{k+1}$ and $\tilde{w}^{k+1}$ are the optimal primal and dual solutions of (87). According to the strong duality theorem of the convex optimization, we have

$$\overline{F}_k(\tilde{\mathbf{x}}^{k+1}) - \frac{\rho}{2}\|\mathbf{A}_2\overline{\mathbf{y}}^k\|^2 = -\overline{\mathbf{b}}^T\tilde{\mathbf{w}}^{k+1} - f^*(-\mathbf{A}_1^T\tilde{\mathbf{w}}^{k+1}) - \frac{1}{2\rho}\|\tilde{\mathbf{w}}^{k+1} - \overline{\mathbf{z}}^k - \rho\mathbf{A}_2\overline{\mathbf{y}}^k\|^2. \tag{94}$$

The pair of sequences $\overline{\mathbf{x}}^{k+1}$ and $\tilde{\mathbf{w}}^{k+1}$ is the primal and dual feasible solution of problem (90). According to the weak duality theorem of the convex optimization, we have

$$\tilde{F}_k(\overline{\mathbf{x}}^{k+1}) \geq -\overline{\mathbf{b}}^T\tilde{\mathbf{w}}^{k+1} - f^*(-\mathbf{A}_1^T\tilde{\mathbf{w}}^{k+1}) - \frac{1}{2\rho}\|\tilde{\mathbf{w}}^{k+1} - \mathbf{s} - \rho\mathbf{A}_2\overline{\mathbf{y}}^k\|^2$$

Further, based on the definition of $\overline{F}_k(\mathbf{x})$ and $\tilde{F}^k(\mathbf{x})$, we have

$$\begin{aligned} \overline{F}_k(\overline{\mathbf{x}}^{k+1}) - \frac{\rho}{2}\|\mathbf{A}_2\overline{\mathbf{y}}^k\|^2 \geq (\overline{\mathbf{z}}^k - \mathbf{s})^T(\mathbf{A}_1\overline{\mathbf{x}}^{k+1} - \overline{\mathbf{b}}) - \overline{\mathbf{b}}^T\tilde{\mathbf{w}}^{k+1} - f^*(-\mathbf{A}_1^T\tilde{\mathbf{w}}^{k+1}) - \\ \frac{1}{2\rho}\|\tilde{\mathbf{w}}^{k+1} - \mathbf{s} - \rho\mathbf{A}_2\overline{\mathbf{y}}^k\|^2 \end{aligned} \tag{95}$$

Combining the results of (94) and (95), we have

$$\begin{aligned} \overline{F}_k(\overline{\mathbf{x}}^{k+1}) - \overline{F}_k(\tilde{\mathbf{x}}^{k+1}) \geq & \frac{1}{2\rho}\|\tilde{\mathbf{w}}^{k+1} - \overline{\mathbf{z}}^k - \rho\mathbf{A}_2\overline{\mathbf{y}}^k\|^2 - \frac{1}{2\rho}\|\tilde{\mathbf{w}}^{k+1} - \mathbf{s} - \rho\mathbf{A}_2\overline{\mathbf{y}}^k\|^2 + \\ & (\overline{\mathbf{z}}^k - \mathbf{s})^T(\mathbf{A}_1\overline{\mathbf{x}}^{k+1} - \overline{\mathbf{b}}) \\ \overset{(a)}{\geq} & \frac{1}{2\rho}\|\overline{\mathbf{w}}^{k+1} - \mathbf{prox}_{\rho\overline{f}}(\overline{\mathbf{z}}^k + \rho\mathbf{A}_2\overline{\mathbf{y}}^k)\|^2. \end{aligned} \tag{96}$$

The above, (a) utilizes the definition of $\overline{\mathbf{w}}^{k+1} = \overline{\mathbf{z}}^k + \rho(\mathbf{A}_1\overline{\mathbf{x}}^{k+1} + \mathbf{A}_2\overline{\mathbf{y}}^k - \overline{\mathbf{b}})$ to substitute $\mathbf{A}_1\overline{\mathbf{x}}^{k+1} - \overline{\mathbf{b}}$, and maximizing the righthand side w.r.t s. Thus, the lemma follows. $\square$

Utilizing the requirement that $\overline{F}_k(\overline{\mathbf{x}}^{k+1}) - \min_{\mathbf{x}} \overline{F}_k(\mathbf{x}) \le \epsilon_k$ in the Algorithm 1 and the result in Lemma 8, we have the following inexact version of proximal operations.

$$\|\overline{\mathbf{w}}^{k+1} - \mathbf{prox}_{\rho\overline{f}}(2\overline{\mathbf{z}}^k - \overline{\mathbf{p}}^k)\| \le \sqrt{2\rho\epsilon_k}, \tag{97}$$

$$\overline{\mathbf{z}}^{k+1} = \mathbf{prox}_{\rho\overline{g}}(\overline{\mathbf{w}}^{k+1} - \overline{\mathbf{z}}^k + \overline{\mathbf{p}}^k), \tag{98}$$

$$\overline{\mathbf{p}}^{k+1} = \overline{\mathbf{p}}^k + \overline{\mathbf{w}}^{k+1} - \overline{\mathbf{z}}^k. \tag{99}$$

The (97)-(99) can be simplified as

$$\|\overline{\mathbf{p}}^{k+1} - \overline{\mathbf{p}}^k + \overline{\mathbf{z}}^k - \mathbf{prox}_{\rho\overline{f}}(2\overline{\mathbf{z}}^k - \overline{\mathbf{p}}^k)\| \le \sqrt{2\rho\epsilon_k}, \tag{100}$$

$$\overline{\mathbf{z}}^{k+1} = \mathbf{prox}_{\rho\overline{g}}(\overline{\mathbf{p}}^{k+1}). \tag{101}$$

Start at $\mathbf{p}^0$ and renumber the iterates of (100) and (101), we have

$$\begin{aligned}
&\overline{\mathbf{z}}^{k+1} = \mathbf{prox}_{\rho\overline{g}}(\overline{\mathbf{p}}^k),\\
&\|\overline{\mathbf{p}}^{k+1} - \overline{\mathbf{p}}^k + \overline{\mathbf{z}}^{k+1} - \mathbf{prox}_{\rho\overline{f}}(2\overline{\mathbf{z}}^{k+1} - \overline{\mathbf{p}}^k)\| \le \sqrt{2\rho\epsilon_k},
\end{aligned}$$

which can be further simplified as

$$\begin{aligned}
&\|\overline{\mathbf{p}}^{k+1} - [\overline{\mathbf{p}}^k - \mathbf{prox}_{\rho\overline{g}}(\overline{\mathbf{p}}^k) + \mathbf{prox}_{\rho\overline{f}}(2\mathbf{prox}_{\rho\overline{g}}(\overline{\mathbf{p}}^k) - \overline{\mathbf{p}}^k)]\| \le \sqrt{2\rho\epsilon_k}\\
\Longleftrightarrow\ &\|\overline{\mathbf{p}}^{k+1} - T(\overline{\mathbf{p}}^k)\| \le \sqrt{2\rho\epsilon_k}.
\end{aligned} \tag{102}$$

Then, we have

$$\begin{aligned}
\|\overline{\mathbf{p}}^{k+1} - \mathbf{p}^{k+1}\| &\le \|\overline{\mathbf{p}}^{k+1} - T(\overline{\mathbf{p}}^k)\| + \|T(\overline{\mathbf{p}}^k) - \mathbf{p}^{k+1}\|\\
&\overset{(a)}{=} \|\overline{\mathbf{p}}^{k+1} - T(\overline{\mathbf{p}}^k)\| + \|T(\overline{\mathbf{p}}^k) - T(\mathbf{p}^k)\|\\
&\overset{(b)}{\le} \|\overline{\mathbf{p}}^{k+1} - T(\overline{\mathbf{p}}^k)\| + \|\overline{\mathbf{p}}^k - \mathbf{p}^k\|\\
&\overset{(c)}{\le} \sqrt{2\rho\epsilon_k} + \|\overline{\mathbf{p}}^k - \mathbf{p}^k\|\\
&\overset{(d)}{\le} \sum_{i=0}^{k} \sqrt{2\rho\epsilon_i}.
\end{aligned} \tag{103}$$

The above, (a) is based on the definition that $T(\mathbf{p}^k) = \mathbf{p}^{k+1}$, (b) utilizes the non-expansiveness of the operator T, which can be derived from the Claim 4 in the proof of Lemma 1, (c) is based on the result in (102), (d) derives from telescoping the above inequality. Then, we can obtain

$$\|\overline{\mathbf{p}}^k - [\overline{\mathbf{p}}^k]_{G^*}\| \le \|\overline{\mathbf{p}}^k - [\mathbf{p}^k]_{G^*}\| \le \|\overline{\mathbf{p}}^k - \mathbf{p}^k\| + \|\mathbf{p}^k - [\mathbf{p}^k]_{G^*}\| \le \sum_{i=0}^{k-1} \sqrt{2\rho\epsilon_i} + \|\mathbf{p}^k - [\mathbf{p}^k]_{G^*}\|. \tag{104}$$

Let the solving accuracy ϵ_k of each iteration k satisfies

$$\epsilon_k = \frac{\epsilon^2}{8\rho K^2}, K = 2\gamma^2 \log\left(\frac{2D_0}{\epsilon}\right), \forall k. \tag{105}$$

Then, we have

$$\|\overline{\mathbf{p}}^k - [\overline{\mathbf{p}}^k]_{G^*}\| \le \frac{\epsilon}{2} + \|\mathbf{p}^k - [\mathbf{p}^k]_{G^*}\|. \tag{106}$$

Combining the result (81) in Lemma 7, we finally arrive

$$\|\overline{\mathbf{p}}^k - [\overline{\mathbf{p}}^k]_{G^*}\| \le \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon, \text{ if } k \ge 2\gamma^2 \log\left(\frac{2D_0}{\epsilon}\right). \tag{107}$$

Utilizing a similar argument in the proof of Lemma 7, we can obtain the accuracy of both $\mathbf{x}^*, \mathbf{y}^*, \mathbf{z}^*$ and the duality gap. Therefore, the Theorem 1 follows.

F Proof of Lemma 6

Based on the analysis in [22], to obtain an ϵ accurate solution, it requires running ACDM by $O(1/\tau \log(1/\epsilon))$ iterations, where

$$\tau = O\left(\frac{\sqrt{\rho}}{\sum\limits_{i=1}^{n}\sqrt{L_i}}\right). \tag{108}$$

According to the form of subproblem (11), the component-wise Lipschitz constant L_i is equal to

$$L_i = \|\mathbf{A}_{1,i}\|^2 = \|\mathbf{A}_i\|^2 + 1. \tag{109}$$

Further,

$$\sum_{i=1}^{n}\sqrt{L_i} = \sum_{i=1}^{n}\sqrt{\|\mathbf{A}_i\|^2 + 1} = O(\|\mathbf{A}\|_{2,1}), \tag{110}$$

where $\|\mathbf{A}\|_{p,q} = (\sum_{j=1}^{n}(\sum_{i=1}^{m}|A_{ij}|^p)^{q/p})^{1/q}$ is the $L_{p,q}$ norm of constraint matrix $\mathbf{A}$. The ρ is defined as the strongly convexity parameter of the function $\overline{F}_k(\mathbf{x})$. Based on the form of Hessian of function $\overline{F}_k(\mathbf{x})$ in (13), we have

$$\rho = \lambda_{min}(\rho(\mathbf{A}^T\mathbf{A} + \mathbf{I})) \geq \lambda_{min}(\rho\mathbf{A}^T\mathbf{A}) + \lambda_{min}(\rho\mathbf{I}) \geq \rho > 0. \tag{111}$$

Therefore, the iteration complexity is obtained by choosing parameter $\beta = 0$ in [22] and utilizing the Theorem 1 in [24] to transform the convergence in expectation to the form of probability.

In each iteration of Algorithm 2, the calculation of coordinate-wise gradient $\nabla_i F_k(\mathbf{u}_t)$ requires a vector product between ith column of matrix $\mathbf{A}$ and $\overline{\mathbf{u}}_t$, and the update of auxillary variables $\overline{\mathbf{u}}, \overline{\mathbf{v}}$ requires subtracting ith column of matrix $\mathbf{A}$. These two steps can be calculated in $O(nnz(\mathbf{A}_i))$ time. Therefore, the complexity of each step of ACDM is $O(nnz(\mathbf{A}_i))$.

Note that

$$\det(\mathbf{M}_t) = [\det(\mathbf{M})]^t = \left(\frac{1-\tau}{1+\eta\rho}\right)^t \tag{112}$$

In analysis in [22], we have

$$\eta = O\left(\frac{1}{\sqrt{\rho}\sum\limits_{i=1}^{n}\sqrt{L_i}}\right), \tag{113}$$

and the total number of iterations of ACDM is $O(1/\tau \log(1/\epsilon))$. Thus we have,

$$\det(\mathbf{M}_t) = \left(\frac{1-\tau}{1+\eta\rho}\right)^{\frac{1}{\tau}} = O\left(\left(1 - \frac{2}{1+\|\mathbf{A}\|_{2,1}}\right)^{\|\mathbf{A}\|_{2,1}\log(1/\epsilon)}\right) = O(\log(1/\epsilon)). \tag{114}$$

Hence, $O(\log(1/\epsilon))$ bits of precision suffice to implement this method.

G Proof of Theorem 2

Based on the result of Lemma 6, we can obtain the iteration complexity to solve each subproblem with a given accuracy ϵ_k and confidence level p. To guarantee that K subproblems are all solved to precision ϵ_k with probability $1-p$, it suffices each of them to hold with probability $1 - p/K$ (by union bound). Combining the above results and the iteration complexity of ACDM, we have the required number of inner iterations in the each outer iteration is

$$O\left(\sum_{i=1}^{n}\|\mathbf{A}_i\|\log\left(\frac{D_0^k K}{\epsilon_k p}\right)\right) = O\left(\sum_{i=1}^{n}\|\mathbf{A}_i\|\log\left(\frac{\rho(D_0^k)^{\frac{1}{3}}\gamma^2}{\epsilon^{\frac{2}{3}}p^{\frac{1}{3}}}\log\left(\frac{2D_0}{\epsilon}\right)\right)\right). \tag{115}$$

Finally, we estimate the worst-case overall complexity of Algorithm 1. In each iteration of the ACDM, the Algorithm 2 samples each coordinate i with probability distribution

$$p_i = \frac{\sqrt{\|\mathbf{A}_i\|^2 + 1}}{\sum_{j=1}^{n} \sqrt{\|\mathbf{A}_j\|^2 + 1}},$$

and corresponding iteration cost is $O(nnz(\mathbf{A}_i))$ (given in Lemma 6). Thus, the complexity of solving each subproblem 1 is

$$O\left(\sum_{i=1}^{n} \|\mathbf{A}_i\| nnz(\mathbf{A}_i) \cdot \log\left(\frac{\rho(D_0^k)^{\frac{1}{3}}\gamma^2}{\epsilon^{\frac{2}{3}} p^{\frac{1}{3}}} \log\left(\frac{\gamma D_0}{\epsilon}\right)\right)\right). \tag{116}$$

Thus, the worst case complexity is

$$\begin{aligned}
&O\left(\gamma^2 \sum_{i=1}^{n} \|\mathbf{A}_i\| nnz(\mathbf{A}_i) \cdot \log(1/\epsilon) \log\left(\frac{\rho(D_0^k)^{\frac{1}{3}}\gamma^2}{\epsilon^{\frac{2}{3}} p^{\frac{1}{3}}} \log\left(\frac{\gamma D_0}{\epsilon}\right)\right)\right) \\
&\overset{(a)}{=} O\left(\gamma^2 \sum_{i=1}^{n} a_{\max} nnz(\mathbf{A}_i) \cdot \log(1/\epsilon) \log\left(\frac{\rho(D_0^k)^{\frac{1}{3}}\gamma^2}{\epsilon^{\frac{2}{3}} p^{\frac{1}{3}}} \log\left(\frac{\gamma D_0}{\epsilon}\right)\right)\right) \\
&\overset{(b)}{=} O(a_m \theta_{S^*}^2 (R_x\|\mathbf{A}\| + R_z)^2 nnz(\mathbf{A}) \log^2(1/\epsilon)).
\end{aligned} \tag{117}$$

The above, (a) utilizes the definition of $a_m = \max_i \|\mathbf{A}_i\|$, (b) is based on the estimation that $\gamma = O(\theta_{S^*}(R_x\|\mathbf{A}\| + R_z))$. Thus, the theorem follows.