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# Spike timing-dependent plasticity as dynamic filter: supplementary material

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In this supplement, we present the calculations that lead to Equation (12) in the main article as well as the formulas that underlie Figure 4.

## 1 The simple case $u = z = 1$

For starters we derive the expression for the simple case of  $u$  and  $z$  being constant and equal to one. The firing rates are given by

$$x_i = C_i + \varepsilon_i \cos(\omega_i t + \varphi_i) \quad (1)$$

for the pre- and postsynaptic neuron,  $i = pre, post$ . To simplify further, both baseline firing rates are equal, as well as the perturbations and modulation frequencies:

$$\begin{aligned} C_{pre} &= C_{post} = x_0 \\ \varepsilon_{pre} &= \varepsilon_{post} = \varepsilon \\ \omega_{pre} &= \omega_{post} = \omega \end{aligned} \quad (2)$$

The only difference between the two firing rates is the phase  $\varphi_i$ . For both traces  $y_i$  the differential equation now reads

$$\dot{y}_i = -\frac{y_i}{\tau_i} + x_i = -\frac{y_i}{\tau_i} + x_0 + \varepsilon \cos(\omega t + \varphi_i) \quad (3)$$

Looking only at the solution for long times, neglecting the transients and using the initial condition  $y_i(t_0) = 0$ , we get

$$y_i(t) = \frac{\tau_i \varepsilon}{1 + \tau_i \omega} \left( \cos(\omega t + \varphi_i) + \tau_i \omega \sin(\omega t + \varphi_i) \right) + \tau_i x_0 \quad (4)$$

For the weight change, we consider  $y_{pre}$  and  $\dot{y}_{post}$ :

$$\begin{aligned} y_{pre} &= \frac{\tau_{pre} \varepsilon}{1 + \tau_{pre}^2 \omega^2} \left( \cos(\omega t + \varphi_{pre}) + \tau_{pre} \omega \sin(\omega t + \varphi_{pre}) \right) + \tau_{pre} x_0 \\ \dot{y}_{post} &= \frac{\tau_{post} \omega \varepsilon}{1 + \tau_{post}^2 \omega^2} \left( \tau_{post} \omega \cos(\omega t + \varphi_{post}) - \sin(\omega t + \varphi_{post}) \right) \end{aligned} \quad (5)$$

The calculations now are straightforward, but rather lengthy. The product of the two functions is

$$\begin{aligned} y_{pre} \dot{y}_{post} &= \frac{\varepsilon^2 \omega \tau_{pre} \tau_{post}}{(1 + \tau_{pre}^2 \omega^2)(1 + \tau_{post}^2 \omega^2)} \left[ \tau_{post} \omega \cos(\omega t + \varphi_{pre}) \cos(\omega t + \varphi_{post}) \right. \\ &\quad - \cos(\omega t + \varphi_{pre}) \sin(\omega t + \varphi_{post}) + \omega^2 \tau_{pre} \tau_{post} \sin(\omega t + \varphi_{pre}) \cos(\omega t + \varphi_{post}) \\ &\quad \left. - \tau_{pre} \omega \sin(\omega t + \varphi_{pre}) \sin(\omega t + \varphi_{post}) \right] \\ &\quad + \frac{x_0 \omega \varepsilon \tau_{pre} \tau_{post}}{1 + \tau_{post}^2 \omega^2} \left( \tau_{post} \omega \cos(\omega t + \varphi_{post}) - \sin(\omega t + \varphi_{post}) \right) \end{aligned} \quad (6)$$

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We now use trigonometric product identities to get rid of the products of sin and cos. Also, we make use of the fact that in an integration over one period an expression like  $\cos(n\omega t + \psi)$  vanishes for integers  $n$ . We calculate  $\Delta W = 1/(T\varepsilon^2) \int_T y_{pre}\dot{y}_{post}dt$ :

$$\frac{1}{T\varepsilon^2} \int_0^T y_{pre}\dot{y}_{post}dt = \frac{1}{2} \frac{\omega\tau_{pre}\tau_{post}}{(1 + \tau_{pre}^2\omega^2)(1 + \tau_{post}^2\omega^2)} \cdot \left( \omega(\tau_{post} - \tau_{pre}) \cos \Delta\varphi + (1 + \omega^2\tau_{pre}\tau_{post}) \sin \Delta\varphi \right) \quad (7)$$

where we introduced the phase difference  $\Delta\varphi = \varphi_{pre} - \varphi_{post}$

Last, we use the addition of two sines to get our Equation (12)

$$\Delta W = \frac{1}{2} \frac{\omega\tau_{pre}\tau_{post} \sqrt{\omega^2(\tau_{post} - \tau_{pre})^2 + (1 + \omega^2\tau_{pre}\tau_{post})^2}}{(1 + \tau_{pre}^2\omega^2)(1 + \tau_{post}^2\omega^2)} \cdot \sin \left( \Delta\varphi + \arctan \frac{\omega(\tau_{post} - \tau_{pre})}{1 + \omega^2\tau_{pre}\tau_{post}} \right) \quad (8)$$

## 2 The contributions of $u$ and $z$

In the following section, we are only interested in small perturbations. So, any  $\varepsilon$  popping up is considered small against the background rate,  $\varepsilon \ll x_0$ . This will allow us to discard terms of higher orders of  $\varepsilon$ .

### 2.1 The postsynaptic activation $z$

Consider the equation of the postsynaptic activation

$$\dot{z} = -\alpha(z - z_0)^2 + c_{act}x_{post}z \quad (9)$$

If the postsynaptic rate is constant,  $x_{post} = x_0$ , we expect the value of  $z$  to reach a constant equilibrium value,  $z^{eq}$ . This is given by

$$z^{eq} = \frac{h}{2} + \sqrt{\frac{h^2}{4} - z_0^2} \quad (10)$$

with  $h = 2z_0 + c_{act}x_0/\alpha$ .  $z^{eq}$  is the solution of a quadratic equation. The minus sign is discarded, because it would lead to values of  $z^{eq}$  less than  $z_0$ , which are unstable and lead to a negative divergence of  $z$ . Also, in any case these values can not be reached.

Now we introduce a small perturbation for  $x_{post} = x_0 + \delta x = x_0 + \varepsilon \exp(i(\omega t + \varphi_{post}))$ . For ease of computation, we write the perturbation in a complex way and switch to real functions when appropriate. We expect  $z$  to behave linear around the equilibrium and write it as  $z = z^{eq} + \delta z$ . We now linearize the differential equation for  $z$ :

$$\dot{z} = \delta\dot{z} = \left. \frac{\partial\dot{z}}{\partial z} \right|_{z^{eq}, x_0} \delta z + \left. \frac{\partial\dot{z}}{\partial x} \right|_{z^{eq}, x_0} \delta x \quad (11)$$

The derivatives are

$$\begin{aligned} \left. \frac{\partial\dot{z}}{\partial z} \right|_{z^{eq}, x_0} &= -2\alpha(z^{eq} - z_0) + c_{act}x_0 = -k \\ \left. \frac{\partial\dot{z}}{\partial x} \right|_{z^{eq}, x_0} &= c_{act}z^{eq} = \frac{m}{\varepsilon} \end{aligned} \quad (12)$$

This gives us the differential equation

$$\delta\dot{z} = -k\delta z + \frac{m}{\varepsilon}\delta x \quad (13)$$

which is solved by

$$\delta z = \frac{m}{\sqrt{k^2 + \omega^2}} e^{i(\omega t + \varphi_{post} - \arctan \omega/k)} \quad (14)$$

## 2.2 The attenuation $u$

For the general case

$$\dot{u}_i = \frac{1 - u_i}{\tau_i^{rec}} - c_i(u_i - u_0^i)x_i \quad (15)$$

we do again a linearization for small perturbations:  $x_i = x_0 + \delta x = x_0 + \varepsilon \exp(i(\omega t + \varphi_i))$ ,  $u_i = u_i^{eq} + \delta u$ . First, with constant background rate, the equilibrium value of  $u_i$  is  $u_i^{eq}$ :

$$u_i^{eq} = \frac{\frac{1}{\tau_i^{rec}} + c_i u_0^i x_0}{\frac{1}{\tau_i^{rec}} + c_i x_0} \quad (16)$$

The derivatives are

$$\begin{aligned} \left. \frac{\partial \dot{u}_i}{\partial u_i} \right|_{u_i^{eq}, x_0} &= - \left( \frac{1}{\tau_i^{rec}} + c_i x_0 \right) = -p_i \\ \left. \frac{\partial \dot{u}_i}{\partial x_i} \right|_{u_i^{eq}, x_0} &= -c_i(u_i^{eq} - u_0) = -\frac{q_i}{\varepsilon} \end{aligned} \quad (17)$$

Which leads us to the differential equation for the variation of  $u_i$ :

$$\delta \dot{u}_i + p_i \delta u_i = -\frac{q_i}{\varepsilon} \delta x \quad (18)$$

And the solution is given by

$$\delta u_i = \frac{-q_i}{\sqrt{p_i^2 + \omega^2}} e^{i(\omega t + \varphi_{post} - \arctan \omega/p_i)} \quad (19)$$

## 2.3 The weight change

Now, we want to calculate the weight change for the given firing rate  $x = x_0 + \delta x$ . The perturbations  $\delta x$ ,  $\delta z$  and  $\delta u_i$  are all in the order of  $\varepsilon$ . For the calculations, we discard terms of the order of  $\varepsilon^2$  or higher. This yields

$$\dot{y}_{pre} = -\frac{y_{pre}}{\tau_{pre}} + (u_{pre}^{eq} + \delta u_{pre})(x_0 + \delta x) = -\frac{y_{pre}}{\tau_{pre}} + u_{pre}^{eq} x_0 + x_0 \delta u_{pre} + u_{pre}^{eq} \delta x + \mathcal{O}(\varepsilon^2) \quad , \quad (20)$$

and similarly for the postsynaptic trace

$$\dot{y}_{post} = -\frac{y_{post}}{\tau_{post}} + u_{post}^{eq} x_0 z^{eq} + u_{post}^{eq} x_0 \delta z + u_{post}^{eq} z^{eq} \delta x + x_0 z^{eq} \delta u_{post} + \mathcal{O}(\varepsilon^2) \quad . \quad (21)$$

The task is now to solve these equations, take the temporal derivative of  $y_{post}$ , take the real parts of both functions, use trigonometric product identities and then finally integrate  $\Delta W = 1/T \int_T y_{pre} \dot{y}_{post} dt$ . Since the calculations do not get shorter and more instructive than in the first section, we just present the result.

$$\begin{aligned} \Delta W &= \frac{1}{T} \int_0^T y_{pre} \dot{y}_{post} dt \\ &= \frac{1}{2} \left[ -A_2 B_2 \sin(\Psi_2 - \Phi_2) - A_2 B_3 \sin(\Psi_3 - \Phi_2) + A_2 B_4 \sin(\Psi_4 - \Phi_2) \right. \\ &\quad \left. + A_3 B_2 \sin(\Psi_2 - \Phi_3) + A_3 B_3 \sin(\Psi_3 - \Phi_3) - A_3 B_4 \sin(\Psi_4 - \Phi_3) \right] \end{aligned} \quad (22)$$

with

$$\begin{aligned}
A_1 &= \tau_{pre} u_{pre}^{eq} x_0 \\
A_2 &= \frac{x_0 q_{pre}}{\sqrt{\frac{1}{\tau_{pre}^2} + \omega^2} \sqrt{p_{pre}^2 + \omega^2}}, \quad \Phi_2 = \varphi_{pre} - \arctan \frac{\omega}{p_{pre}} - \arctan \omega \tau_{pre} \\
A_3 &= \frac{u_{pre}^{eq} \varepsilon}{\sqrt{\frac{1}{\tau_{pre}^2} + \omega^2}}, \quad \Phi_3 = \varphi_{pre} - \arctan \omega \tau_{pre} \\
B_1 &= \tau_{post} u_{post}^{eq} x_0 z^{eq} \\
B_2 &= \frac{\omega u_{post}^{eq} x_0 m}{\sqrt{\frac{1}{\tau_{post}^2} + \omega^2} \sqrt{k^2 + \omega^2}}, \quad \Psi_2 = \varphi_{post} - \arctan \frac{\omega}{k} - \arctan \omega \tau_{post} \\
B_3 &= \frac{\omega u_{post}^{eq} z^{eq} \varepsilon}{\sqrt{\frac{1}{\tau_{post}^2} + \omega^2}}, \quad \Psi_3 = \varphi_{post} - \arctan \omega \tau_{post} \\
B_4 &= \frac{\omega x_0 z^{eq} q_{post}}{\sqrt{\frac{1}{\tau_{post}^2} + \omega^2} \sqrt{p_{post}^2 + \omega^2}}, \quad \Psi_4 = \varphi_{post} - \arctan \frac{\omega}{p_{post}} - \arctan \omega \tau_{post}
\end{aligned} \tag{23}$$