Supplementary Material for Effects of Synaptic Weight Diffusion on Learning in Decision Making Networks

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Here, we present the derivation of the ensemble averages that appeared in Equation (9) with omitting constants X_0 and σ_p.

1 Derivation of $\langle h^a | a \rangle$

By definition,

$$\begin{aligned} p_A \langle h^A | A \rangle &= p_A \frac{\int_{-\infty}^{\infty} dh^B p(h^B) \int_{h^B}^{\infty} h^A dh^A p(h^A)}{\int_{-\infty}^{\infty} dh^B p(h^B) \int_{h^B}^{\infty} dh^A p(h^A)} \\ &= \int_{-\infty}^{\infty} dh^B p(h^B) \int_{h^B}^{\infty} h^A dh^A p(h^A) \\ &= \int_{-\infty}^{\infty} dh^B p(h^B) \left[\frac{1}{\sqrt{2\pi} l_A} \int_{h^B}^{\infty} h^A dh^A \exp\left(-\frac{(h^A - \bar{J}_A)}{2 l_A^2} \right) \right]. \end{aligned} \tag{1}$$

Replacing the variable as $z = \frac{(h^A - \bar{J}_A)}{l_A}$, and hence $h^A = l_A z + \bar{J}_A, dh^A = l_A dz$, we calculate the second integral as

$$\begin{aligned} &\frac{1}{\sqrt{2\pi} l_A} \int_{h^B}^{\infty} h^A dh^A \exp\left(-\frac{(h^A - \bar{J}_A)}{2 l_A^2} \right) \\ =&\frac{1}{\sqrt{2\pi}} \int_{\frac{h^B - \bar{J}_A}{l_A}}^{\infty} dz\, l_A z \exp\left(-\frac{z^2}{2} \right) + \frac{\bar{J}^A}{\sqrt{2\pi}} \int_{\frac{h^B - \bar{J}_A}{l_A}}^{\infty} dz \exp\left(-\frac{z^2}{2} \right). \end{aligned} \tag{2}$$

The first term in Equation (2) is

$$\frac{1}{\sqrt{2\pi}} \int_{\frac{h^B - \bar{J}_A}{l_A}}^{\infty} dz\, l_A z \exp\left(-\frac{z^2}{2} \right) = \frac{l_A}{\sqrt{2\pi}} \exp\left(-\frac{(h^B - \bar{J}_A)^2}{2 l_A^2} \right). \tag{3}$$

Multiplying this by $\int_{-\infty}^{\infty} dh^B p(h^B)$ leads to

$$\frac{l_A^2}{\sqrt{2\pi (l_A^2 + l_B^2)}} \exp\left(-\frac{(\bar{J}_A - \bar{J}_B)^2}{2 (l_A^2 + l_B^2)} \right). \tag{4}$$

Multiplying the second term in Equation 2 by $\int_{-\infty}^{\infty} dh^B p(h^B)$ leads to

$$\frac{\bar{J}_A}{2} \mathrm{erfc}\left(-\frac{\bar{J}_A - \bar{J}_B}{\sqrt{2 (l_A^2 + l_B^2)}} \right) = \bar{J}_A p_A. \tag{5}$$

Taken together, we obtain

$$p_A\langle h^A|A\rangle = \frac{l_A^2}{\sqrt{2\pi(l_A^2+l_B^2)}}\exp\left(-\frac{(\bar{J}_A-\bar{J}_B)^2}{2(l_A^2+l_B^2)}\right)+p_A\bar{J}_A, \tag{6}$$

from which we obtain the first equation of (9) in main text.

2 Deviation of $\langle \tilde{\boldsymbol{x}}_a|a\rangle$

Since $\tilde{\boldsymbol{x}}^A$ is a summation of N random variables that independently obey a Gaussian distribution with mean $1/N$ and variance $1/N$, it obeys a Gaussian distribution whose mean is one, and variance is one. Using Bayes' theorem,

$$\begin{aligned} p_A\langle \tilde{\boldsymbol{x}}_A|A\rangle &= p_A\frac{\int_{-\infty}^{\infty} d\tilde{\boldsymbol{x}}^A\, \tilde{\boldsymbol{x}}^A p(\tilde{\boldsymbol{x}}^A|a=A)}{\int_{-\infty}^{\infty} d\tilde{\boldsymbol{x}}^A\, p(\tilde{\boldsymbol{x}}^A|a=A)} \\ &= p_A\frac{\int_{-\infty}^{\infty} d\tilde{\boldsymbol{x}}^A\, \tilde{\boldsymbol{x}}^A p(a=A|\tilde{\boldsymbol{x}}^A)p(\tilde{\boldsymbol{x}}^A)/p(a=A)}{\int_{-\infty}^{\infty} d\tilde{\boldsymbol{x}}^A\, p(a=A|\tilde{\boldsymbol{x}}^A)p(\tilde{\boldsymbol{x}}^A)/p(a=A)} \\ &= \int_{-\infty}^{\infty} d\tilde{\boldsymbol{x}}^A\, \tilde{\boldsymbol{x}}^A p(a=A|\tilde{\boldsymbol{x}}^A)p(\tilde{\boldsymbol{x}}^A) \end{aligned} \tag{7}$$

Since the variance of h^A given the value of $\tilde{\boldsymbol{x}}^A$ is $l_A^2-\bar{J}_A^2$, $p(a=A|\tilde{\boldsymbol{x}}^A)$ is calculated as

$$\begin{aligned} p(a=A|\tilde{\boldsymbol{x}}^A) &= \Pr[h^A>h^B|\tilde{\boldsymbol{x}}^A] \\ &= \frac{1}{2}\mathrm{erfc}\left(-\frac{\bar{J}_A-\bar{J}_B\tilde{\boldsymbol{x}}^A}{\sqrt{2(l_A^2-\bar{J}_A^2+l_B^2)}}\right). \end{aligned} \tag{8}$$

Substituting this into Equation 7, we arrive at the second equation of (9) in main text.