

Appendix 1: Computing Clebsch-Gordan Coefficients

One way to compute Clebsch-Gordan coefficients is to naively view it as a similarity matrix recovery problem, with the twist that the similarity matrix must be consistent over all group elements. We first cast the problem of recovering a similarity matrix as a nullspace computation.

Proposition 1. *Let A, B, C be matrices such that AC and CB are defined. Let $K = I \otimes A - B^T \otimes I$. Then $AC = CB$ if and only if $\text{vec}(C) \in \text{Nullspace}(K)$.*

Proof. A well known matrix identity ([1]) states that if A, B, C are matrices such that the product ABC is defined, then $\text{vec}(ABC) = (C^T \otimes A) \text{vec}(B)$. Applying the identity,

$$\begin{aligned} \text{vec}(ACI) &= \text{vec}(ICB) \\ (I \otimes A) \text{vec}(C) &= (B^T \otimes I) \text{vec}(C) \\ (I \otimes A - B^T \otimes I) \text{vec}(C) &= 0 \end{aligned}$$

□

For each $\sigma \in S_n$, the matrix K_σ constructed using the above proposition yields the space of matrices C_{ij} such that

$$(\rho_i \otimes \rho_j(\sigma)) \cdot C = C \cdot \oplus_k \oplus_{\ell=1}^{z_{ijk}} \rho_k(\sigma)$$

To find a C_{ij} which is consistent across all group elements, we need to find the intersection: $\cap_\sigma \text{Nullspace}(K_\sigma)$. At first glance, it seems that this might require looking at $n!$ nullspaces, but as luck would have it, most of these nullspaces are extraneous, as we now show.

Definition 2. We say that a group G is *generated* by a set of *generators* $S = \{g_1, \dots, g_m\}$ if every element of G can be written as a finite product of elements in S and their inverses.

To ensure a consistent similarity matrix for all group elements, we use the follow proposition which says that it suffices to be consistent on only a set of generators of the group.

Proposition 3. *Let ρ and τ be representations of G . Suppose that G is generated by the elements g_1, \dots, g_m . If there exists a linear map C such that $\rho(g_i) \cdot C = C \cdot \tau(g_i)$ for each $i \in \{1, \dots, m\}$, then ρ and τ are equivalent as representations with C as the equivalence map.*

Proof. We just need to show that C is a similarity transform for any other element of G as well. Let x be any element of G and write it as a product of generators: $x = \prod_{i=1}^n g_i$. It follows that:

$$\begin{aligned} C^{-1} \cdot \rho(x) \cdot C &= C^{-1} \cdot \rho \left(\prod_i g_i \right) \cdot C = C^{-1} \cdot \left(\prod_i \rho(g_i) \right) \cdot C \\ &= \prod_i (C^{-1} \cdot \rho(g_i) \cdot C) = \prod_i \tau(g_i) = \tau \left(\prod_i g_i \right) = \tau(x) \end{aligned}$$

Since this holds for every $x \in G$, we have shown C to be an equivalence map between representations. □

The good news is that despite having order $n!$, S_n can be generated by just two elements, and so the problem reduces to solving for the intersection of two nullspaces, which can be done using standard numerical methods. Computationally, it is often useful to use sparse nullspace algorithms since the matrices happen to be quite sparse. Most of sparse nullspace algorithms require an initial estimate for the dimension of the nullspace, and for our particular problem of finding a similarity transform between two tensor product representations, there exists an analytical expression for the dimension of this nullspace.

Theorem 4. *If K is constructed for finding the Clebsch-Gordan coefficients for $\rho_i \otimes \rho_j$, the nullspace of the matrix K is $\left(\sum_k z_{ijk}^2 \right)$ -dimensional, where z_{ijk} is the Clebsch-Gordan series.*

Proof. The result follows directly from some basic results about endomorphism algebras and Schur's lemma (ch 1.7 from [2]). □

There is a second algorithm for finding Clebsch-Gordan coefficients which is more efficient than the one described here; we refer the reader to [3] for details on the *Eigenfunction method*.

Appendix 2: Proof of Proposition 3

We use the following matrix identities:

1. Let A be an $n \times n$ matrix, and C an invertible $n \times n$ matrix. Then $\text{Tr } A = \text{Tr } (C^{-1}AC)$.
2. Let A be an $n \times n$ matrix and B_i be matrices of size $m_i \times m_i$ where $\sum_i m_i = n$. Then $\text{Tr } (A \cdot (\bigoplus_i B_i)) = \sum_i \text{Tr } (A_i \cdot B_i)$, where A_i is the block of A corresponding to block B_i in the matrix $(\bigoplus_i B_i)$.
3. If A and B are square, $\text{Tr } (A \otimes B) = (\text{Tr } A) \cdot (\text{Tr } B)$
4. $(A \otimes B) \cdot (C \otimes D) = AC \otimes BD$

We also use the fact that C_{ij} is orthogonal for all pairs (ρ_i, ρ_j) : $C_{ij}^T \cdot C_{ij} = I$.

$$\begin{aligned}
f(\sigma) \cdot g(\sigma) &= \left[\frac{1}{|G|} \sum_i d_{\rho_i} \text{Tr} \left(\hat{f}_{\rho_i}^T \cdot \rho_i(\sigma) \right) \right] \cdot \left[\frac{1}{|G|} \sum_j d_{\rho_j} \text{Tr} \left(\hat{g}_{\rho_j}^T \cdot \rho_j(\sigma) \right) \right] \\
&= \left(\frac{1}{|G|} \right)^2 \sum_{i,j} d_{\rho_i} d_{\rho_j} \left[\text{Tr} \left(\hat{f}_{\rho_i}^T \cdot \rho_i(\sigma) \right) \cdot \text{Tr} \left(\hat{g}_{\rho_j}^T \cdot \rho_j(\sigma) \right) \right] \\
&= \left(\frac{1}{|G|} \right)^2 \sum_{i,j} d_{\rho_i} d_{\rho_j} \left[\text{Tr} \left(\left(\hat{f}_{\rho_i}^T \cdot \rho_i(\sigma) \right) \otimes \left(\hat{g}_{\rho_j}^T \cdot \rho_j(\sigma) \right) \right) \right] \quad (\text{by Property 3}) \\
&= \left(\frac{1}{|G|} \right)^2 \sum_{i,j} d_{\rho_i} d_{\rho_j} \text{Tr} \left(\left(\hat{f}_{\rho_i} \otimes \hat{g}_{\rho_j} \right)^T \cdot (\rho_i(\sigma) \otimes \rho_j(\sigma)) \right) \quad (\text{by Property 4}) \\
&= \left(\frac{1}{|G|} \right)^2 \sum_{i,j} d_{\rho_i} d_{\rho_j} \text{Tr} \left(C_{ij}^T \cdot \left(\hat{f}_{\rho_i} \otimes \hat{g}_{\rho_j} \right)^T \cdot C_{ij} \cdot (\rho_i(\sigma) \otimes \rho_j(\sigma)) \cdot C_{ij} \right) \\
&\quad (\text{by Property 1}) \\
&= \left(\frac{1}{|G|} \right)^2 \sum_{i,j} d_{\rho_i} d_{\rho_j} \text{Tr} \left(A_{ij}^T \cdot \left(\bigoplus_k \bigoplus_{\ell=1}^{z_{ijk}} \rho_k(\sigma) \right) \right) \quad (\text{by definition of } C_{ij} \text{ and } A_{ij}) \\
&= \frac{1}{|G|^2} \sum_{ij} d_{\rho_i} d_{\rho_j} \sum_k d_{\rho_k} \sum_{\ell=1}^{z_{ijk}} \text{Tr} \left((d_{\rho_k}^{-1} A_{ij}^{k\ell})^T \rho_k(\sigma) \right) \quad (\text{by Property 2}) \\
&= \frac{1}{|G|} \sum_k d_{\rho_k} \text{Tr} \left[\left(\sum_{ij} \sum_{\ell=1}^{z_{ijk}} \frac{d_{\rho_i} d_{\rho_j}}{d_{\rho_k} |G|} A_{ij}^{k\ell} \right)^T \rho_k(\sigma) \right] \quad (\text{rearranging terms})
\end{aligned}$$

References

- [1] Charles F. van Loan. The ubiquitous kronecker product. *J. Comput. Appl. Math.*, 123(1-2):85–100, 2000.
- [2] Bruce E. Sagan. *The Symmetric Group*. Springer, April 2001.
- [3] Jin-Quan Chen. *Group Representation Theory for Physicists*. World Scientific, 1989.