Appendices for PLUGIn: A simple algorithm for inverting generative models with recovery guarantees

A Some Results on Gaussian Matrices

Here we state some results on Gaussian Matrices, which will be used in the proofs later.

Lemma 2 (21, 22). Let \( \sigma : \mathbb{R} \to \mathbb{R} \) be a positively homogeneous activation function. Let \( A \in \mathbb{R}^{m \times n} \) have i.i.d. \( \mathcal{N}(0, \frac{1}{m}) \) entries. Then for any \( x \in \mathbb{R}^n \),
\[
\mathbb{E} A \sigma(Ax) = \lambda x,
\]
where \( \lambda := \mathbb{E} g \cdot \sigma(g) \) with \( g \sim \mathcal{N}(0, 1) \). In particular, \( \lambda = \frac{1}{2} \) when \( \sigma \) is ReLU.

Proof. Since \( \sigma \) is positively homogeneous, we can assume (without loss of generality) \( x \in \mathbb{S}^{n-1} \). Denote by \( a_j \) the \( j \)-th row of \( A \). Then
\[
\mathbb{E} A \sigma(Ax) = \mathbb{E} \sum_{j=1}^m (a_j \cdot x) a_j = m \mathbb{E} \sigma(a_j^T x) a_1 = \mathbb{E} \sigma(a^T x) a
\]
where \( a := \sqrt{m} a_1 \sim \mathcal{N}(0, I_n) \). Take an orthogonal matrix \( U \) such that \( Ux = \|x\|e_1 = e_1 \) where \( e_1 = (1, 0, \ldots, 0)^T \). Note that by rotation invariance for standard Gaussian, \( Ua \) and \( a \) have the same distribution \( \mathcal{N}(0, I_n) \), thus
\[
\mathbb{E} \sigma(a^T x) a = \mathbb{E} \sigma(a^T U^T e_1) U^T Ua = \mathbb{E} \sigma(a^T e_1) U^T a = U^T \mathbb{E} \sigma(a^T e_1) a = \lambda U^T e_1 = \lambda x.
\]

The following theorem is the concentration of (Gaussian) measure inequality for Lipschitz functions. Here we only state a one-sided version, though it is more commonly stated with a two-sided one, i.e.,
\[
P(|f(g) - \mathbb{E} f(g)| \geq t) \leq 2 \exp \left(-\frac{t^2}{2L_f^2}\right).
\]

Theorem 2. Let \( f : \mathbb{R}^n \to \mathbb{R} \) be a Lipschitz function with Lipschitz constant \( L_f \). Let \( g \in \mathbb{R}^n \) be a random vector with independent \( \mathcal{N}(0, 1) \) entries. Then, for all \( t > 0 \),
\[
P(f(g) - \mathbb{E} f(g) \geq t) \leq \exp \left(-\frac{t^2}{2L_f^2}\right).
\]

A proof of Theorem 2 can be found in 30 Chap. 8]. Based on this theorem, it is easy to prove the following results.

Lemma 3. Let \( A \in \mathbb{R}^{m \times n} \) have i.i.d. \( \mathcal{N}(0, 1) \) entries.

(a) For any fixed point \( s \in \mathbb{R}^n \), we have
\[
P\left(\|As\| \geq \sqrt{m} \|s\| + \sqrt{t}\|s\|\right) \leq e^{-t/2}, \quad \forall t > 0.
\]

(b) For any fixed \( k \)-dimensional subspace \( S \subseteq \mathbb{R}^n \), we have
\[
P\left(\|A\|_S \geq \sqrt{m} + \sqrt{k} + \sqrt{t}\right) \leq e^{-t/2}, \quad \forall t > 0.
\]

Proof. (a) Without loss of generality, assume \( \|s\| = 1 \). Then \( As \sim \mathcal{N}(0, I_m) \) and by Jensen’s inequality, \( \mathbb{E} \|As\| \leq \sqrt{\mathbb{E} \|As\|^2} = \sqrt{m} \). The result follows immediately from Theorem 2 (with \( f(g) = \|g\| \) and \( g = As \)).

(b) Let \( U \) be an orthogonal matrix such that \( U^T S = \text{span}\{e_1, \ldots, e_k\} =: S_0 \), then \( \|A\|_S = \|AU\|_{S_0} \).

Also, since \( AU \) has the same distribution as \( A \) (by rotation invariance), we get
\[
P\left(\|A\|_{S_0} \geq \sqrt{m} + \sqrt{k} + \sqrt{t}\right) = P\left(\|A\|_{S_0} \geq \sqrt{m} + \sqrt{k} + \sqrt{t}\right).
\]
Notice that \( \|A\|_{\mathcal{S}_p} \) is the operator norm for a particular sub-matrix (obtained by taking first \( k \)-columns) of \( A \), so without loss of generality, we can assume \( k = n \).

Let \( f(A) = \|A\| \). Since \( |f(A) - f(A')| \leq \|A - A'\|_p \), \( f \) is 1-Lipschitz when viewed as a mapping from \( \mathbb{R}^{m \times n} \) to \( \mathbb{R} \). By Theorem 2

\[
\mathbb{P} \left( f(A) \geq \mathbb{E} f(A) + \sqrt{t} \right) \leq e^{-t/2}, \quad \forall t > 0.
\]

The result follows since \( \mathbb{E}\|A\| \leq \sqrt{m} + \sqrt{n} \) (see, e.g., [31] Section 7.3).

\[\square\]

**B Preliminaries and Proof for Lemma 1**

**Preliminaries**

For \( \alpha \geq 1 \), the \( \psi_\alpha \)-norm of a random variable \( X \) is defined as

\[
\|X\|_{\psi_\alpha} := \inf \{ t > 0 : \mathbb{E} \exp(|X|^\alpha/t^\alpha) \leq 2 \}.
\]

We say \( X \) is sub-Gaussian if \( \|X\|_{\psi_2} < \infty \) and sub-exponential if \( \|X\|_{\psi_1} < \infty \). The \( \psi_2 \) and \( \psi_1 \) norms are also called sub-Gaussian and sub-exponential norms respectively. Loosely speaking, a sub-Gaussian (or a sub-exponential) random variable has tail dominated by the tail of a Gaussian (or an exponential) random variable.

For independent, mean zero, sub-exponential random variables \( X_1, \ldots, X_m \), their sum concentrates around zero. In particular, the following *Bernstein’s Inequality* [31] Section 2.8 holds:

\[
\mathbb{P} \left( \left| \sum_{i=1}^m X_i \right| \geq t \right) \leq 2 \exp \left[ -c \min \left( \frac{t^2}{\sum_{i=1}^m \|X_i\|_{\psi_1}^2}, \frac{t}{\max_i \|X_i\|_{\psi_1}} \right) \right].
\]

The above inequality also suggests that \( \sum_{i=1}^m X_i \) has a mixed tail, i.e., a tail consisting of both a sub-Gaussian part and a sub-exponential part. In our proof, we will use the following result from generic chaining for mixed tail processes.

**Theorem 3** (Theorem 3.5 [24]). If \( (X_t)_{t \in T} \) has a mixed tail with respect to metric pair \((d_1, d_2)\), i.e.

\[
\mathbb{P} \left( |X_t - X_s| \geq \sqrt{ud_2(t, s) + ud_1(t, s)} \right) \leq 2e^{-u}, \quad \forall u \geq 0.
\]

Then there are constants \( c, C > 0 \) such that for any \( u \geq 1 ,

\[
\mathbb{P} \left( \sup_{t \in T} |X_t - X_{t_0}| \geq C(\gamma_2(T, d_2) + \gamma_1(T, d_1)) + c(\sqrt{u\Delta_{d_1}(T)} + u\Delta_{d_2}(T)) \right) \leq e^{-u}.
\]

Here \( t_0 \) is any fixed point in \( T \), \( \gamma_\alpha(T, d) \) is the \( \gamma_\alpha \)-functional and \( \Delta_{d_i} \) is the diameter given by \( \Delta_{d_i}(T) = \sup_{s, t \in T} d_i(s, t) \).

The \( \gamma_\alpha \)-functional of \( (T, d) \) is defined as

\[
\gamma_\alpha(T, d) := \inf \sup_{(T_n) \in T} \sum_{n=0}^{\infty} 2^{n/\alpha} d(t, T_n),
\]

where the infimum is taken with respect to all admissible sequences. A sequence \( (T_n)_{n \geq 0} \) of subsets of \( T \) is called admissible if \( |T_0| = 1 \) and \( |T_n| \leq 2^m \) for all \( n \geq 1 \).

For our proof, we will use the following estimate on \( \gamma_\alpha(T, d) \), which involves the generalized Dudley’s integral [32] [24].

\[
\gamma_\alpha(T, d) \leq C(\alpha) \int_{0}^{\Delta_1(T)} \left( \log N(T, d, \varepsilon) \right)^{1/\alpha} d\varepsilon,
\]

where \( C(\alpha) \) is a constant depending only on \( \alpha \) and \( N(T, d, \varepsilon) \) is the covering number, i.e., the smallest number of balls (in metric \( d \) and with radius \( \varepsilon \)) needed to cover set \( T \).
Proof for Lemma 1

We recall the statement of Lemma 1 below.

**Lemma 1** Let \( \sigma = \text{ReLU} \). Fix \( w \in \mathbb{R}^n \) and let \( A \in \mathbb{R}^{m \times n} \) have i.i.d. \( \mathcal{N}(0, \frac{1}{m}) \) entries. Define

\[
Z(u, v; w) := \langle Au, \sigma(Av) - \sigma(Aw) \rangle - \frac{1}{2} \langle u, v - w \rangle.
\]

Suppose \( \mathcal{T}_1, \mathcal{T}_2 \) are sets (not depending on \( A \)) such that

\[
\mathcal{T}_1 = \mathcal{S}_1 \cap B^n(0, \alpha) \quad \text{and} \quad \mathcal{T}_2 = \mathcal{S}_2 \cap B(w, \alpha r)
\]

for some \( q \)-dimensional (affine) subspaces \( \mathcal{S}_1, \mathcal{S}_2 \subseteq \mathbb{R}^n \) and real numbers \( \alpha, r > 0 \). Then for any \( t \geq 1 \),

\[
\sup_{u \in \mathcal{T}_1, v \in \mathcal{T}_2} |Z(u, v; w)| \leq C_1 \alpha^2 r \left( \sqrt{\frac{q}{m}} + \frac{q}{m} + \sqrt{\frac{t}{m}} + \frac{t}{m} \right)
\]

with probability at least \( 1 - e^{-t} \). Here \( C_1 > 0 \) is an absolute constant.

**Proof.** First, we establish that \( Z(u, v; w) \) has a mixed tail.

Let \( a_i^T \) be the \( i \)-th row of \( A \), then \( a_i \sim \mathcal{N}(0, I_n/m) \). For \( u \in B^n(0, \alpha) \) and \( v \in B(w, \alpha r) \), define random variables

\[
Z_{i,u,v} := \langle a_i, u \rangle [\sigma(\langle a_i, v \rangle) - \sigma(\langle a_i, w \rangle)] - \frac{1}{2m} \langle u, v - w \rangle, \quad i \in [m].
\]

We have \( \mathbb{E}Z_{i,u,v} = 0 \) by Lemma 2 and

\[
Z_{u,v} := \sum_{i=1}^m Z_{i,u,v} = \langle Au, \sigma(Av) - \sigma(Aw) \rangle - \frac{1}{2} \langle u, v - w \rangle = Z(u, v; w).
\]

For the increments of \( Z_{i,u,v} \), we have

\[
Z_{i,u,v} - Z_{i,u',v'} = \langle a_i, u \rangle \sigma(a_i^T v) - \frac{1}{2m} \langle u, v \rangle - \langle a_i, u' \rangle \sigma(a_i^T v') + \frac{1}{2m} \langle u', v' \rangle
\]

\[
- \langle a_i, u - u' \rangle \sigma(a_i^T w) + \frac{1}{2m} \langle u - u', w \rangle
\]

\[
= \langle a_i, u \rangle \sigma(a_i^T v) - \frac{1}{2m} \langle u, v \rangle - [\langle a_i, u \rangle \sigma(a_i^T v') - \frac{1}{2m} \langle u, v' \rangle]
\]

\[
+ [\langle a_i, u \rangle \sigma(a_i^T v') - \frac{1}{2m} \langle u, v' \rangle] - (a_i, u') \sigma(a_i^T v') + \frac{1}{2m} \langle u', v' \rangle
\]

\[
- \langle a_i, u - u' \rangle \sigma(a_i^T w) + \frac{1}{2m} \langle u - u', w \rangle
\]

\[
= \langle a_i, u \rangle [\sigma(a_i^T v) - \sigma(a_i^T v')] - \frac{1}{2m} \langle u, v - v' \rangle
\]

\[
+ \langle a_i, u - u' \rangle [\sigma(a_i^T v') - \sigma(a_i^T w)] - \frac{1}{2m} \langle u - u', v' - v \rangle
\]

We can estimate its sub-exponential norm from Lemma 3 which gives

\[
\|Z_{i,u,v} - Z_{i,u',v'}\|_{\psi_1} \leq C_2 m^{-1} (\|u\|\|v - v'\| + \|u - u'\|\|v' - w\|)
\]

\[
\leq C_2 m^{-1} (r\|u - u'\| + \|v - v'\|).
\]

By Bernstein’s inequality,

\[
\mathbb{P}(\|Z_{u,v} - Z_{u',v'}\| \geq t) \leq 2 \exp \left(-c \min \left(\frac{t^2}{d_2^2}, \frac{t}{d_1} \right)\right)
\]

where the metrics \( d_i \) are given by

\[
d_2^2 = \frac{\alpha^2}{m} \left(r\|u - u'\| + \|v - v'\|\right)^2 \quad \text{and} \quad d_1 = \frac{\alpha}{m} (r\|u - u'\| + \|v - v'\|).
\]

Therefore \( (Z_{u,v})_{(u,v) \in \mathcal{T}} \) has a mixed tail with respect to the metric pair \((C d_1, C d_2)\) for some absolute constant \(C\).

Next, we bound the supremum of \( Z(u, v; w) \). Without loss of generality, we will assume that \( q \geq 1 \). (In fact, if \( q = 0 \), then \( \mathcal{T}_1, \mathcal{T}_2 \) are either empty set or singleton, in which case the result is trivial or follows directly from Bernstein’s inequality.)
Denote $T := T_1 \times T_2$ and define a metric $d$ on $T$ as
\[ d((u, v), (u', v')) := r\|u - u'\| + \|v - v'\|. \]

It is easy to see that $d_2 = \frac{\alpha}{\sqrt{m}} d$ and $d_1 = \frac{\alpha}{m} d$. Also note that $\gamma_i(T, td) = t \gamma_i(T, d)$ from definition (10). We can assume that $S_i$ is a subspace then $Z_{0,v} = 0$ for $v \in T_2$. Thus by Theorem 3 we have
\[ \sup_{(u,v) \in T} \left| Z_{u,v} \right| \leq \frac{\alpha}{\sqrt{m}} \gamma_2(T, d) + \frac{\alpha}{m} \gamma_1(T, d) + \sqrt{\frac{4\alpha^2 r}{\sqrt{m}}} + \frac{4\alpha^2 r}{m} \]
with probability at least $1 - e^{-t}$. It remains to estimate $\gamma_i(T, d)$.

From (11) we have
\[ \gamma_i(T, d) \leq C_3 \int_0^{\Delta_e(T)} (\log N(T, d, \varepsilon))^{1/i} \, d\varepsilon, \quad i = 1, 2. \]

Let $d_{E^2}$ be the Euclidean metric. Note that one can always obtain a $\varepsilon$-covering on $T$ (with metric $d$) from the product set of a $\varepsilon/2$-covering on $T_1$ (with metric $r d_{E^2}$) and a $\varepsilon/2$-covering on $T_2$ (with metric $d_{E^2}$). Moreover, note that $T_1$ is contained in a $q$-dimensional ball of radius $\alpha$ and $T_2$ is contained in a $q$-dimensional ball of radius $\sqrt{r}$. Hence
\[ N(T, d, \varepsilon) \leq N(T_1, rd_{E^2}, \varepsilon/2) \cdot N(T_2, d_{E^2}, \varepsilon/2) \]
\[ \leq N(\alpha B^q, rd_{E^2}, \varepsilon/2) \cdot N(\sqrt{r} B^q, d_{E^2}, \varepsilon/2) \]
\[ = N(\mathbb{B}^q, d_{E^2}, \varepsilon/2\alpha) \cdot N(\mathbb{B}^q, d_{E^2}, \varepsilon/2\sqrt{r}) \]
\[ \leq \left(1 + \frac{4\alpha r}{\varepsilon}\right)^{2q}. \]

Here the last line uses estimate $N(\mathbb{B}^q, d_{E^2}, \varepsilon) \leq (1 + \frac{2}{\varepsilon})^q$ for the covering number of unit balls (see e.g., [11] Section 4.2).

Note the estimate $\int_0^a \log\left(\frac{2a}{x}\right) \, dx = a(\log 2 + 1) < 2a$, we get
\[ \gamma_1(T, d) \leq C_3 \int_0^{4\alpha r} 2q \log\left(1 + \frac{4\alpha r}{\varepsilon}\right) \, d\varepsilon \leq 2C_3 q \int_0^{4\alpha r} \log\left(\frac{8\alpha r}{\varepsilon}\right) \, d\varepsilon \leq 16C_3 \alpha r q. \]

Also note the inequality $\sqrt{\log(1 + x)} < \sqrt{2} \log(1 + x)$ for $x \geq 1$, we have
\[ \gamma_2(T, d) \leq C_3 \int_0^{4\alpha r} \sqrt{2q} \log\left(1 + \frac{4\alpha r}{\varepsilon}\right) \, d\varepsilon \]
\[ \leq 2C_3 \sqrt{q} \int_0^{4\alpha r} \log\left(1 + \frac{4\alpha r}{\varepsilon}\right) \, d\varepsilon \]
\[ \leq 2C_3 \sqrt{q} \int_0^{4\alpha r} \log\left(\frac{8\alpha r}{\varepsilon}\right) \, d\varepsilon \]
\[ \leq 16C_3 \alpha r \sqrt{q}. \]

Therefore with probability at least $1 - e^{-t}$,
\[ \sup_{(u,v) \in T} |Z_{u,v}| \leq C_1 \alpha^2 r \left(\sqrt{\frac{q}{m}} + \frac{q}{m} + \sqrt{\frac{t}{m}} + \frac{t}{m}\right). \]

\[ \square \]

Footnotes:
1. If $S_1$ is an affine subspace, let $q' = q + 1$ and let $S'_1$ be the $q'$-dimensional subspace containing $S_1$ (and origin). One can proceed with $S'_1$ and $q'$ for the proof. Finally, notice that $\sqrt{\frac{q}{m}} + \frac{q}{m} \leq 2 \left(\sqrt{\frac{m}{q}} + \frac{m}{q}\right)$, so this will give the same result with only a different absolute constant. (In fact, in our application of Lemma 10 for the multi-layer proof, $S_1$ is chosen as range($A_1 \cdot \cdot \cdot A_1$), which is always a subspace.)
2. This comes from the indefinite integral $\int \log\left(\frac{x}{2}\right) \, dx = x \log\left(\frac{x}{2}\right) + x + C$. 

Lemma 4. Let $\sigma = \text{ReLU}$. For $u, x, y \in \mathbb{R}^n$ and $g \sim \mathcal{N}(0, I_n)$, the (mean zero) random variable
\[ Z^g := \langle g, u \rangle [\sigma(g^T x) - \sigma(g^T y)] - \frac{1}{2} \langle u, x - y \rangle \]
has sub-exponential norm $\|Z^g\|_{\psi_1} \leq C_2 \|u\|_{\psi_1} \|x - y\|_{\psi_1}$, where $C_2$ is an absolute constant.

Proof. It is easy to see that $Z^g$ is mean zero from Lemma 2. Also from the following two properties of $\psi_1, \psi_2$-norms (see \cite{1}, Section 2.7):
\[ \|X - \mathbb{E}X\|_{\psi_1} \lesssim \|X\|_{\psi_1} \quad \text{and} \quad \|XY\|_{\psi_1} \lesssim \|X\|_{\psi_1} \|Y\|_{\psi_2}, \]
we have (note that $\sigma$ is 1-Lipschitz)
\[ \|Z^g\|_{\psi_1} \lesssim \|\langle g, u \rangle\|_{\psi_2} \|\sigma(g^T x) - \sigma(g^T y)\|_{\psi_2} \lesssim \|\langle g, u \rangle\|_{\psi_2} \|g, x - y\|_{\psi_2}. \]
The result follows by noting that $\|\langle g, u \rangle\|_{\psi_2} = \|g_1\|_{\psi_2} \|u\|$ where $g_1 \sim \mathcal{N}(0, 1)$.

C Proof for Theorem 1

Additional notations: We use $\mathbb{P}_{A_i}$ to denote that the probability is taken only with respect to $A_i$. In neural network $\mathcal{G}: \mathbb{R}^m \rightarrow \mathbb{R}^n$, let $\mathcal{G}_i: \mathbb{R}^m \rightarrow \mathbb{R}^{n_i}$ be the mapping that corresponds to the first $i$ layers, i.e. $\mathcal{G}_i(x) = \sigma(A_i \ldots \sigma(A_1 x) \ldots)$. For its weight matrices, let $A_0 = I_{n_0}$ and $\bar{A}_i = A_i A_{i-1} \ldots A_1$ for $i \in [d]$.

Proof of Theorem 1. First we write
\[ x^{k+1} - x^* = \theta \left( x^k - x^* - 2d \bar{A}_d^T [\mathcal{G}(x^k) - y] \right) + (1 - \theta)(x^k - x^*). \]
For any fixed $r > 0$, using triangle inequality and Lemma 3 (with events $\mathcal{E}_i$ defined as in Lemma 5), we can conclude that if $\|x^k - x^*\| \leq r$, then with probability at least $1 - \mathbb{P}(\mathcal{E}_1) - \mathbb{P}(\mathcal{E}_2) - 2e^{-10n_0}$,
\[ \|x^{k+1} - x^*\| \leq \frac{\theta}{2} \left( r + 30 \cdot 2^d \sqrt{\frac{n_0}{n_d}} \|\epsilon\| \right) + |1 - \theta| r = \alpha(r + \beta \epsilon) \quad (12) \]
where
\[ \alpha = \frac{\theta}{2} + |1 - \theta|, \quad \beta = \frac{\theta/2}{|1 - \theta| + \theta/2}, \quad \epsilon = 30 \cdot 2^d \sqrt{n_0/n_d} \|\epsilon\|. \]
Now define a sequence $\{r_k\}_{k \in \mathbb{N}}$ such that $r_{k+1} = \alpha(r_k + \beta \epsilon)$ and $r_0 = R$. We can find its general formula as follow:
\[ r_{k+1} - \frac{\alpha \beta}{1 - \alpha} \epsilon = \alpha \left( r_k - \frac{\alpha \beta}{1 - \alpha} \epsilon \right) \quad \Rightarrow \quad r_k = \alpha^k \left( R - \frac{\alpha \beta}{1 - \alpha} \epsilon \right) + \frac{\alpha \beta}{1 - \alpha} \epsilon. \]
Next, by induction on $k$ (i.e., apply (12) with $r = r_k$ for $k = 0, 1, 2, \ldots$) we get
\[ \|x^k - x^*\| \leq r_k \leq \alpha^k R + \frac{\alpha \beta}{1 - \alpha} \epsilon, \quad k \in \mathbb{N}. \quad (13) \]
Notice that the events $\mathcal{E}_1, \mathcal{E}_2$ remain unchanged throughout iterations, so (13) holds with probability at least $1 - \mathbb{P}(\mathcal{E}_1) - \mathbb{P}(\mathcal{E}_2) - 2e^{-10n_0}$.
Lastly, from Lemma 6 and Lemma 8 we know $\mathbb{P}(\mathcal{E}_i) \leq 3e^{-10n_0}$ and $\|\mathcal{G}(x^k) - \mathcal{G}(x^*)\| \leq 3 \|x^k - x^*\|$ on $\mathcal{E}_2$. This completes the proof.

Lemma 5. Fix $r > 0$ and assume assumptions A1-A4 hold. If $\|x^k - x^*\| \leq r$, then after one iteration according to 5, with step size $\eta = 2^d$, we have
\[ \|x^{k+1} - x^*\| \leq \frac{1}{2} \left( r + 30 \cdot 2^d \sqrt{\frac{n_0}{n_d}} \|\epsilon\| \right) \]
with probability at least $1 - \mathbb{P}(\mathcal{E}_1) - \mathbb{P}(\mathcal{E}_2) - 2e^{-10n_0}$.
Here $\mathcal{E}_1, \mathcal{E}_2$ are the events
\[ \mathcal{E}_1 := \{\|\bar{A}_d^T \epsilon\| > 15 \sqrt{n_0/n_d} \|\epsilon\|\} \quad \text{and} \quad \mathcal{E}_2 := \{\max(\mathbb{L}_{\bar{A}_i}, \mathbb{L}_{\mathcal{G}_i}) > 3 \text{ for all } i \in [d]\} \]
where $\mathbb{L}_{\mathcal{G}_i}$ and $\mathbb{L}_{\bar{A}_i}$ denote the Lipschitz constants of $\mathcal{G}_i, \bar{A}_i: \mathbb{R}^{n_0} \rightarrow \mathbb{R}^{n_i}$ respectively.
Proof. For $x \in \mathbb{R}^{n_0}$, denote $x_0 = x$ and $x_i = G_i(x)$ for $i \in [d]$. Then

\[
x^{k+1} - x^* = x^k - x^* - 2d \tilde{A}_i^T [G(x^k) - G(x^*) - \epsilon]
\]

\[
= (x_i - x_0) - 2 \tilde{A}_i^T (x^k - x_1)
\]

\[
+ 2 \tilde{A}_i^T [(x^k_1 - x_1) - 2 \tilde{A}_2^T (x_2 - x_2)]
\]

\[
+ \ldots
\]

\[
+ 2^{d-1} \tilde{A}_{d-1}^T [(x_{d-1}^k - x_{d-1}^*) - 2 \tilde{A}_d^T (x_d^k - x_d^*)]
\]

\[
+ 2^d \tilde{A}_d^T \epsilon
\]

thus we can write

\[
\|x^{k+1} - x^*\| = \sup_{u \in S^{n_0-1}} \sum_{i=0}^{d-1} \left(2^i + \sup_{u \in S^{n_0-1}} Z_{i+1} \left( \tilde{A}_i u, x_i^k \right) \right)
\]

where

\[
Z_j(u, v) := \langle A_{j} u, \sigma(A_{j} v) - \sigma(A_{j} x_{j-1}^*) \rangle - \frac{1}{2} \langle u, v - x_{j-1}^* \rangle, \quad j \in [d].
\]

On event $\mathcal{E}_2$, $\forall i \in [d - 1]$ we have

\[
\tilde{A}_i S^{n_0-1} \subseteq \text{range}(\tilde{A}_i) \cap B(n_0, 0, 3) =: T^i_1.
\]

\[
x_i^k \in \text{range}(\tilde{G}_i) \cap B(x_i^*, 3r) =: T^i_2.
\]

By Lemma 7 there are $N_{G_i}$ many $n_0$-dimensional affine subspaces $\{S_{i,j}\}$ such that

\[
T^i_2 \subseteq \bigcup_{j \in [N_{G_i}]} T^i_{2,j} \quad \text{where} \quad T^i_{2,j} = S_{i,j} \cap B(x_i^*, 3r) \subseteq \mathbb{R}^{n_i} \quad \text{and} \quad N_{G_i} \leq \Phi_i := \prod_{j=1}^{i} \left( \left( \frac{en_j}{n_0} \right) \right)^{n_0}.
\]

For $i \in [d - 1]$, apply Lemma on $T^i_1 \times T^i_{2,j}$ followed by a union bound over $j \in [N_{G_i}]$, we get

\[
\sup_{T^i_1 \times T^i_{2,j}} Z_{i+1}(u, v) \leq C_1(2r) \left( \sqrt{\frac{n_0}{n_{i+1}}} + \sqrt{\frac{n_0}{n_{i+1}}} + \sqrt{\frac{t_{i+1}}{n_{i+1}}} + \sqrt{\frac{t_{i+1}}{n_{i+1}}} \right)
\]

with probability (over $A_{i+1}$ and conditioning on $\{A_j\}_{j \in [i]}$) at least $1 - \Phi_i e^{-t_{i+1}}$.

Choose $t_{i+1} = 2 \log \Phi_i = 2n_0 \sum_{j=1}^{i} \log \left( \frac{en_j}{n_0} \right)$, then we get

\[
\mathbb{P}_{A_{i+1}} \left( \sup_{T^i_1 \times T^i_{2,j}} Z_{i+1}(u, v) \leq 9C_1 r \cdot 4 \sqrt{2 \log \Phi_i \frac{n_{i+1}}{n_{i+1}}} \right) \geq 1 - e^{-\log \Phi_i}, \quad \forall i \in [d - 1].
\]

Also for $i = 0$, applying Lemma on $B^{n_0}(0, 1) \times B(x^*, r)$, we get

\[
\sup_{u \in B^{n_0}(0, 1)} Z_1(u, v) \leq C_1 r \cdot 4 \sqrt{10n_0 \frac{10n_0}{n_1}}
\]

with probability (over $A_1$) at least $1 - e^{-10n_0}$.

Therefore under assumption A3 (with $C_0 \geq 160 \cdot 722\cdot C_2^2$), we have

\[
\sum_{i=0}^{d-1} 2^{i+1} \sup_{u \in S^{n_0-1}} Z_{i+1} \left( \tilde{A}_i u, x_i^k \right) \leq \frac{r}{72} + \sum_{i=1}^{d-1} 2^{i+1}, \frac{r}{2} \sqrt{\frac{2}{160 \cdot 5^{i+1}}}
\]
\[
\begin{align*}
&= \frac{r}{72} + \frac{r}{2} \cdot \frac{1}{10} \sum_{i=1}^{d-1} \left( \frac{2}{\sqrt{5}} \right)^i \\
&< \frac{r}{2} \cdot \frac{1}{10} \sum_{i=0}^{\infty} \left( \frac{2}{\sqrt{5}} \right)^i \\
&< \frac{r}{2}
\end{align*}
\]

with probability at least \(1 - \mathbb{P}(\mathcal{E}_2) - e^{-10n_0} - \sum_{i=1}^{d-1} e^{-\log \Phi_i}\).

The result follows by noting that (assume \(C_0 \geq 160 \cdot 72^2\))

\[
\log \Phi_i = n_0 \sum_{j=1}^{i} \log \left( \frac{en_j}{n_0} \right) \geq n_0 \log (eC_0) > 11n_0i,
\]

so \(\sum_{i \geq 1} e^{-\log \Phi_i} \leq \frac{e^{-11n_0}}{1-e^{-11n_0}} < e^{-10n_0}\). Also note that on \(\mathcal{E}_1^c\),

\[
2^d \|A_d^T \epsilon\| \leq 15 \cdot 2^d \sqrt{n_0/nd} \|\epsilon\|.
\]

\[\square\]

Lemma 6. Under assumptions A2-A4, we have

\[
\mathbb{P} \left( \|A_1^T A_2^T \cdots A_d^T \epsilon\| \geq 15 \sqrt{\frac{n_0}{nd}} \|\epsilon\| \right) \leq 3e^{-10n_0}.
\]

Proof. Denote \(s_i := A_i^T \epsilon\) for \(i \in [d-1]\), and \(s_d := \epsilon\).

For \(i \in [d]\), by Lemma 3(a) we have

\[
\mathbb{P}_{A_i} \left( \|s_i\| \leq \sqrt{n_{i-1}} \|s_{i-1}\| + \|s_{i}\| \right) \geq 1 - e^{-t_i/2}, \quad \forall t_i > 0.
\]

Choose \(t_1 = 20n_0\) and \(t_j = n_{j-1}/4^{j-1}\) for \(j > 1\), we get

\[
\mathbb{P}_{A_i} \left( \|A_1^T s_1\| \leq (1 + \sqrt{20}) \sqrt{\frac{n_0}{n_1}} \|s_1\| \right) \geq 1 - e^{-10n_0},
\]

\[
\mathbb{P}_{A_i} \left( \|A_i^T s_i\| \leq (1 + 2^{-i+1}) \sqrt{\frac{n_{i-1}}{n_i}} \|s_i\| \right) \geq 1 - e^{-n_{i-1}/4^i}, \quad i > 1.
\]

Thus with probability at least \(1 - e^{-10n_0} - \sum_{i=2}^d e^{-n_{i-1}/4^i}\),

\[
\|A_1^T A_2^T \cdots A_d^T \epsilon\| \leq \left(1 + \sqrt{20}\right) \sqrt{\frac{n_0}{n_1}} \prod_{i=2}^d \left(1 + \frac{1}{2^{i-1}}\right) \sqrt{\frac{n_{i-1}}{n_i}}
\]

\[
\leq \left(1 + \sqrt{20}\right) \frac{n_0}{n_d} \prod_{i=1}^\infty \left(1 + \frac{1}{2^i}\right)
\]

\[
< 15 \sqrt{n_0/n_d}
\]

where the last inequality uses estimate \(\prod_{i=1}^\infty \left(1 + \frac{1}{2^i}\right) \leq e\) and \((1 + \sqrt{20})e < 15\).

It remains to show \(\sum_{i=2}^d e^{-n_{i-1}/4^i} \leq 2e^{-10n_0}\) for the desired probability bound. Note that by assumption A3 (assume \(C_0 \geq 40\),

\[
\frac{n_i}{4^{i+1}} \geq \frac{1}{4} C_0 n_0 \sum_{j=0}^{i-1} \log \left( \frac{en_j}{n_0} \right) \geq 10n_0i.
\]

Hence

\[
\sum_{i=2}^d e^{-n_{i-1}/4^i} \leq \sum_{i=2}^d e^{-10n_0(i-1)} < \sum_{i=1}^\infty e^{-10n_0i} = \frac{e^{-10n_0}}{1 - e^{-10n_0}} < 2e^{-10n_0}.
\]

\[\square\]

\(^6\)For \(\alpha > 0\), estimate \(\sum_{j=1}^\infty \log \left(1 + \alpha 2^{-j}\right) \leq \sum_{j=1}^\infty \alpha 2^{-j} = \alpha\) holds, thus \(\prod_{i=1}^\infty \left(1 + \frac{\alpha}{2^i}\right) \leq e^\alpha\).
With ReLU (or positively homogeneous) activation functions, the range of neural network (in each layer) is contained in a union of affine subspaces. The following lemma, which is based on ideas and results in [1], gives a precise statement of this.

**Lemma 7.** Assume $A_1$ and $\min_{j \in [d]} \{ n_j \} \geq n_0$, then for $i \in [d]$, $\text{range}(G_i)$ is contained in a union of affine subspaces. Precisely,

$$\text{range}(G_i) \subseteq \bigcup_{j \in [N_{G_i}]} S_{i,j}$$

where $N_{G_i} \leq \prod_{j=1}^{i} \left( \frac{en_j}{n_0} \right)^{n_0}$. Here each $S_{i,j}$ is some $n_0$-dimensional affine subspace (which depends on $\{A_i\}_{i \in [d]}$) in $\mathbb{R}^{n_i}$.

**Proof.** The theory on hyperplane arrangements [25 Chapter 6.1] tells us that $n$ hyperplanes in $\mathbb{R}^k$ (assume $n \geq k$) partition the space $\mathbb{R}^k$ into at most $\binom{n}{k}$ regions.

Also for $k \in [n]$,

$$\sum_{j=0}^{k} \binom{n}{j} \leq \sum_{j=0}^{k} \frac{n^j}{j!} \leq \sum_{j=0}^{k} \frac{k^j}{j!} \left( \frac{n}{k} \right)^j \leq \left( \frac{n}{k} \right)^k \sum_{j=0}^{\infty} \frac{k^j}{j!} = \left( \frac{en}{k} \right)^k.$$

So consider $\text{range}(G_1) = \{ \sigma(A_1 x) : x \in \mathbb{R}^{n_0} \}$. Denote by $a_1^j$ ($j \in [n_1]$) the rows of $A_1$ and let $H$ be the set of hyperplanes $H := \bigcup_{j \in [n_1]} \{ x : a_1^j x = 0 \}$. Then $H$ partitions $\mathbb{R}^{n_0}$ into at most $(en_1/n_0)^{n_0}$ regions. Note that $\sigma$ is linear in each of these regions (thus the mapping $G_1$ is linear in each region), so $\text{range}(G_1)$ is contained in at most $(en_1/n_0)^{n_0}$ many $n_0$-dimensional (affine) subspaces.

The result then follows by induction. \hfill \Box

The following lemma shows that the network $G$ in our model is Lipschitz with high probability. This may be an interesting result on its own.

**Lemma 8.** For mappings $\hat{G}_i, \hat{A}_i : \mathbb{R}^{n_0} \rightarrow \mathbb{R}^{n_i}$, let $L_{\hat{G}_i}$ and $L_{\hat{A}_i}$ be their Lipschitz constants respectively. Under assumptions $A1$-$A3$, we have

$$\mathbb{P} \left( \max\{ L_{\hat{A}_i}, L_{\hat{G}_i} \} \leq 3 \text{ for all } i \in [d] \right) \geq 1 - 3e^{-10n_0}.$$

**Proof.** Denote $\hat{R}_0 = R_0 = \mathbb{R}^{n_0}$ and

$$\hat{R}_j = \text{range}(G_j) - \text{range}(G_j), \quad \hat{R}_j = R_j \cup \text{range}(\hat{A}_j), \quad j \in [d].$$

Note that $\hat{A}_j$ is linear, so $\text{range}(\hat{A}_j)$ is a subspace in $\mathbb{R}^{n_i}$ with dimension at most $n_0$.

Since $\sigma$ is 1-Lipschitz, we have

$$\| \hat{G}_i(x) - \hat{G}_i(x') \| = \| \sigma(A_i \hat{G}_{i-1}(x)) - \sigma(A_i \hat{G}_{i-1}(x')) \| \\
\leq \| A_i (\hat{G}_{i-1}(x) - \hat{G}_{i-1}(x')) \| \\
\leq \| A_i \| \| \hat{G}_{i-1}(x) - \hat{G}_{i-1}(x') \| .$$

Hence

$$\| \hat{G}_i(x) - \hat{G}_i(x') \| \leq \left( \prod_{l=1}^{i} \| A_l \| \hat{R}_{i-1} \right) \| x - x' \|, \quad \forall i \in [d].$$

Similarly,

$$\| \hat{A}_i x - \hat{A}_i x' \| \leq \left( \prod_{l=1}^{i} \| A_l \| \hat{R}_{i-1} \right) \| x - x' \|, \quad \forall i \in [d].$$

By Lemma 7, $\text{range}(G_i)$ is contained in a union of $N_{G_i}$ many $n_0$-dimensional affine subspaces, so $\hat{R}_i$ is contained in a union of at most $N_{G_i}^2$ many $2n_0$-dimensional affine subspaces. Since every

---

7Such regions are also called $k$-faces or $k$-cells. Relative to each of the $n$ hyperplanes, all points inside a region are on the same side.
2\(n_0\)-dimensional affine subspaces in \(\mathbb{R}^{n_i}\) is also contained in a \((2n_0 + 1)\)-dimensional subspace, we can further write this as

\[
\tilde{R}_i = \mathcal{R}_i \cup \text{range}(\tilde{A}_i) \subseteq \bigcup_{j \in [N_{G_i}^2 + 1]} S_{i,j} \quad \text{where} \quad N_{G_i} \leq \Phi_i := \prod_{j=1}^{i} \left( \frac{en_j}{n_0} \right)^{n_0},
\]

and each \(S_{i,j}\) is a \((2n_0 + 1)\)-dimensional subspace in \(\mathbb{R}^{n_i}\).

Thus by Lemma 3(b) and union bound we have, for \(i \leq d - 1\),

\[
P_{A_{i+1}} \left( \|A_{i+1}\|_{\tilde{R}_i} \geq \sqrt{n_{i+1}} + \sqrt{2n_0 + 1 + \sqrt{t_i}} \right) \leq \left( \Phi_i^2 + 1 \right) e^{-t_i/2}, \quad \forall t_i > 0.
\]

Choose \(t_i = 26 \log \Phi_i = 26n_0 \sum_{j=1}^{i} \log \left( \frac{en_j}{n_0} \right) > 2n_0 + 1\) we get

\[
P_{A_{i+1}} \left( \|A_{i+1}\|_{\tilde{R}_i} \geq 1 + 2 \sqrt{ \frac{26 \log \Phi_i}{n_{i+1}} } \right) \leq e^{-10 \log \Phi_i}.
\]

Under assumption A3 (with \(C_0 \geq 2^2 \cdot 26\)), this implies

\[
P_{A_{i+1}} \left( \|A_{i+1}\|_{\tilde{R}_i} \geq 1 + \frac{1}{2^{i+1}} \right) \leq e^{-10 \log \Phi_i}, \quad i \in [d - 1].
\]

Also by Lemma 3(b) with \(t = 20n_0\) and assumption A3 (assume \(C_0 \geq 2^2 \cdot 26\), we have

\[
P_{A_{i+1}} \left( \|A_{i+1}\|_{R_i} \geq 1 + \frac{1}{2} \right) \leq e^{-10n_0}.
\]

Therefore with probability at least \(1 - e^{-10n_0} - \sum_{i=1}^{d-1} e^{-10 \log \Phi_i}\),

\[
\forall i \in [d], \quad \prod_{l=1}^{i} \|A_l\|_{\tilde{R}_{i-1}} \leq \prod_{l=1}^{i} \left( 1 + \frac{1}{2^l} \right) \leq \prod_{l=1}^{\infty} \left( 1 + \frac{1}{2^l} \right) < 3.
\]

Finally, note that \(\log \Phi_i \geq in_0\), so we have \(\sum_{i=1}^{d-1} e^{-10 \log \Phi_i} \leq \sum_{i=1}^{\infty} e^{-10n_0i} < 2e^{-10n_0}\). This completes the proof. \[\square\]

## D An Example of \(n_i\)

Here we show if \(n_i = \beta C_0 5^d n_0 d(2d - i)\) where \(\beta\) is any fixed number such that \(\beta C_0 \in \mathbb{N}\) and \(\beta \geq 4 + \log C_0\), then \(n_i\) satisfy \(\mathbb{F}_4\).

In fact, note that \(2 \log d < d\) and \(\log(2\beta) < \beta\), we have

\[
\log \left( \prod_{j=0}^{i-1} \frac{en_j}{n_0} \right) = 1 + \log \left( \frac{en_i}{n_0} \right)
\]

\[
\leq 1 + (d-1) \log (e\beta C_0 5^d \cdot 2d^2) = 1 + (d-1)(d \log 5 + 2 \log d + \log(eC_0)) + (d - 1) \log(2\beta)
\]

\[
< 1 + d(d-1)(\log 5 + 1 + \log(eC_0)) + (d - 1)\beta \leq \beta + d(d-1)\beta + (d - 1)\beta = \beta d^2.
\]

Since \(n_i \geq C_0 5^d n_0 (\beta d^2)\), it is easy to see that \(n_i\) satisfy \(\mathbb{F}_4\).

**Remark:** A similar argument as above can also show that \(n_i = \beta C_0 5^i n_0^2\) satisfy \(\mathbb{F}_4\).

## E Code Link

Codes for numerical experiments are available at https://github.com/babhrujoshi/PLUGIn.
F  NeurIPS Paper Checklist

1. For all authors...
   (a) Do the main claims made in the abstract and introduction accurately reflect the paper’s contributions and scope? [Yes]
   (b) Did you describe the limitations of your work? [Yes] In Section 5
   (c) Did you discuss any potential negative societal impacts of your work? [N/A] We are not aware of such impact.
   (d) Have you read the ethics review guidelines and ensured that your paper conforms to them? [Yes]

2. If you are including theoretical results...
   (a) Did you state the full set of assumptions of all theoretical results? [Yes]
   (b) Did you include complete proofs of all theoretical results? [Yes] Proofs are included in appendices.

3. If you ran experiments...
   (a) Did you include the code, data, and instructions needed to reproduce the main experimental results (either in the supplemental material or as a URL)? [Yes] See Appendix E.
   (b) Did you specify all the training details (e.g., data splits, hyperparameters, how they were chosen)? [Yes] See Section 4.
   (c) Did you report error bars (e.g., with respect to the random seed after running experiments multiple times)? [Yes] See Section 4.
   (d) Did you include the total amount of compute and the type of resources used (e.g., type of GPUs, internal cluster, or cloud provider)? [Yes] We used Google Colaboratory to conduct the experiments included the paper, see Section 4.

4. If you are using existing assets (e.g., code, data, models) or curating/releasing new assets...
   (a) If your work uses existing assets, did you cite the creators? [Yes] We used MNIST dataset [29], which is cited in the paper.
   (b) Did you mention the license of the assets? [N/A] MNIST dataset is made available under the terms of the Creative Commons Attribution-Share Alike 3.0 license.
   (c) Did you include any new assets either in the supplemental material or as a URL? [N/A]
   (d) Did you discuss whether and how consent was obtained from people whose data you’re using/curating? [N/A] MNIST dataset is made available under the terms of the Creative Commons Attribution-Share Alike 3.0 license.
   (e) Did you discuss whether the data you are using/curating contains personally identifiable information or offensive content? [N/A]

5. If you used crowdsourcing or conducted research with human subjects...
   (a) Did you include the full text of instructions given to participants and screenshots, if applicable? [N/A]
   (b) Did you describe any potential participant risks, with links to Institutional Review Board (IRB) approvals, if applicable? [N/A]
   (c) Did you include the estimated hourly wage paid to participants and the total amount spent on participant compensation? [N/A]