Supplementary material:
Global Convergence of Online Optimization for Nonlinear Model Predictive Control

A Expression of Newton System

For future references, we explicitly write out each component of (3). For stage k, we let $H_k(\boldsymbol{z}_k, \boldsymbol{\lambda}_k) = \nabla^2_{\boldsymbol{z}_k}(g_k(\boldsymbol{z}_k) - \boldsymbol{\lambda}_k^T f_k(\boldsymbol{z}_k))$, $A_k(\boldsymbol{z}_k) = \nabla^T_{\boldsymbol{x}_k} f_k(\boldsymbol{z}_k)$ and $B_k(\boldsymbol{z}_k) = \nabla^T_{\boldsymbol{u}_k} f_k(\boldsymbol{z}_k)$. Then, we have

$$H^t(\tilde{\boldsymbol{z}}_t, \tilde{\boldsymbol{\lambda}}_t) = \text{diag}\left(H_t, \ldots, H_{M_t-1}, \nabla^2_{\boldsymbol{x}_{M_t}} g_{M_t}(\boldsymbol{x}_{M_t}, \boldsymbol{0}) + \mu I\right) \tag{12}$$

with $H_k = H_k(\boldsymbol{z}_k, \boldsymbol{\lambda}_k)$ for $k \in [t, M_t - 1]$, and have

$$G^t(\tilde{\boldsymbol{z}}_t) = \begin{pmatrix} I & & & & & & \\ -A_t & -B_t & I & & & & \\ & & -A_{t+1} & -B_{t+1} & I & & \\ & & & & \ddots & \ddots & \\ & & & & -A_{M_t-1} & -B_{M_t-1} & I \end{pmatrix} \tag{13}$$

with $A_k = A_k(\boldsymbol{z}_k)$ and $B_k = B_k(\boldsymbol{z}_k)$. The gradient of Lagrangian $\mathcal{L}^t(\cdot)$ on the right side of (3) can be expressed as

$$\begin{aligned} \nabla_{\tilde{\boldsymbol{z}}_t}\mathcal{L}^t(\tilde{\boldsymbol{z}}_t, \tilde{\boldsymbol{\lambda}}_t; \bar{\boldsymbol{x}}_t) &= \begin{pmatrix} \nabla_{\boldsymbol{x}_t} g_t(\boldsymbol{z}_t) + \boldsymbol{\lambda}_{t-1} - A_t^T(\boldsymbol{z}_t)\boldsymbol{\lambda}_t \\ \nabla_{\boldsymbol{u}_t} g_t(\boldsymbol{z}_t) - B_t^T(\boldsymbol{z}_t)\boldsymbol{\lambda}_t \\ \vdots \\ \nabla_{\boldsymbol{x}_{M_t-1}} g_{M_t-1}(\boldsymbol{z}_{M_t-1}) + \boldsymbol{\lambda}_{M_t-2} - A^T_{M_t-1}(\boldsymbol{z}_{M_t-1})\boldsymbol{\lambda}_{M_t-1} \\ \nabla_{\boldsymbol{u}_{M_t-1}} g_{M_t-1}(\boldsymbol{z}_{M_t-1}) - B^T_{M_t-1}(\boldsymbol{z}_{M_t-1})\boldsymbol{\lambda}_{M_t-1} \\ \nabla_{\boldsymbol{x}_{M_t}} g_{M_t}(\boldsymbol{x}_{M_t}, \boldsymbol{0}) + \boldsymbol{\lambda}_{M_t-1} + \mu \boldsymbol{x}_{M_t} \end{pmatrix}, \\ \nabla_{\tilde{\boldsymbol{\lambda}}_t}\mathcal{L}^t(\tilde{\boldsymbol{z}}_t, \tilde{\boldsymbol{\lambda}}_t; \bar{\boldsymbol{x}}_t) &= \begin{pmatrix} \boldsymbol{x}_t - \bar{\boldsymbol{x}}_t \\ \boldsymbol{x}_{t+1} - f_t(\boldsymbol{z}_t) \\ \vdots \\ \boldsymbol{x}_{M_t} - f_{M_t-1}(\boldsymbol{z}_t) \end{pmatrix}. \end{aligned} \tag{14}$$

We also explicitly write out the gradient of the augmented Lagrangian (5) by

$$\begin{pmatrix} \nabla_{\tilde{\boldsymbol{z}}_t}\mathcal{L}^t_\eta \\ \nabla_{\tilde{\boldsymbol{\lambda}}_t}\mathcal{L}^t_\eta \end{pmatrix} = \begin{pmatrix} I + \eta_2 H^t & \eta_1 (G^t)^T \\ \eta_2 G^t & I \end{pmatrix} \begin{pmatrix} \nabla_{\tilde{\boldsymbol{z}}_t}\mathcal{L}^t \\ \nabla_{\tilde{\boldsymbol{\lambda}}_t}\mathcal{L}^t \end{pmatrix}. \tag{15}$$

B Proof of Theorem 4.4

We first have a simple observation: by Assumptions 4.1, 4.2, for any $(\tilde{\boldsymbol{z}}_t, \tilde{\boldsymbol{\lambda}}_t) \in \mathcal{Z} \otimes \Lambda$ (by $(\tilde{\boldsymbol{z}}_t, \tilde{\boldsymbol{\lambda}}_t) \in \mathcal{Z} \otimes \Lambda$ we mean $(\tilde{\boldsymbol{z}}_{k,t}, \tilde{\boldsymbol{\lambda}}_{k,t}) \in \mathcal{Z} \times \Lambda$ for all stages k of the t-th subproblem), $\|G^t(\tilde{\boldsymbol{z}}_t)\| \leq 1 + 2\Upsilon$, $\|H^t(\tilde{\boldsymbol{z}}_t, \tilde{\boldsymbol{\lambda}}_t)\| \leq \Upsilon' + \mu$, and

$$\|\nabla((G^t)^T \nabla_{\tilde{\boldsymbol{\lambda}}_t}\mathcal{L}^t)(\tilde{\boldsymbol{z}}_t, \tilde{\boldsymbol{\lambda}}_t; \bar{\boldsymbol{x}}_t)\| \leq \Upsilon', \qquad \|\nabla(H^t \nabla_{\tilde{\boldsymbol{z}}_t}\mathcal{L}^t)(\tilde{\boldsymbol{z}}_t, \tilde{\boldsymbol{\lambda}}_t; \bar{\boldsymbol{x}}_t)\| \leq \Upsilon' + \mu^2 \tag{16}$$

for some constant Υ' not depending on μ. This is from the definitions (12)-(14) and noting that only the last block of H^t and the last row of $\nabla_{\tilde{\boldsymbol{z}}_t}\mathcal{L}^t$ contain μ. We can also replace Υ in Assumption 4.2 by $\Upsilon \leftarrow (1 + 2\Upsilon) \vee \Upsilon' \vee \delta$ and require $\mu \geq \Upsilon$. Then we have $\|G^t\| \leq \Upsilon$, $\|B^t\| \vee \|H^t\| \leq 2\mu$, $\|\nabla((G^t)^T \nabla_{\tilde{\boldsymbol{\lambda}}_t}\mathcal{L}^t)\| \leq \Upsilon$, and $\|\nabla(H^t \nabla_{\tilde{\boldsymbol{z}}_t}\mathcal{L}^t)\| \leq 2\mu^2$. By the definition of H^t in (12), without loss of generality we let the last block of B^t be μI.

We then provide a formula for the KKT matrix inverse. We suppress the index t since the results hold for any $t \geq 0$.

Lemma B.1. Let $G^T = YK$ where Y has orthonormal columns that span $\text{Im}(G^T)$ and K is a nonsingular square matrix (since G^T has full column rank), and let Z have orthonormal columns that span Ker(G). If $Z^T B Z$ is invertible, then

$$S := \begin{pmatrix} B & G^T \\ G & \boldsymbol{0} \end{pmatrix}^{-1} = \begin{pmatrix} S_1 & S_2^T \\ S_2 & S_3 \end{pmatrix}$$

where

$$S_1 = Z(Z^TBZ)^{-1}Z^T,$$
$$S_2 = K^{-1}Y^T(I - BZ(Z^TBZ)^{-1}Z^T),$$
$$S_3 = K^{-1}Y^T(BZ(Z^TBZ)^{-1}Z^TB - B)YK^{-1}.$$

Under Assumption 4.2, we have $\|S\| \le 5\Upsilon^2\mu^2/\gamma_{RH}$.

Given Lemma B.1, we apply (3) and (15) and have

$$\begin{pmatrix} \nabla_{\tilde{\boldsymbol{z}}}\mathcal{L}_\eta^0 \\ \nabla_{\tilde{\boldsymbol{\lambda}}}\mathcal{L}_\eta^0 \end{pmatrix}^T \begin{pmatrix} \Delta\tilde{\boldsymbol{z}} \\ \Delta\tilde{\boldsymbol{\lambda}} \end{pmatrix} = -\begin{pmatrix} \nabla_{\tilde{\boldsymbol{z}}}\mathcal{L}^0 \\ \nabla_{\tilde{\boldsymbol{\lambda}}}\mathcal{L}^0 \end{pmatrix}^T \begin{pmatrix} B & G^T \\ G & \mathbf{0} \end{pmatrix}^{-1} \begin{pmatrix} I + \eta_2 H & \eta_1 G^T \\ \eta_2 G & I \end{pmatrix} \begin{pmatrix} \nabla_{\tilde{\boldsymbol{z}}}\mathcal{L}^0 \\ \nabla_{\tilde{\boldsymbol{\lambda}}}\mathcal{L}^0 \end{pmatrix}.$$

By Lemma B.1, we define $W = I - Z(Z^TBZ)^{-1}Z^TB$ and have

$$\begin{aligned} &\begin{pmatrix} B & G^T \\ G & \mathbf{0} \end{pmatrix}^{-1} \begin{pmatrix} I + \eta_2 H & \eta_1 G^T \\ \eta_2 G & I \end{pmatrix} \\ &\qquad = \begin{pmatrix} \eta_2 I + Z(Z^TBZ)^{-1}Z^T\{I + \eta_2(H - B)\} & WY(K^{-1})^T \\ K^{-1}Y^TW^T\{I + \eta_2(H - B)\} & \eta_1 I - K^{-1}Y^TBWY(K^{-1})^T \end{pmatrix} \\ &\qquad =: W_1 + W_2 + W_3, \end{aligned} \tag{17}$$

where

$$W_1 = \begin{pmatrix} \frac{\eta_2}{2} I & \mathbf{0} \\ \mathbf{0} & \frac{\eta_1}{2} I \end{pmatrix},$$
$$W_2 = \begin{pmatrix} \frac{\eta_2}{2} I & WY(K^{-1})^T \\ K^{-1}Y^TW^T & \frac{\eta_1}{2} I - K^{-1}Y^TBWY(K^{-1})^T \end{pmatrix},$$
$$W_3 = \begin{pmatrix} Z(Z^TBZ)^{-1}Z^T\{I + \eta_2(H - B)\} & \mathbf{0} \\ \eta_2 K^{-1}Y^TW^T(H - B) & \mathbf{0} \end{pmatrix}.$$

We deal with each term separately. First, we have

$$\begin{aligned} &\begin{pmatrix} \nabla_{\tilde{\boldsymbol{z}}}\mathcal{L}^0 \\ \nabla_{\tilde{\boldsymbol{\lambda}}}\mathcal{L}^0 \end{pmatrix}^T W_3 \begin{pmatrix} \nabla_{\tilde{\boldsymbol{z}}}\mathcal{L}^0 \\ \nabla_{\tilde{\boldsymbol{\lambda}}}\mathcal{L}^0 \end{pmatrix} \\ &\qquad = \nabla_{\tilde{\boldsymbol{z}}}^T\mathcal{L}^0 Z(Z^TBZ)^{-1}Z^T\nabla_{\tilde{\boldsymbol{z}}}\mathcal{L}^0 + \eta_2\nabla_{\tilde{\boldsymbol{z}}}^T\mathcal{L}^0 Z(Z^TBZ)^{-1}Z^T(H - B)\nabla_{\tilde{\boldsymbol{z}}}\mathcal{L}^0 \\ &\qquad\quad + \eta_2\nabla_{\tilde{\boldsymbol{\lambda}}}^T\mathcal{L}^0 K^{-1}Y^TW^T(H - B)\nabla_{\tilde{\boldsymbol{z}}}\mathcal{L}^0 \\ &\qquad = (\Delta\tilde{\boldsymbol{z}})^TBZ(Z^TBZ)^{-1}Z^TB\Delta\tilde{\boldsymbol{z}} - \eta_2(\Delta\tilde{\boldsymbol{z}})^TBZ(Z^TBZ)^{-1}Z^T(H - B)\nabla_{\tilde{\boldsymbol{z}}}\mathcal{L}^0 \\ &\qquad\quad - \eta_2(\Delta\tilde{\boldsymbol{z}})^TYY^TW^T(H - B)\nabla_{\tilde{\boldsymbol{z}}}\mathcal{L}^0 \\ &\qquad = (\Delta\tilde{\boldsymbol{z}})^TBZ(Z^TBZ)^{-1}Z^TB\Delta\tilde{\boldsymbol{z}} - \eta_2(\Delta\tilde{\boldsymbol{z}})^T(I - W^T)(H - B)\nabla_{\tilde{\boldsymbol{z}}}\mathcal{L}^0 \\ &\qquad\quad - \eta_2(\Delta\tilde{\boldsymbol{z}})^TYY^TW^T(H - B)\nabla_{\tilde{\boldsymbol{z}}}\mathcal{L}^0 \\ &\qquad = (\Delta\tilde{\boldsymbol{z}})^TBZ(Z^TBZ)^{-1}Z^TB\Delta\tilde{\boldsymbol{z}} - \eta_2(\Delta\tilde{\boldsymbol{z}})^T(H - B)\nabla_{\tilde{\boldsymbol{z}}}\mathcal{L}^0. \end{aligned} \tag{18}$$

Here, the second equality is due to the KKT system (3) and the fact that $GZ = \mathbf{0}$; the third equality is due to the definition of W; and the fourth equality is due to $YY^TW^T = W^T$. Let us decompose $\Delta\tilde{\boldsymbol{z}} = \Delta\tilde{\boldsymbol{v}} + \Delta\tilde{\boldsymbol{u}}$, where $\Delta\tilde{\boldsymbol{v}} = Z\Delta\boldsymbol{v}$ is a vector in Im(Z), and $\Delta\tilde{\boldsymbol{u}} = G^T\Delta\boldsymbol{u}$ is a vector in Im(G^T). Since $G\Delta\tilde{\boldsymbol{z}} = -\nabla_{\tilde{\boldsymbol{\lambda}}}\mathcal{L}^0$ from (3), we know $\Delta\boldsymbol{u} = -(GG^T)^{-1}\nabla_{\tilde{\boldsymbol{\lambda}}}\mathcal{L}^0$ and hence $\Delta\tilde{\boldsymbol{u}} = -G^T(GG^T)^{-1}\nabla_{\tilde{\boldsymbol{\lambda}}}\mathcal{L}^0 = -Y(K^{-1})^T\nabla_{\tilde{\boldsymbol{\lambda}}}\mathcal{L}^0$. Plugging the decomposition into (18), we have

$$\begin{aligned} &\begin{pmatrix} \nabla_{\tilde{\boldsymbol{z}}}\mathcal{L}^0 \\ \nabla_{\tilde{\boldsymbol{\lambda}}}\mathcal{L}^0 \end{pmatrix}^T W_3 \begin{pmatrix} \nabla_{\tilde{\boldsymbol{z}}}\mathcal{L}^0 \\ \nabla_{\tilde{\boldsymbol{\lambda}}}\mathcal{L}^0 \end{pmatrix} \\ &\qquad = (\Delta\boldsymbol{v})^TZ^TBZ\Delta\boldsymbol{v} - 2(\Delta\boldsymbol{v})^TZ^TBY(K^{-1})^T\nabla_{\tilde{\boldsymbol{\lambda}}}\mathcal{L}^0 - \eta_2(\Delta\tilde{\boldsymbol{z}})^T(H - B)\nabla_{\tilde{\boldsymbol{z}}}\mathcal{L}^0 \\ &\qquad\quad + \nabla_{\tilde{\boldsymbol{\lambda}}}^T\mathcal{L}^0K^{-1}Y^TBZ(Z^TBZ)^{-1}Z^TBY(K^{-1})^T\nabla_{\tilde{\boldsymbol{\lambda}}}\mathcal{L}^0 \\ &\qquad \ge \gamma_{RH}\|\Delta\boldsymbol{v}\|^2 - 4\mu\Upsilon\|\Delta\boldsymbol{v}\|\|\nabla_{\tilde{\boldsymbol{\lambda}}}\mathcal{L}^0\| - \eta_2\delta\|\Delta\tilde{\boldsymbol{z}}\|\|\nabla_{\tilde{\boldsymbol{z}}}\mathcal{L}^0\| \end{aligned}$$

$$
\begin{aligned}
&\geq \frac{\gamma_{RH}}{2}\|\Delta \boldsymbol{v}\|^2 - \frac{8\mu^2\Upsilon^2}{\gamma_{RH}}\|\nabla_{\tilde{\boldsymbol{\lambda}}}\mathcal{L}^0\|^2 - \eta_2\delta^2\|\Delta\tilde{\boldsymbol{z}}\|^2 - \frac{\eta_2}{4}\|\nabla_{\tilde{\boldsymbol{z}}}\mathcal{L}^0\|^2 \\
&= \frac{\gamma_{RH}}{2}\|\Delta \boldsymbol{v}\|^2 - \frac{8\mu^2\Upsilon^2}{\gamma_{RH}}\|\nabla_{\tilde{\boldsymbol{\lambda}}}\mathcal{L}^0\|^2 - \eta_2\delta^2(\|\Delta\boldsymbol{v}\|^2 + \|\Delta\tilde{\boldsymbol{u}}\|^2) - \frac{\eta_2}{4}\|\nabla_{\tilde{\boldsymbol{z}}}\mathcal{L}^0\|^2 \\
&\geq \left(\frac{\gamma_{RH}}{2} - \eta_2\delta^2\right)\|\Delta\boldsymbol{v}\|^2 - \left(\frac{8\mu^2\Upsilon^2}{\gamma_{RH}} + \eta_2\delta^2\Upsilon^2\right)\|\nabla_{\tilde{\boldsymbol{\lambda}}}\mathcal{L}^0\|^2 - \frac{\eta_2}{4}\|\nabla_{\tilde{\boldsymbol{z}}}\mathcal{L}^0\|^2,
\end{aligned}
$$

where the second and fifth inequalities are due to Assumption 4.2, which implies $\|K^{-1}\| \leq \Upsilon$, $\|B\| \vee \|H\| \leq 2\mu$; the third inequality is due to Young's inequality; and the fourth equality is due to $\|\Delta\tilde{\boldsymbol{z}}\|^2 = \|\Delta\tilde{\boldsymbol{v}}\|^2 + \|\Delta\tilde{\boldsymbol{u}}\|^2 = \|\Delta\boldsymbol{v}\|^2 + \|\Delta\tilde{\boldsymbol{u}}\|^2$. Using the above display and supposing

$$
\frac{\gamma_{RH}}{2} - \eta_2\delta^2 \geq 0 \iff \eta_2 \leq \frac{\gamma_{RH}}{2\delta^2}, \tag{19}
$$

we further have

$$
\begin{aligned}
\begin{pmatrix}\nabla_{\tilde{\boldsymbol{z}}}\mathcal{L}^0 \\ \nabla_{\tilde{\boldsymbol{\lambda}}}\mathcal{L}^0\end{pmatrix}^T W_3 \begin{pmatrix}\nabla_{\tilde{\boldsymbol{z}}}\mathcal{L}^0 \\ \nabla_{\tilde{\boldsymbol{\lambda}}}\mathcal{L}^0\end{pmatrix} &\geq -\left(\frac{8\mu^2\Upsilon^2}{\gamma_{RH}} + \frac{\gamma_{RH}\Upsilon^2}{2}\right)\|\nabla_{\tilde{\boldsymbol{\lambda}}}\mathcal{L}^0\|^2 - \frac{\eta_2}{4}\|\nabla_{\tilde{\boldsymbol{z}}}\mathcal{L}^0\|^2 \\
&\geq -\frac{9\mu^2\Upsilon^2}{\gamma_{RH}}\|\nabla_{\tilde{\boldsymbol{\lambda}}}\mathcal{L}^0\|^2 - \frac{\eta_2}{4}\|\nabla_{\tilde{\boldsymbol{z}}}\mathcal{L}^0\|^2. \qquad (20)
\end{aligned}
$$

Let us now deal with W_2. By Schur complement, in order to show $W_2 \succeq \mathbf{0}$, we only have to let

$$
\frac{\eta_1}{2}I - K^{-1}Y^TBWY(K^{-1})^T - \frac{2}{\eta_2}K^{-1}Y^TW^TWY(K^{-1})^T \succeq \mathbf{0}. \tag{21}
$$

Note that $-K^{-1}Y^TBWY(K^{-1})^T \succeq -K^{-1}Y^TBY(K^{-1})^T$ and

$$
\begin{aligned}
\|K^{-1}Y^TBY(K^{-1})^T + \frac{2}{\eta_2}K^{-1}Y^TW^TWY(K^{-1})^T\| &\leq 2\mu\Upsilon^2 + \frac{2\Upsilon^2}{\eta_2}\|W\|^2 \\
&\leq 2\mu\Upsilon^2 + \frac{2\Upsilon^2}{\eta_2}\left(1 + \frac{2\mu}{\gamma_{RH}}\right)^2 = 2\mu\Upsilon^2 + \frac{2\Upsilon^2}{\eta_2} + \frac{8\mu\Upsilon^2}{\eta_2\gamma_{RH}} + \frac{8\mu^2\Upsilon^2}{\eta_2\gamma_{RH}^2} \\
&\leq \frac{12\mu\Upsilon^2}{\eta_2\gamma_{RH}} + \frac{8\mu^2\Upsilon^2}{\eta_2\gamma_{RH}^2} \leq \frac{16\mu^2\Upsilon^2}{\eta_2\gamma_{RH}^2},
\end{aligned}
$$

where the fifth inequality supposes $\gamma_{RH} \leq \sqrt{2}\delta$ (without loss of generality, since δ is upper bound and γ_{RH} is lower bound in Assumption 4.2) so that $\eta_2\gamma_{RH} \leq 1$; and the last inequality uses $\mu \geq 2\gamma_{RH}$. Thus, we only have to let

$$
\frac{\eta_1}{2} \geq \frac{16\mu^2\Upsilon^2}{\eta_2\gamma_{RH}^2} \iff \eta_1\eta_2 \geq \frac{32\mu^2\Upsilon^2}{\gamma_{RH}^2}, \tag{22}
$$

then (21) is satisfied and $W_2 \succeq \mathbf{0}$. Combining (17), (20), and noting that W_1 is a diagonal matrix, we obtain that under (19) and (22),

$$
\begin{pmatrix}\nabla_{\tilde{\boldsymbol{z}}}\mathcal{L}_\eta^0 \\ \nabla_{\tilde{\boldsymbol{\lambda}}}\mathcal{L}_\eta^0\end{pmatrix}^T \begin{pmatrix}\Delta\tilde{\boldsymbol{z}} \\ \Delta\tilde{\boldsymbol{\lambda}}\end{pmatrix} \leq -\begin{pmatrix}\nabla_{\tilde{\boldsymbol{z}}}\mathcal{L}^0 \\ \nabla_{\tilde{\boldsymbol{\lambda}}}\mathcal{L}^0\end{pmatrix}^T \begin{pmatrix}\frac{\eta_2}{4} & \mathbf{0} \\ \mathbf{0} & \frac{\eta_1}{2} - \frac{9\mu^2\Upsilon^2}{\gamma_{RH}}\end{pmatrix} \begin{pmatrix}\nabla_{\tilde{\boldsymbol{z}}}\mathcal{L}^0 \\ \nabla_{\tilde{\boldsymbol{\lambda}}}\mathcal{L}^0\end{pmatrix}.
$$

Using $\gamma_{RH} \leq 6\mu\delta\Upsilon$, we can easily check that, as long as $\eta = (\eta_1, \eta_2)$ satisfies

$$
\eta_1 \geq \frac{25\mu^2\Upsilon^2}{\gamma_{RH}} =: \tau_1, \quad \eta_2 \leq \frac{\gamma_{RH}}{2\delta^2} =: \tau_2, \quad \eta_1\eta_2 \geq \frac{32\mu^2\Upsilon^2}{\gamma_{RH}^2} =: \tau_3,
$$

we have

$$
\begin{pmatrix}\nabla_{\tilde{\boldsymbol{z}}}\mathcal{L}_\eta^0 \\ \nabla_{\tilde{\boldsymbol{\lambda}}}\mathcal{L}_\eta^0\end{pmatrix}^T \begin{pmatrix}\Delta\tilde{\boldsymbol{z}} \\ \Delta\tilde{\boldsymbol{\lambda}}\end{pmatrix} \leq -\frac{\eta_2}{4}\left\|\begin{pmatrix}\nabla_{\tilde{\boldsymbol{z}}}\mathcal{L}^0 \\ \nabla_{\tilde{\boldsymbol{\lambda}}}\mathcal{L}^0\end{pmatrix}\right\|^2.
$$

This completes the proof of the first part of the statement. For the second part of the statement. We note that $\eta_2^0 = 1$ and each While loop decreases η_2^0 by ρ. Thus, to satisfy $\eta_2 \leq \tau_2$, the number of the required While loop iterations $\mathcal{T}$ only need satisfy $\rho^{\mathcal{T}} \geq 1/\tau_2$. For the similar reason, we require $\rho^{\mathcal{T}} \geq \tau_3/\mu^2$ and $\rho^{\mathcal{T}} \geq \sqrt{\tau_1/\mu^2}$. Combining them together, we know if $\mathcal{T}$ satisfies

$$\rho^{\mathcal{T}} \geq \left(\frac{1}{\tau_2} \vee \frac{\tau_3}{\mu^2} \vee \sqrt{\frac{\tau_1}{\mu^2}}\right) = \frac{32\Upsilon^2}{\gamma_{RH}^2},$$

then no other iterations will go into the While loop again. Thus, we know $\rho^{\mathcal{T}} \leq \frac{32\Upsilon^2\rho}{\gamma_{RH}^2}$. Moreover,

$$\bar{\eta}_2 = 1/\rho^{\mathcal{T}} \geq \frac{\gamma_{RH}}{32\Upsilon^2\rho}, \quad \text{and} \quad \bar{\eta}_1 = \mu^2(\rho^{\mathcal{T}})^2 \leq \frac{32^2\rho^2\mu^2\Upsilon^4}{\gamma_{RH}^4}.$$

This completes the second part of the statement.

C Proof of Lemma B.1

We note that $YY^T + ZZ^T = I$. Thus, $YY^T(I - BZ(Z^TBZ)^{-1}Z^T) = I - BZ(Z^TBZ)^{-1}Z^T$. Using this observation, the formula of S can be verified directly by checking $SS^{-1} = I$. Moreover, under Assumption 4.2, we know

$$\|(Z^TBZ)^{-1}\| \leq 1/\gamma_{RH}, \quad \|K^{-1}\| \leq \Upsilon, \quad \text{and} \quad \|B\| \leq 2\mu.$$

Therefore,

$$\|S\| \leq \|S_1\| + 2\|S_2\| + \|S_3\| \leq \frac{1}{\gamma_{RH}} + 2\Upsilon(1 + \frac{2\mu}{\gamma_{RH}}) + \Upsilon^2\left(\frac{4\mu^2}{\gamma_{RH}} + 2\mu\right).$$

Without loss of generality, we suppose $\Upsilon \geq 4$ and $\mu \geq 2(\gamma_{RH} + 1)$. Then

$$\|S\| \leq \frac{1}{\gamma_{RH}} + \frac{6\Upsilon\mu}{\gamma_{RH}} + 2\mu\Upsilon^2 + \frac{4\Upsilon^2\mu^2}{\gamma_{RH}} \leq \frac{\Upsilon^2\mu^2}{\gamma_{RH}} + \frac{4\Upsilon^2\mu^2}{\gamma_{RH}} \leq \frac{5\Upsilon^2\mu^2}{\gamma_{RH}}.$$

This completes the proof.

D Proof of Theorem 4.5

We drop off the index t for simplicity. By the definition of $\mathcal{L}_\eta(\cdot)$ in (5), we have

$$\nabla^2\mathcal{L}_\eta(\tilde{\boldsymbol{z}}, \tilde{\boldsymbol{\lambda}}; \bar{\boldsymbol{x}}) = \begin{pmatrix} H + \eta_2\nabla_{\tilde{\boldsymbol{z}}}(H\nabla_{\tilde{\boldsymbol{z}}}\mathcal{L}) + \eta_1\nabla_{\tilde{\boldsymbol{z}}}(G^T\nabla_{\tilde{\boldsymbol{\lambda}}}\mathcal{L}) & \eta_2\nabla_{\tilde{\boldsymbol{\lambda}}}^T(H\nabla_{\tilde{\boldsymbol{z}}}\mathcal{L}) + G^T \\ \eta_2\nabla_{\tilde{\boldsymbol{\lambda}}}(H\nabla_{\tilde{\boldsymbol{z}}}\mathcal{L}) + G & \eta_2 GG^T \end{pmatrix}.$$

Using Assumption 4.2, (16), and Theorem 4.4, we know

$$\|\nabla^2\mathcal{L}_\eta(\tilde{\boldsymbol{z}}, \tilde{\boldsymbol{\lambda}}; \bar{\boldsymbol{x}})\| \leq 4\bar{\eta}_1\Upsilon \leq \frac{32^2\rho^2\mu^2\Upsilon^5}{\gamma_{RH}} =: \mu^2\Upsilon'.$$

Therefore, by Taylor expansion

$$\mathcal{L}_{\bar{\eta}}^1 \leq \mathcal{L}_{\bar{\eta}}^0 + \alpha\begin{pmatrix}\nabla_{\tilde{\boldsymbol{z}}}\mathcal{L}_{\bar{\eta}}^0 \\ \nabla_{\tilde{\boldsymbol{\lambda}}}\mathcal{L}_{\bar{\eta}}^0\end{pmatrix}^T\begin{pmatrix}\Delta\tilde{\boldsymbol{z}} \\ \Delta\tilde{\boldsymbol{\lambda}}\end{pmatrix} + \frac{\mu^2\Upsilon'\alpha^2}{2}\left\|\begin{pmatrix}\Delta\tilde{\boldsymbol{z}} \\ \Delta\tilde{\boldsymbol{\lambda}}\end{pmatrix}\right\|^2. \tag{23}$$

Moreover, by Lemma B.1 and the condition (7), we further have

$$\left\|\begin{pmatrix}\Delta\tilde{\boldsymbol{z}} \\ \Delta\tilde{\boldsymbol{\lambda}}\end{pmatrix}\right\|^2 \leq \frac{25\mu^4\Upsilon^4}{\gamma_{RH}^2}\left\|\begin{pmatrix}\nabla_{\tilde{\boldsymbol{z}}}\mathcal{L}^0 \\ \nabla_{\tilde{\boldsymbol{\lambda}}}\mathcal{L}^0\end{pmatrix}\right\|^2 \leq -\frac{100\mu^4\Upsilon^4}{\bar{\eta}_2\gamma_{RH}}\begin{pmatrix}\nabla_{\tilde{\boldsymbol{z}}}\mathcal{L}_{\bar{\eta}}^0 \\ \nabla_{\tilde{\boldsymbol{\lambda}}}\mathcal{L}_{\bar{\eta}}^0\end{pmatrix}^T\begin{pmatrix}\Delta\tilde{\boldsymbol{z}} \\ \Delta\tilde{\boldsymbol{\lambda}}\end{pmatrix}.$$

Plugging the above display into (23),

$$\mathcal{L}_{\bar{\eta}}^1 \leq \mathcal{L}_{\bar{\eta}}^0 + \alpha\left(1 - \frac{50\mu^6\Upsilon'\Upsilon^4}{\bar{\eta}_2\gamma_{RH}}\alpha\right)\begin{pmatrix}\nabla_{\tilde{\boldsymbol{z}}}\mathcal{L}_{\bar{\eta}}^0 \\ \nabla_{\tilde{\boldsymbol{\lambda}}}\mathcal{L}_{\bar{\eta}}^0\end{pmatrix}^T\begin{pmatrix}\Delta\tilde{\boldsymbol{z}} \\ \Delta\tilde{\boldsymbol{\lambda}}\end{pmatrix}.$$

Thus, as long as

$$1 - \frac{50\mu^6\Upsilon'\Upsilon^4}{\bar{\eta}_2\gamma_{RH}}\alpha \geq \beta \Longleftrightarrow \alpha \leq \frac{(1-\beta)\bar{\eta}_2\gamma_{RH}}{50\mu^6\Upsilon'\Upsilon^4} \Longleftarrow \alpha \leq \frac{(1-\beta)\gamma_{RH}^2}{32\cdot 50\mu^6\Upsilon'\Upsilon^6} =: \bar{\alpha}',$$

then Armijo condition (6) is satisfied. Thus, if we use backtracking line search, the selected stepsize $\alpha \geq \nu\bar{\alpha}' =: \bar{\alpha}$ for some $\nu \in (0, 1)$. Moreover, by Armijo condition,

$$\mathcal{L}_{\bar{\eta}}^1 \leq \mathcal{L}_{\bar{\eta}}^0 + \alpha\beta\begin{pmatrix}\nabla_{\tilde{\boldsymbol{z}}}\mathcal{L}_{\bar{\eta}}^0 \\ \nabla_{\tilde{\boldsymbol{\lambda}}}\mathcal{L}_{\bar{\eta}}^0\end{pmatrix}^T\begin{pmatrix}\Delta\tilde{\boldsymbol{z}} \\ \Delta\tilde{\boldsymbol{\lambda}}\end{pmatrix} \leq \mathcal{L}_{\bar{\eta}}^0 - \frac{\bar{\eta}_2\bar{\alpha}\beta}{4}\left\|\begin{pmatrix}\nabla_{\tilde{\boldsymbol{z}}}\mathcal{L}^0 \\ \nabla_{\tilde{\boldsymbol{\lambda}}}\mathcal{L}^0\end{pmatrix}\right\|^2.$$

This completes the proof.

E Proof of Lemma 4.6

By the definition (5), we know

$$
\begin{aligned}
&\mathcal{L}_{\bar{\eta}}^{t,1} - \mathcal{L}_{\bar{\eta}}^{t+1,0} \\
&= \mathcal{L}^{t,1} - \mathcal{L}^{t+1,0} + \frac{\bar{\eta}_1}{2}\left(\left\|\nabla_{\tilde{\boldsymbol{\lambda}}_t}\mathcal{L}^{t,1}\right\|^2 - \left\|\nabla_{\tilde{\boldsymbol{\lambda}}_{t+1}}\mathcal{L}^{t+1,0}\right\|^2\right) + \frac{\bar{\eta}_2}{2}\left(\left\|\nabla_{\tilde{\boldsymbol{z}}_t}\mathcal{L}^{t,1}\right\|^2 - \left\|\nabla_{\tilde{\boldsymbol{z}}_{t+1}}\mathcal{L}^{t+1,0}\right\|^2\right) \\
&=: Term_1 + Term_2 + Term_3. \qquad (24)
\end{aligned}
$$

Let us deal with each term separately. For $Term_1$, we apply the definition of Lagrangian function, the transition (8), and the fact that $g(\mathbf{0}, \mathbf{0}) = 0$. Then

$$
\begin{aligned}
\mathcal{L}^{t+1,0} = &\sum_{k=t+1}^{M_t}\left\{g_k(\boldsymbol{z}_{k,t+1}^0) + (\boldsymbol{\lambda}_{k-1,t+1}^0)^T\boldsymbol{x}_{k,t+1}^0 - (\boldsymbol{\lambda}_{k,t+1}^0)^T f_k(\boldsymbol{z}_{k,t+1}^0)\right\} + g_{M_t+1}(\boldsymbol{x}_{M_t+1,t+1}^0, \mathbf{0}) \\
&+ \frac{\mu}{2}\|\boldsymbol{x}_{M_t+1,t+1}^0\| + (\boldsymbol{\lambda}_{M_t,t+1}^0)^T\boldsymbol{x}_{M_t+1,t+1}^0 - (\boldsymbol{\lambda}_{t,t+1}^0)^T\bar{\boldsymbol{x}}_{t+1} \\
= &\sum_{k=t+1}^{M_t-1}\left\{g_k(\boldsymbol{z}_{k,t}^1) + (\boldsymbol{\lambda}_{k-1,t}^1)^T\boldsymbol{x}_{k,t}^1 - (\boldsymbol{\lambda}_{k,t}^1)^T f_k(\boldsymbol{z}_{k,t}^1)\right\} + g_{M_t}(\boldsymbol{x}_{M_t,t}^1, \mathbf{0}) + (\boldsymbol{\lambda}_{M_t-1,t}^1)^T\boldsymbol{x}_{M_t,t}^1 \\
&- (\boldsymbol{\lambda}_{t,t}^1)^T f_t(\boldsymbol{z}_{t,t}^1).
\end{aligned}
$$

Using the above display, we further have

$$
\begin{aligned}
Term_1 =& \mathcal{L}^{t,1} - \mathcal{L}^{t+1,0} \\
=& \sum_{k=t}^{M_t-1}\left\{g_k(\boldsymbol{z}_{k,t}^1) + (\boldsymbol{\lambda}_{k-1,t}^1)^T\boldsymbol{x}_{k,t}^1 - (\boldsymbol{\lambda}_{k,t}^1)^T f_k(\boldsymbol{z}_{k,t}^1)\right\} + g_{M_t}(\boldsymbol{x}_{M_t,t}^1, \mathbf{0}) + \frac{\mu}{2}\|\boldsymbol{x}_{M_t,t}^1\|^2 \\
&+ (\boldsymbol{\lambda}_{M_t-1,t}^1)^T\boldsymbol{x}_{M_t,t}^1 - (\boldsymbol{\lambda}_{t-1,t}^1)^T\bar{\boldsymbol{x}}_t - \mathcal{L}^{t+1,0} \\
=& g_t(\boldsymbol{z}_{t,t}^1) + (\boldsymbol{\lambda}_{t-1,t}^1)^T(\boldsymbol{x}_{t,t}^1 - \bar{\boldsymbol{x}}_t) + \frac{\mu}{2}\|\boldsymbol{x}_{M_t,t}^1\|^2 \\
\geq& -\|\boldsymbol{\lambda}_{t-1,t}^1\|\|\boldsymbol{x}_{t,t}^1 - \bar{\boldsymbol{x}}_t\| + \frac{\mu}{2}\|\boldsymbol{x}_{M_t,t}^1\|^2 \\
\geq& -C\|\boldsymbol{x}_{t,t}^1 - \bar{\boldsymbol{x}}_t\|^2 + \frac{\mu}{2}\|\boldsymbol{x}_{M_t,t}^1\|^2, \qquad (25)
\end{aligned}
$$

where the last inequality is due to Assumption 4.3(ii). For $Term_2$, we apply the formula (14) and the transition (8). We have

$$
\begin{aligned}
\left\|\nabla_{\tilde{\boldsymbol{\lambda}}_{t+1}}\mathcal{L}^{t+1,0}\right\|^2 &= \sum_{k=t+1}^{M_t}\|\boldsymbol{x}_{k+1,t+1}^0 - f_k(\boldsymbol{z}_{k,t+1}^0)\|^2 + \|\boldsymbol{x}_{t+1,t+1}^0 - \bar{\boldsymbol{x}}_{t+1}\|^2 \\
&= \sum_{k=t+1}^{M_t-1}\|\boldsymbol{x}_{k+1,t}^1 - f_k(\boldsymbol{z}_{k,t}^1)\|^2 + \|f_{M_t}(\boldsymbol{x}_{M_t,t}^1, \mathbf{0})\|^2 + \|\boldsymbol{x}_{t+1,t}^1 - f_t(\boldsymbol{z}_{t,t}^1)\|^2 \\
&= \sum_{k=t}^{M_t-1}\|\boldsymbol{x}_{k+1,t}^1 - f_k(\boldsymbol{z}_{k,t}^1)\|^2 + \|f_{M_t}(\boldsymbol{x}_{M_t,t}^1, \mathbf{0})\|^2.
\end{aligned}
$$

Using the above display, we further have

$$
\begin{aligned}
Term_2 =& \frac{\bar{\eta}_1}{2}\left(\left\|\nabla_{\tilde{\boldsymbol{\lambda}}_t}\mathcal{L}^{t,1}\right\|^2 - \left\|\nabla_{\tilde{\boldsymbol{\lambda}}_{t+1}}\mathcal{L}^{t+1,0}\right\|^2\right) = \frac{\bar{\eta}_1}{2}\|\boldsymbol{x}_{t,t}^1 - \bar{\boldsymbol{x}}_t\|^2 - \frac{\bar{\eta}_1}{2}\|f_{M_t}(\boldsymbol{x}_{M_t,t}^1, \mathbf{0})\|^2 \\
\geq& \frac{\bar{\eta}_1}{2}\|\boldsymbol{x}_{t,t}^1 - \bar{\boldsymbol{x}}_t\|^2 - \frac{\bar{\eta}_1\Upsilon^2}{2}\|\boldsymbol{x}_{M_t,t}^1\|^2, \qquad (26)
\end{aligned}
$$

where the last inequality is due to Assumption 4.2. Last, for $Term_3$, we apply the formula (14) and the transition (8). We have

$$\begin{aligned}\|\nabla_{\tilde{\boldsymbol{z}}_{t+1}}\mathcal{L}^{t+1,0}\|^2 = &\sum_{k=t+1}^{M_t}\left\|\begin{pmatrix}\nabla_{\boldsymbol{x}_k} g_k(\boldsymbol{z}_{k,t+1}^0)+\boldsymbol{\lambda}_{k-1,t+1}^0 - A_k^T(\boldsymbol{z}_{k,t+1}^0)\boldsymbol{\lambda}_{k,t+1}^0\\ \nabla_{\boldsymbol{u}_k} g_k(\boldsymbol{z}_{k,t+1}^0) - B_k^T(\boldsymbol{z}_{k,t+1}^0)\boldsymbol{\lambda}_{k,t+1}^0\end{pmatrix}\right\|^2\\
&+\|\nabla_{\boldsymbol{x}_{M_t+1}} g_{M_t+1}(\boldsymbol{x}_{M_t+1,t+1}^0,\boldsymbol{0})+\boldsymbol{\lambda}_{M_t,t+1}^0+\mu\boldsymbol{x}_{M_t+1,t+1}^0\|^2\\
=&\sum_{k=t+1}^{M_t-1}\left\|\begin{pmatrix}\nabla_{\boldsymbol{x}_k} g_k(\boldsymbol{z}_{k,t}^1)+\boldsymbol{\lambda}_{k-1,t}^1 - A_k^T(\boldsymbol{z}_{k,t}^1)\boldsymbol{\lambda}_{k,t}^1\\ \nabla_{\boldsymbol{u}_k} g_k(\boldsymbol{z}_{k,t}^1) - B_k^T(\boldsymbol{z}_{k,t}^1)\boldsymbol{\lambda}_{k,t}^1\end{pmatrix}\right\|^2\\
&+\left\|\begin{pmatrix}\nabla_{\boldsymbol{x}_{M_t}} g_{M_t}(\boldsymbol{x}_{M_t,t}^1,\boldsymbol{0})+\boldsymbol{\lambda}_{M_t-1,t}^1\\ \nabla_{\boldsymbol{u}_{M_t}} g_{M_t}(\boldsymbol{x}_{M_t,t}^1,\boldsymbol{0})\end{pmatrix}\right\|^2.\end{aligned}$$

Using the above display, we further have

$$\begin{aligned}Term_3 =&\frac{\bar{\eta}_2}{2}\left(\|\nabla_{\tilde{\boldsymbol{z}}_t}\mathcal{L}^{t,1}\|^2-\|\nabla_{\tilde{\boldsymbol{z}}_{t+1}}\mathcal{L}^{t+1,0}\|^2\right)\\
\geq&\frac{\bar{\eta}_2}{2}\left(\|\nabla_{\boldsymbol{x}_{M_t}} g_{M_t}(\boldsymbol{x}_{M_t,t}^1,\boldsymbol{0})+\boldsymbol{\lambda}_{M_t-1,t}^1+\mu\boldsymbol{x}_{M_t,t}^1\|^2-\left\|\begin{pmatrix}\nabla_{\boldsymbol{x}_{M_t}} g_{M_t}(\boldsymbol{x}_{M_t,t}^1,\boldsymbol{0})+\boldsymbol{\lambda}_{M_t-1,t}^1\\ \nabla_{\boldsymbol{u}_{M_t}} g_{M_t}(\boldsymbol{x}_{M_t,t}^1,\boldsymbol{0})\end{pmatrix}\right\|^2\right)\\
\geq&\frac{\bar{\eta}_2(\mu^2-\Upsilon^2)}{2}\|\boldsymbol{x}_{M_t,t}^1\|^2+\bar{\eta}_2\mu\left((\boldsymbol{x}_{M_t,t}^1)^T\nabla_{\boldsymbol{x}_{M_t}} g_{M_t}(\boldsymbol{x}_{M_t,t}^1,\boldsymbol{0})+(\boldsymbol{x}_{M_t,t}^1)^T\boldsymbol{\lambda}_{M_t-1,t}^1\right)\\
\geq&\frac{\bar{\eta}_2(\mu^2-\Upsilon^2-2\mu\Upsilon)}{2}\|\boldsymbol{x}_{M_t,t}^1\|^2+\bar{\eta}_2\mu(\boldsymbol{x}_{M_t,t}^1)^T\boldsymbol{\lambda}_{M_t-1,t}^1,\end{aligned}$$

where the second inequality is due to the definition of $\nabla_{\tilde{\boldsymbol{z}}_t}\mathcal{L}^{t,1}$; and the third and the fourth inequalities are due to Assumption 4.2, which implies $\|\nabla_{\boldsymbol{z}_{M_t}} g_{M_t}(\boldsymbol{x}_{M_t,t}^1,\boldsymbol{0})\|\leq\Upsilon\|\boldsymbol{x}_{M_t,t}^1\|$. Noting that $\boldsymbol{\lambda}_{M_t-1,t}^0=\boldsymbol{0}$ and, by (3),

$$\mu\Delta\tilde{\boldsymbol{x}}_{M_t,t}+\Delta\tilde{\boldsymbol{\lambda}}_{M_t-1,t}=-\left(\nabla_{\boldsymbol{x}_{M_t}} g_{M_t}(\boldsymbol{x}_{M_t,t}^0,\boldsymbol{0})+\boldsymbol{\lambda}_{M_t-1,t}^0+\mu\boldsymbol{x}_{M_t,t}^0\right)=\boldsymbol{0},$$

we then have

$$(\boldsymbol{x}_{M_t,t}^1)^T\boldsymbol{\lambda}_{M_t-1,t}^1=-\alpha_t\mu(\boldsymbol{x}_{M_t,t}^1)^T\Delta\tilde{\boldsymbol{x}}_{M_t,t}=-\mu\|\boldsymbol{x}_{M_t,t}^1\|^2.$$

Suppose $\mu\geq 4\Upsilon$, then $\mu^2-\Upsilon^2-2\mu\Upsilon\geq\mu^2/2$. Together with the above three displays,

$$Term_3\geq-\bar{\eta}_2\mu^2\|\boldsymbol{x}_{M_t,t}^1\|^2. \tag{27}$$

Combining (24), (25), (26), and (27), and noting that $\bar{\eta}_2\mu^2\leq\mu^2\leq\bar{\eta}_1\Upsilon^2/2$, we have

$$\begin{aligned}\mathcal{L}_{\bar{\eta}}^{t,1}-\mathcal{L}_{\bar{\eta}}^{t+1,0}\geq&\left(\frac{\bar{\eta}_1}{2}-C\right)\|\boldsymbol{x}_{t,t}^1-\bar{\boldsymbol{x}}_t\|^2+\left(\frac{\mu}{2}-\frac{\bar{\eta}_1\Upsilon^2}{2}-\bar{\eta}_2\mu^2\right)\|\boldsymbol{x}_{M_t,t}^1\|^2\\
\geq&\left(\frac{\mu^2}{2}-C\right)\|\boldsymbol{x}_{t,t}^1-\bar{\boldsymbol{x}}_t\|^2-\bar{\eta}_1\Upsilon^2\|\boldsymbol{x}_{M_t,t}^1\|^2\geq-\bar{\eta}_1\Upsilon^2\|\boldsymbol{x}_{M_t,t}^1\|^2,\end{aligned}$$

where the last inequality holds if $C\leq\mu^2/2$. By Lemma B.1, Theorem 4.4 and Assumption 4.3(i),

$$\begin{aligned}\bar{\eta}_1\Upsilon^2\|\boldsymbol{x}_{M_t,t}^1\|^2\leq&\frac{32^2\rho^2\Upsilon^6}{\gamma_{RH}^4}\mu^2\alpha_t^2\|\Delta\tilde{\boldsymbol{x}}_{M_t,t}\|^2=\frac{32^2\rho^2\Upsilon^6}{\gamma_{RH}^4}\alpha_t^2\|\Delta\tilde{\boldsymbol{\lambda}}_{M_t-1,t}\|^2\\
\leq&\frac{32^2\rho^2\Upsilon^6}{\gamma_{RH}^4}c^2\|(\Delta\tilde{\boldsymbol{z}}_t,\Delta\tilde{\boldsymbol{\lambda}}_t)\|^2\leq\frac{32^2\rho^2\Upsilon^6c^2}{\gamma_{RH}^4}\|\nabla\mathcal{L}^{t,0}\|^2.\end{aligned}$$

We require

$$\begin{aligned}\frac{32^2\rho^2\Upsilon^6c^2}{\gamma_{RH}^4}\leq\frac{\bar{\eta}_2\bar{\alpha}\beta}{8}\Longleftarrow&\frac{32^2\rho^2\Upsilon^6c^2}{\gamma_{RH}^4}\leq\frac{\beta\gamma_{RH}\bar{\alpha}}{8\times 32\rho\Upsilon^2}\\
\Longleftarrow&\frac{32^2\rho^2\Upsilon^6c^2}{\gamma_{RH}^4}\leq\frac{\beta(1-\beta)\gamma_{RH}^3}{20^2\times 32^2\rho\mu^6\Upsilon'\Upsilon^8}\\
\Longleftrightarrow&c^2\lesssim\frac{\gamma_{RH}^2}{\kappa^6}.\end{aligned}$$

where the first implication is due to Theorem 4.4; and the second implication is due to Theorem 4.5. Then, we have

$$\mathcal{L}_{\bar{\eta}}^{t,1} - \mathcal{L}_{\bar{\eta}}^{t+1,0} \geq -\frac{\bar{\eta}_2\bar{\alpha}\beta}{8}\|\nabla\mathcal{L}^{t,0}\|^2.$$

This completes the proof.

F Proof of Theorem 4.7

Summing over t from τ to ∞ on both sides of (11), we have

$$\frac{\bar{\eta}_2\bar{\alpha}\beta}{8}\sum_{t=\tau}^{\infty}\|\nabla\mathcal{L}^{t,0}\|^2 \leq \mathcal{L}_{\bar{\eta}}^{0,\tau} - \min_{\mathcal{Z}\otimes\Lambda}\mathcal{L}_{\bar{\eta}}(\tilde{\boldsymbol{z}}, \tilde{\boldsymbol{\lambda}}; \bar{\boldsymbol{x}}) < \infty.$$

Thus, $\|\nabla\mathcal{L}^{t,0}\|^2 \to 0$ as $t \to \infty$. We complete the proof.