A Supplementary materials

A.1 Exact mixed-integer formulations

In this section we present the exact mixed-integer programming formulations that can be used for solving the proposed models in one of the available academic or commercial solvers. We use SCIP (Gamrath et al., 2020) which can handle mixed-integer nonlinear programs (MINLP's) with constraints that can be written as "expressions" with certain operations. It is preferable, however, that the constraints are quadratic or linear.[1]

Note that a general nonlinear objective $f(x)$ can be replaced by a linear objective y with an auxiliary variable y and an additional constraint $y \geq f(x)$ (for a minimization problem).

A.1.1 Two-way global problem

We start with the two-way global problem which was formulated in the following way in the main part of the paper:

$$\begin{aligned}
\min_{\{D_i, w_i\}_{i=1}^N} \quad & \frac{1}{T} \sum_{t=1}^{T} \left(\sum_{i=1}^{N} w_i D_i Y_{it} - \sum_{i=1}^{N} w_i (1 - D_i) Y_{it} \right)^2 + \lambda \sum_{i=1}^{N} w_i^2 \\
\text{s.t.} \quad & w_i \geq 0,\ D_i \in \{0, 1\} \ \text{ for } i = 1, \dots, N, \\
& \sum_{i=1}^{N} D_i = K, \\
& \sum_{i=1}^{N} w_i D_i = 1, \quad \sum_{i=1}^{N} w_i (1 - D_i) = 1
\end{aligned}$$

This can be rewritten in a slightly different way that utilizes auxiliary variables $z_t = \sum_{i=1}^{N} (2 w_i D_i - 1) Y_{it}$ and $q_i = w_i D_i$.

$$\begin{aligned}
\min_{\{D_i, w_i, q_i\}_{i=1}^N,\, \{z_t\}_{t=1}^T} \quad & \frac{1}{T} \sum_{t=1}^{T} z_t^2 + \lambda \sum_{i=1}^{N} w_i^2 \\
\text{s.t.} \quad & w_i \geq 0,\ q_i \geq 0,\ D_i \in \{0, 1\} \ \text{ for } i = 1, \dots, N, \\
& \sum_{i=1}^{N} D_i = K, \\
& \sum_{i=1}^{N} q_i = 1, \quad \sum_{i=1}^{N} w_i = 2, \\
& q_i \leq D_i, \quad q_i \leq w_i, \quad q_i \geq w_i - (1 - D_i) \ \text{ for } i = 1, \dots, N, \\
& z_t = \sum_{i=1}^{N} (2 q_i - 1) Y_{it} \ \text{ for } t = 1, \dots, T,
\end{aligned}$$

where the inequality constraints on variables q_i enforce the nonlinear equality constraints $q_i = w_i D_i$. Indeed, when $D_i = 0$, q_i has to be equal to 0 given the constraints $q_i \geq 0$ and $q_i \leq D_i = 0$ while the constraints $q_i \leq w_i$ and $q_i \geq w_i - 1$ are non-binding. When $D_i = 1$, on the one hand, $q_i \leq w_i$ and $q_i \geq w_i - (1 - D_i) = w_i$ while $q_i \leq D_i = 1$ and $q_i \geq 0$ are non-binding.

The problem above has a quadratic objective with a positive semi-definite (diagonal, in fact) Hessian and linear constraints.

Note that the constraint on the number of treated units $\sum_{i=1}^{N} D_i = K$ can be safely removed without complicating any of the other constraints.

[1] See `https://www.scipopt.org/doc/html/FAQ.php#minlptypes` for more details.

A.1.2 One-way global problem

If the constraint on the number of treated units, $\sum_{i=1}^N D_i = K$, is imposed, the problem above becomes the one-way global problem as soon as we impose additional constraints

$$q_i = \frac{D_i}{K} \text{ for } i = 1, \dots, N.$$

Indeed, both sides are equal to zero when $D_i = 0$ and when $D_i = 1$ the constraint is equivalent to $w_i = 1/K$.

The problem becomes more complicated when there is no constraint on the number of treated units. We want to impose

$$q_i = \frac{D_i}{\sum_{j=1}^N D_j} \text{ for } i = 1, \dots, N$$

which are nonlinear.[2] These constraints can be rewritten as linear by multiplying both sides by the denominator of the right-had side and introducing additional variables $r_{ij} = q_i D_j$ for $i, j = 1, \dots, N$ and enforcing this equality in the same way that we used for $q_i = w_i D_i$. This, however, substantially increases the number of required variables from $O(N)$ to $O(N^2)$.

A.1.3 Per-unit problem

The per-unit problem was formulated as

$$\begin{aligned} \min_{\{D_i, \{w_j^i\}_{j=1}^N\}_{i=1}^N} \quad & \frac{1}{K} \sum_{i=1}^N D_i \left[\frac{1}{T} \sum_{t=1}^T \left(Y_{it} - \sum_{j=1}^N w_j^i (1 - D_j) Y_{jt} \right)^2 + \lambda \sum_{j=1}^N \left(w_j^i \right)^2 \right] \\ \text{s.t.} \quad & w_j^i \geq 0, \ D_i \in \{0, 1\} \text{ for } i, j = 1, \dots, N, \\ & \sum_{i=1}^N D_i = K, \\ & \sum_{i=1}^N w_j^i (1 - D_j) = 1 \text{ for } i = 1, \dots, N \text{ such that } D_i = 1 \end{aligned}$$

which can be rewritten using techniques similar to the ones we used for the two-way problem by introducing auxiliary variables $q_{ij} = w_j^i (1 - D_j)$ for $i, j = 1, \dots, N$, but it also requires two additional observations.

First, since $D_i \in \{0, 1\}$, $D_i^2 = D_i$ and therefore D_i's can be carried inside the parentheses. Second, w_j^i need to be able to take nonzero values only for such i's that have $D_i = 1$. By imposing additional constraints $w_j^i \leq D_i$ for $i, j = 1, \dots, N$ we can use w_j^i instead of $w_j^i D_i$.

[2] We do not need to worry about $\sum_{j=1}^N D_j = 0$ which is ruled out by other constraints.

As a result, the per-unit problem can be written as

$$\begin{aligned}
\min_{\{D_i,\{w_j^i,q_{ij}\}_{j=1}^N,\{z_{it}\}_{t=1}^T\}_{i=1}^N} \quad & \frac{1}{KT}\sum_{i=1}^N\sum_{t=1}^T z_{it}^2 + \frac{\lambda}{K}\sum_{i=1}^N\sum_{i=1}^N \left(w_j^i\right)^2 \\
\text{s.t.} \quad & w_j^i \geq 0,\ q_{ij}\geq 0,\ D_i\in\{0,1\}\ \text{ for } i,j=1,\dots,N, \\
& \sum_{i=1}^N D_i = K, \\
& \sum_{j=1}^N q_{ij} = D_i \ \text{ for } i=1,\dots,N, \\
& q_{ij}\leq 1-D_j, \quad q_{ij}\leq w_j^i, \quad q_{ij}\geq w_j^i - D_j \ \text{ for } i,j=1,\dots,N, \\
& w_j^i \leq D_i \ \text{ for } i,j=1,\dots,N, \\
& z_{it} = Y_{it}D_i - \sum_{j=1}^N q_{ij}Y_{jt} \ \text{ for } i=1,\dots,N \text{ and } t=1,\dots,T
\end{aligned}$$

which has a quadratic objective with a positive semi-definite Hessian and linear constraints.

The constraint on the number of treated units, $\sum_{i=1}^N D_i = K$ can be removed from the per-unit problem too if we recall the technique used for formulating general nonlinear objectives. Suppose that the objective written above is denoted as $f\left(\{\{w_j^i\}_{j=1}^N, \{z_i\}_{t=1}^T\}_{i=1}^N\right)/K$. Let us introduce an auxiliary variable y and two additional constraints: $y\sum_{i=1}^N D_i \geq f\left(\{\{w_j^i\}_{j=1}^N, \{z_i\}_{t=1}^T\}_{i=1}^N\right)$ (which is quadratic as long as f is quadratic) and $\sum_{i=1}^N D_i \geq 1$ which ensures that we are not normalizing by zero in the objective that we are actually trying to minimize.

B Proof of Theorem 1

In this section we provide a proof of Theorem 1. To derive the formulas presented in this theorem, we solve for the optimal set of weights in each optimization problem.

B.1 Two-way global problem

In this case, we can define the Lagrangian of the relaxed objective—recall that we allowed weights to be negative—as follows

$$\mathcal{L}(w,\lambda_1,\lambda_2) = \left(\sum_{i\in I} a_i w_i - \sum_{j\in \bar{I}} a_j w_j\right)^2 + \sigma^2\sum_{i=1}^n w_i^2 - \lambda_1\left(\sum_{i\in I} w_i - 1\right) - \lambda_2\left(\sum_{j\in\bar{I}} w_j - 1\right).$$

Taking a derivative with respect to w_l where $l\in I$ and equating it to zero gives us

$$\frac{\partial\mathcal{L}(w,\lambda_1,\lambda_2)}{\partial w_l} = 2a_l\left(\sum_{i\in I} a_i w_i - \sum_{j\in\bar{I}} a_j w_j\right) + 2\sigma^2 w_l - \lambda_1 = 0. \tag{1}$$

Similarly, for $l\in\bar{I}$ we have:

$$\frac{\partial\mathcal{L}(w,\lambda_1,\lambda_2)}{\partial w_l} = 2a_l\left(\sum_{j\in\bar{I}} a_j w_j - \sum_{i\in I} a_i w_i\right) + 2\sigma^2 w_l - \lambda_2 = 0. \tag{2}$$

These equations together with $\sum_{i\in I} w_i = 1$ and $\sum_{j\in\bar{I}} w_j = 1$ lead to a (linear) system of equations with $N+2$ variables which can be solved. We claim that the solution to this system is given by:

$$\begin{aligned}
w_l^* &= \frac{1}{K} + \frac{(\bar{a}_I - \bar{a}_{\bar{I}})(\bar{a}_I - a_l)}{\sigma^2 + V_I^2 + V_{\bar{I}}^2} \quad \text{for } l\in I, \\
w_l^* &= \frac{1}{N-K} - \frac{(\bar{a}_I - \bar{a}_{\bar{I}})(\bar{a}_{\bar{I}} - a_l)}{\sigma^2 + V_I^2 + V_{\bar{I}}^2} \quad \text{for } l\in\bar{I}.
\end{aligned}$$

It is easy to show that $\sum_{l\in I} w_l^* = \sum_{l\in\bar{I}} w_l^* = 1$. Now it remains to show that for a suitable choice of λ_1 and λ_2, Eq. (1) and (2) are satisfied. For any $l \in I$, we can write

$$\begin{aligned}
&2a_l\left(\sum_{i\in I} a_i w_i - \sum_{j\in\bar{I}} a_j w_j\right) + 2\sigma^2 w_l \\
&= 2a_l\left(\bar{a}_I - \bar{a}_{\bar{I}}\right)\left(1 + \frac{1}{\sigma^2 + V_I^2 + V_{\bar{I}}^2}\left[\sum_{i\in I}(\bar{a}_I - a_i)a_i + \sum_{j\in\bar{I}}(\bar{a}_{\bar{I}} - a_j)a_j\right]\right) \\
&\quad + 2\sigma^2\left(\frac{1}{K} + \frac{(\bar{a}_I - \bar{a}_{\bar{I}})(\bar{a}_I - a_l)}{\sigma^2 + V_I^2 + V_{\bar{I}}^2}\right).
\end{aligned}$$

For simplicity, denote $\bar{a}_I - \bar{a}_{\bar{I}} = A$ and $\sigma^2 + V_I^2 + V_{\bar{I}}^2 = B$. Then noting that

$$\begin{aligned}
\sum_{i\in I}(\bar{a}_I - a_i)a_i &= -V_I^2, \\
\sum_{j\in\bar{I}}(\bar{a}_{\bar{I}} - a_j)a_j &= -V_{\bar{I}}^2,
\end{aligned}$$

the above is equal to:

$$\begin{aligned}
2a_l A\left(1 - \frac{V_I^2 + V_{\bar{I}}^2}{B}\right) + \frac{2\sigma^2}{K} + \frac{(2\sigma^2 A)(\bar{a}_I - a_l)}{B} &= 2a_l A(\frac{\sigma^2}{B}) + \frac{2\sigma^2}{K} + \frac{2\sigma^2 A}{B}\bar{a}_I - 2a_l A\frac{\sigma^2}{B} \\
&= 2\sigma^2\left(\frac{1}{K} + \bar{a}_I\frac{A}{B}\right),
\end{aligned}$$

which is independent of the index l. In particular, if we let

$$\lambda_1 = 2\sigma^2\left(\frac{1}{K} + \bar{a}_I\frac{A}{B}\right),$$

then for all $l \in I$, the condition in Eq. (1) is satisfied. Similarly, it can be shown that for

$$\lambda_2 = 2\sigma^2\left(\frac{1}{N-K} - \bar{a}_{\bar{I}}\frac{A}{B}\right),$$

Eq. (2) is satisfied for all $l \in \bar{I}$. Given the computed sets of weights, we can now calculate the optimal objective value as follows:

$$\begin{aligned}
J_{\text{2-way}}(I) = &\left(\sum_{i\in I} a_i w_i^* - \sum_{j\in\bar{I}} a_j w_j^*\right)^2 + \sigma^2\sum_{i=1}^{n} {w_i^*}^2 \\
&= (\bar{a}_I - \bar{a}_{\bar{I}})^2\left(1 + \frac{1}{\sigma^2 + V_I^2 + V_{\bar{I}}^2}\left[\sum_{i\in I}(\bar{a}_I - a_i)a_i + \sum_{j\in\bar{I}}(\bar{a}_I - a_j)a_j\right]\right)^2 \\
&\quad + \sigma^2\left(\frac{1}{K} + \sum_{i\in I}\frac{(\bar{a}_I - \bar{a}_{\bar{I}})^2(\bar{a}_I - a_i)^2}{(\sigma^2 + V_I^2 + V_{\bar{I}}^2)^2}\right) \\
&\quad + \sigma^2\left(\frac{1}{N-K} + \sum_{j\in\bar{I}}\frac{(\bar{a}_I - \bar{a}_{\bar{I}})^2(\bar{a}_{\bar{I}} - a_j)^2}{(\sigma^2 + V_I^2 + V_{\bar{I}}^2)^2}\right) \\
&= \sigma^2\left(\frac{1}{K} + \frac{1}{N-K}\right) + A^2\left[\left(\frac{\sigma^2}{B}\right)^2 + \sigma^2\left(\frac{V_I^2 + V_{\bar{I}}^2}{B^2}\right)\right] \\
&= \sigma^2\left(\frac{1}{K} + \frac{1}{N-K} + \frac{A^2}{B}\right) \\
&= \sigma^2\left(\frac{1}{K} + \frac{1}{N-K} + \frac{(\bar{a}_I - \bar{a}_{\bar{I}})^2}{\sigma^2 + V_I^2 + V_{\bar{I}}^2}\right)
\end{aligned}$$

where we used $A = \bar{a}_I - \bar{a}_{\bar{I}}$ and $B = \sigma^2 + V_I^2 + V_{\bar{I}}^2$. This completes the proof for the two-way problem.

B.2 One-way global problem

The derivation here is similar to the two-way problem. Indeed, the Lagrangian can be written as

$$\mathcal{L}(w, \lambda) = \left(\bar{a}_I - \sum_{j \in \bar{I}} a_j w_j\right)^2 + \frac{\sigma^2}{K} + \sigma^2 \sum_{j \in \bar{I}} w_j^2 - \lambda \left(\sum_{j \in \bar{I}} w_j - 1\right).$$

For any $l \in \bar{I}$, the weights need to satisfy

$$\frac{\partial \mathcal{L}(w, \lambda)}{\partial w_l} = 2a_l \left(\sum_{j \in \bar{I}} a_j w_j - \sum_{i \in I} a_i w_i\right) + 2\sigma^2 w_l - \lambda = 0. \tag{3}$$

This, together with the condition that $\sum_{l \in \bar{I}} w_l = 1$, leads to a system of equations that can be solved. Similarly to the two-way problem, we claim that the following set of weights satisfy the first-order conditions.

$$\begin{aligned} w_l^* &= \frac{1}{K} \quad \text{for } l \in I, \\ w_l^* &= \frac{1}{N-K} - \frac{(\bar{a}_I - \bar{a}_{\bar{I}})(\bar{a}_{\bar{I}} - a_l)}{\sigma^2 + V_{\bar{I}}^2} \quad \text{for } l \in \bar{I}\,. \end{aligned}$$

It is straightforward to verify that $\sum_{l \in \bar{I}} w_l^* = 1$. Also, for any $l \in \bar{I}$ we can write:

$$\begin{aligned} &2a_l \left(\sum_{j \in \bar{I}} a_j w_j - \bar{a}_I\right) + 2\sigma^2 w_l \\ &= 2a_l \left(\bar{a}_{\bar{I}} - \bar{a}_I\right) \left(1 + \frac{1}{\sigma^2 + V_{\bar{I}}^2} \left[\sum_{j \in \bar{I}} (\bar{a}_{\bar{I}} - a_j) a_j\right]\right) + 2\sigma^2 \left(\frac{1}{N-K} + \frac{(\bar{a}_{\bar{I}} - \bar{a}_I)(\bar{a}_{\bar{I}} - a_l)}{\sigma^2 + V_{\bar{I}}^2}\right) \\ &= 2\left(\bar{a}_{\bar{I}} - \bar{a}_I\right) \left[\left(\frac{\sigma^2}{\sigma^2 + V_{\bar{I}}^2} a_l\right) + \sigma^2 \frac{\bar{a}_{\bar{I}} - a_l}{\sigma^2 + V_{\bar{I}}^2}\right] + \frac{2\sigma^2}{N-K} \\ &= 2\sigma^2 \left[\frac{(\bar{a}_{\bar{I}} - \bar{a}_I)\bar{a}_{\bar{I}}}{\sigma^2 + V_{\bar{I}}^2} + \frac{1}{N-K}\right], \end{aligned}$$

which is independent of l. Hence, for

$$\lambda = 2\sigma^2 \left[\frac{(\bar{a}_{\bar{I}} - \bar{a}_I)\bar{a}_{\bar{I}}}{\sigma^2 + V_{\bar{I}}^2} + \frac{1}{N-K}\right]$$

Eq. (3) is satisfied for all $l \in \bar{I}$. Substituting these optimal weights into the formula for the one-way objective yields

$$
\begin{aligned}
J_{\text{1-way}}(I) = &\left(\sum_{i\in I} a_i w_i^* - \sum_{j\in \bar{I}} a_j w_j^*\right)^2 + \sigma^2 \sum_{i=1}^{n} {w_i^*}^2 \\
&= (\bar{a}_I - \bar{a}_{\bar{I}})^2 \left(1 + \frac{1}{\sigma^2 + V_{\bar{I}}^2}\left[\sum_{j\in\bar{I}} (\bar{a}_I - a_j) a_j\right]\right)^2 + \sigma^2\left(\frac{1}{K}\right) \\
&\quad + \sigma^2\left(\frac{1}{N-K} + \sum_{j\in\bar{I}} \frac{(\bar{a}_I - \bar{a}_{\bar{I}})^2 (\bar{a}_{\bar{I}} - a_j)^2}{(\sigma^2 + V_{\bar{I}}^2)^2}\right) \\
&= \sigma^2\left(\frac{1}{K} + \frac{1}{N-K}\right) + (\bar{a}_I - \bar{a}_{\bar{I}})^2 \left[\left(\frac{\sigma^2}{\sigma^2 + V_{\bar{I}}^2}\right)^2 + \sigma^2 \frac{V_{\bar{I}}^2}{(\sigma^2 + V_{\bar{I}}^2)^2}\right] \\
&= \sigma^2\left(\frac{1}{K} + \frac{1}{N-K} + \frac{(\bar{a}_I - \bar{a}_{\bar{I}})^2}{\sigma^2 + V_{\bar{I}}^2}\right),
\end{aligned}
$$

which completes the proof for the one-way problem.

B.3 Per-unit problem

The per-unit problem can be thought of as solving K separate two-way (or one-way) global problems where in each sub-problem, a single treated unit is selected as set I and all $N - K$ control units are in the set $\bar{I}$. Hence, using our derivation for the one-way global problem, in each sub-problem with units in P and $\bar{P}$ where $P = \{i\}$ for $i \in I$ and $\bar{P} = \bar{I}$ the optimal weights are given by

$$
w_j^i = \frac{1}{N-K} - \frac{(a_i - \bar{a}_{\bar{I}})(\bar{a}_{\bar{I}} - a_j)}{\sigma^2 + V_{\bar{I}}^2}.
$$

Note that we can also use our derivation for the one-way global problem to calculate the optimal objective within each sub-problem, with a single change that in the per-unit objective we do not penalize the weight of the single treated unit in P. In other words, the term $\sigma^2/1$ of the objective will not show up in the calculations. Hence, denoting J_i^* as the optimal value of this sub-problem, we have

$$
J_i^* = \sigma^2\left(\frac{1}{N-K} + \frac{(a_i - \bar{a}_{\bar{I}})^2}{\sigma^2 + V_{\bar{I}}^2}\right).
$$

Furthermore, for the per-unit objective we can write $J_{\text{per-unit}}(I) = \sum_{i\in I} J_i^*/K$ which implies

$$
\begin{aligned}
J_{\text{per-unit}}(I) = \frac{1}{K}\sum_{i\in I} J_i^* &= \frac{1}{K}\sum_{i\in I} \sigma^2 \left(\frac{1}{N-K} + \frac{(a_i - \bar{a}_{\bar{I}})^2}{\sigma^2 + V_{\bar{I}}^2}\right) \\
&= \frac{\sigma^2}{N-K} + \frac{\sigma^2}{\sigma^2 + V_{\bar{I}}^2} \cdot \frac{\sum_{i\in I}(a_i - \bar{a}_{\bar{I}})^2}{K} \\
&= \frac{\sigma^2}{N-K} + \frac{\sigma^2}{\sigma^2 + V_{\bar{I}}^2} \cdot \frac{\sum_{i\in I}(a_i - \bar{a}_I + \bar{a}_I - \bar{a}_{\bar{I}})^2}{K} \\
&= \frac{\sigma^2}{N-K} + \frac{\sigma^2}{\sigma^2 + V_{\bar{I}}^2}\left[(\bar{a}_I - \bar{a}_{\bar{I}})^2 + \frac{V_I^2}{K}\right] \\
&= \sigma^2\left(\frac{1}{N-K} + \frac{(\bar{a}_I - \bar{a}_{\bar{I}})^2 + K^{-1} V_I^2}{\sigma^2 + V_{\bar{I}}^2}\right).
\end{aligned}
$$

C Hardness

We prove that the underlying optimization problems for the optimal design are indeed NP-Hard. We do so by providing a formal reduction from a variant of the partitioning problem that is known to be NP-hard. In an instance of the equal-size partitioning problem, we are given a set of numbers $B = \{b_1, b_2, \cdots, b_n\}$. Let $T = \sum_{i=1}^{n} b_i$. The equal-size partitioning problem is to decide if there exists a subset $S \subset B$ of $\frac{n}{2}$ items with total sum of elements equal to $\frac{T}{2}$. This decision problem is NP-complete (Cieliebak et al., 2008). We show that we can distinguish between a "YES" and "NO" instance of this problem using an optimal algorithm for our optimization problems, hence our problem is also NP-hard.

Given an instance $(B, \frac{T}{2})$ of the partitioning problem, consider an instance of the $J_{\text{2-way}}(I)$ or $J_{\text{1-way}}(I)$ problem where we set $N = \{a_i | 1 \leq i \leq n\}$ with $a_i = b_i + T$. We note that with this transformation, there exists a subset I with a total sum of $\frac{nT+T}{2}$ iff there exists a subset $S \subset B$ of $\frac{n}{2}$ items with total sum of $\frac{T}{2}$. Furthermore, for such a subset I, since $|I| = |N \backslash I|$, the average of items in I will be the same as the average of $\bar{I} = N \backslash I$ and equivalently, $(\overline{a}_I - \overline{a})^2 = 0$. We also observe that the optimal solution for both $J_{\text{2-way}}(I)$ and $J_{\text{1-way}}(I)$ is at least $\frac{1}{K} + \frac{1}{N-K}$. As a result, the optimal solution for problems $J_{\text{2-way}}$ or $J_{\text{1-way}}(I)$ is $\frac{1}{K} + \frac{1}{N-K}$ if and only if we can find a subset of size $\frac{n}{2}$ of N with total sum of $\frac{nT+T}{2}$ or equivalently iff there exists a subset $S \subset B$ of $\frac{n}{2}$ items of B with total sum $\frac{T}{2}$. In other words, determining if optimal solution is $\frac{1}{K} + \frac{1}{N-K}$ corresponds to having a "YES" instance of the equal-size partitioning problem $(B, \frac{T}{2})$. Therefore, finding such an optimal solution is NP-hard.

References

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