1 Proof of the lower tail inequality for weakly self-bounded functions

Since the result is not presented in the literature in the form we need, we reproduce the standard argument. Let \( Z = f(X_1, \ldots, X_n) \), \( Z_i = f(X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_n) \) be such that \( Z \leq Z_i \) almost surely, and the weakly self-bounding property holds, that is, \( \sum_{i=1}^n (Z_i - Z)^2 \leq aZ + b \). Let us first apply a modified logarithmic Sobolev inequality [1, Theorem 6.6]. We have

\[
\lambda \mathbb{E}[Z e^{\lambda Z}] - \mathbb{E}[e^{\lambda Z}] \log \mathbb{E}[e^{\lambda Z}] \leq \mathbb{E}\left(e^{\lambda Z} \sum_{i=1}^n \phi(-\lambda(Z - Z_i))\right),
\]

where \( \phi(x) = e^x - x - 1 \). Since for \( x \geq 0, \phi(-x) \leq x^2/2 \) and \( Z - Z_i \leq 0 \) for all \( i = 1, \ldots, n \), we have for any \( \lambda \leq 0 \),

\[
\lambda \mathbb{E}[Z e^{\lambda Z}] - \mathbb{E}[e^{\lambda Z}] \log \mathbb{E}[e^{\lambda Z}] \leq \frac{\lambda^2}{2} \mathbb{E}\left(e^{\lambda Z} \sum_{i=1}^n (Z - Z_i)^2\right) \leq \frac{\lambda^2}{2} (a\mathbb{E}[Ze^{\lambda Z}] + b\mathbb{E}[e^{\lambda Z}]).
\]

Define \( G(\lambda) = \log \mathbb{E}[e^{\lambda Z}] \), so that \( G'(\lambda) = \mathbb{E}[Ze^{\lambda Z}] / \mathbb{E}[e^{\lambda Z}] \). Dividing both sides of the last display by \( \mathbb{E}[e^{\lambda Z}] \), we obtain

\[
\lambda G'(\lambda) - G(\lambda) \leq \frac{\lambda^2}{2} (aG'(\lambda) + b), \quad \lambda \leq 0.
\]

For \( \lambda < 0 \), we have

\[
\left(1 - \frac{a}{2}\right) G'(\lambda) = \left(1 - \frac{a}{2}\right) G'(\lambda) - \frac{\lambda^2}{2} \leq \frac{b}{2}.
\]

Finally, we integrate this inequality. Observe that \( G(0) = 0 \) and we also have \( G'(0) = \mathbb{E}Z \). Hence, as \( \lambda \to 0 \), by Taylor’s theorem \((1/\lambda - a/2) G(\lambda) = (1/\lambda - a/2)(\lambda \mathbb{E}Z + o(\lambda)) = \mathbb{E}Z + o(1)\). Integrating the last display over the interval \([\lambda, 0]\), we obtain

\[
\mathbb{E}Z - \left(1 - \frac{a}{2}\right) G(\lambda) \leq \left(-\lambda\right) \frac{b}{2}.
\]

After some rearrangements this leads to the following inequality (recall that \( \lambda < 0 \)),

\[
\log \mathbb{E}[e^{\lambda(Z - \mathbb{E}Z)}] = G(\lambda) - \lambda \mathbb{E}Z \leq \frac{\lambda^2(a\mathbb{E}Z + b)}{2(1 - \lambda a/2)} \leq \frac{\lambda^2}{2} (a\mathbb{E}Z + b).
\]

It remains to use the Markov inequality (recall again that \( \lambda < 0 \)) to show that

\[
\mathbb{P}(Z < \mathbb{E}Z - t) = \mathbb{P}(\lambda(Z - \mathbb{E}Z) > -\lambda t) = \mathbb{P}\left(e^{\lambda(Z - \mathbb{E}Z + t)} > 1\right) \leq \exp\left(\frac{-\lambda^2}{2}(a\mathbb{E}Z + b)\right).
\]

The latter exponent is minimized by \( \lambda = -t/(a\mathbb{E}Z + b) < 0 \) implying the statement. \( \Box \)
References