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# Appendix

## Certifying Confidence via Randomized Smoothing

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### A Proof of Theorem 1

We first prove a slightly modified version of the Neyman-Pearson lemma.

**Lemma 1** (Neyman & Pearson, 1933). *Let  $X$  and  $Y$  be random variables in  $\mathbb{R}^d$  with densities  $\mu_X$  and  $\mu_Y$ . Let  $h : \mathbb{R}^d \rightarrow (a, b)$  be a function. Then:*

1. *If  $S = \left\{ z \in \mathbb{R}^d \mid \frac{\mu_Y(z)}{\mu_X(z)} \leq t \right\}$  for some  $t > 0$  and  $\mathbb{E}[h(X)] \geq (b - a)\mathbb{P}(X \in S) + a$ , then  $\mathbb{E}[h(Y)] \geq (b - a)\mathbb{P}(Y \in S) + a$ .*
2. *If  $S = \left\{ z \in \mathbb{R}^d \mid \frac{\mu_Y(z)}{\mu_X(z)} \geq t \right\}$  for some  $t > 0$  and  $\mathbb{E}[h(X)] \leq (b - a)\mathbb{P}(X \in S) + a$ , then  $\mathbb{E}[h(Y)] \leq (b - a)\mathbb{P}(Y \in S) + a$ .*

*Proof.* Let  $S^c$  be the complement set of  $S$ .

$$\begin{aligned}
\mathbb{E}[h(Y)] - (b - a)\mathbb{P}(Y \in S) - a &= \mathbb{E}[h(Y)] - b\mathbb{P}(Y \in S) - a(1 - \mathbb{P}(Y \in S)) \\
&= \mathbb{E}[h(Y)] - b\mathbb{P}(Y \in S) - a\mathbb{P}(Y \notin S) \\
&= \int_{\mathbb{R}^d} h(z)\mu_Y(z)dz - b \int_S \mu_Y(z)dz - a \int_{S^c} \mu_Y(z)dz \\
&= \left[ \int_{S^c} h(z)\mu_Y(z)dz + \int_S h(z)\mu_Y(z)dz \right] - b \int_S \mu_Y(z)dz - a \int_{S^c} \mu_Y(z)dz \\
&= \int_{S^c} (h(z) - a)\mu_Y(z)dz - \int_S (b - h(z))\mu_Y(z)dz \\
&\geq t \left[ \int_{S^c} (h(z) - a)\mu_X(z)dz - \int_S (b - h(z))\mu_X(z)dz \right] \\
&\hspace{15em} \text{(since } a < h(z) < b) \\
&= t \left[ \int_{\mathbb{R}^d} h(z)\mu_X(z)dz - b \int_S \mu_X(z)dz - a \int_{S^c} \mu_X(z)dz \right] \\
&= t [\mathbb{E}[h(X)] - b\mathbb{P}(X \in S) - a\mathbb{P}(X \notin S)] \\
&= t [\mathbb{E}[h(X)] - b\mathbb{P}(X \in S) - a(1 - \mathbb{P}(X \in S))] \\
&= t [\mathbb{E}[h(X)] - (b - a)\mathbb{P}(X \in S) - a] \geq 0
\end{aligned}$$

The second statement can be proven similarly by switching  $\geq$  and  $\leq$ . □

In the first statement of the lemma, set  $h$  to  $f_i$ ,  $\mu_X$  to  $\mathcal{N}(x, \sigma^2 I)$  and  $\mu_Y$  to  $\mathcal{N}(x', \sigma^2 I)$ , and find a  $t$ , such that,  $\mathbb{P}(X \in S) = \underline{p}_i(x)$ . Now, since  $\mu_X$  and  $\mu_Y$  are isometric Gaussians with the same variance,

$$\frac{\mu_Y(z)}{\mu_X(z)} \leq t \iff (x' - x)^T z \leq \beta$$

for some  $\beta \in \mathbb{R}$ . Therefore, the set  $S$  is a half-space defined by a hyper-plane orthogonal to the perturbation  $x' - x$ . So, if  $\|x' - x\|_2 \leq R$ , then  $\mathbb{P}(Y \in S) \geq \Phi_\sigma(\Phi_\sigma^{-1}(\underline{p}_i(x)) - R)$ .

$$\begin{aligned} \bar{f}_i(x') &= \mathbb{E}[f_i(Y)] \\ &\geq (b - a)\mathbb{P}(Y \in S) + a && \text{(from the above lemma)} \\ &\geq (b - a)\Phi_\sigma(\Phi_\sigma^{-1}(\underline{p}_i(x)) - R) + a \\ &= b\Phi_\sigma(\Phi_\sigma^{-1}(\underline{p}_i(x)) - R) + a(1 - \Phi_\sigma(\Phi_\sigma^{-1}(\underline{p}_i(x)) - R)) \end{aligned}$$

The upper bound on  $\bar{f}_i(x')$  can be derived similarly by applying the second statement of the above lemma.

### A.1 Alternate proof

Theorem 1 can also be proved for  $\sigma = 1$  using Lemma 2 from Salman et al. in [25]. This lemma states that for any function  $g : \mathbb{R}^d \rightarrow (0, 1)$ ,  $\Phi^{-1}(\bar{g})$  is 1-Lipschitz, where  $\Phi^{-1}$  is the inverse CDF of the standard Gaussian distribution. Set  $g(\cdot)$  to be  $\frac{f_i(\cdot) - a}{b - a}$  for an arbitrary class  $i$ . Then,  $\bar{g}(x) = \frac{\bar{f}_i(x) - a}{b - a}$  is upper and lower bounded by  $\bar{p}_i(x)$  and  $\underline{p}_i(x)$  respectively. Due to the Lipschitz condition, we have,

$$\begin{aligned} \Phi^{-1}(\bar{g}(x)) - \Phi^{-1}(\bar{g}(x')) &\leq \|x - x'\|_2 \leq R \\ \Phi^{-1}(\bar{g}(x')) &\geq \Phi^{-1}(\bar{g}(x)) - R \geq \Phi^{-1}(\underline{p}_i(x)) - R \\ \bar{g}(x') &\geq \Phi(\Phi^{-1}(\underline{p}_i(x)) - R) \end{aligned}$$

Substituting  $\bar{g}(x) = \frac{\bar{f}_i(x) - a}{b - a}$  and rearranging terms appropriately gives us the first bound in theorem 1. The second bound can be derived similarly.

## B Proof of Lemma 3

Let  $S^c$  be the complement set of  $S$ .

$$\begin{aligned} \mathbb{P}(h(Y) \geq s) - \mathbb{P}(Y \in S) &= \int_{\mathbb{R}^d} \mathbf{1}\{h(z) \geq s\} \mu_Y(z) dz - \int_S \mu_Y(z) dz \\ &= \left[ \int_{S^c} \mathbf{1}\{h(z) \geq s\} \mu_Y(z) dz + \int_S \mathbf{1}\{h(z) \geq s\} \mu_Y(z) dz \right] - \int_S \mu_Y(z) dz \\ &= \int_{S^c} \mathbf{1}\{h(z) \geq s\} \mu_Y(z) dz - \int_S (1 - \mathbf{1}\{h(z) \geq s\}) \mu_Y(z) dz \\ &\geq t \left[ \int_{S^c} \mathbf{1}\{h(z) \geq s\} \mu_X(z) dz - \int_S (1 - \mathbf{1}\{h(z) \geq s\}) \mu_X(z) dz \right] \\ &\hspace{15em} \text{(since } 0 \leq \mathbf{1}\{h(z) \geq s\} \leq 1) \\ &= t \left[ \int_{\mathbb{R}^d} \mathbf{1}\{h(z) \geq s\} \mu_X(z) dz - \int_S \mu_X(z) dz \right] \\ &= t [\mathbb{P}(h(X) \geq s) - \mathbb{P}(X \in S)] \geq 0 \end{aligned}$$

The second statement of the lemma can be proven similarly by switching  $\geq$  and  $\leq$ .

## C Additional Experiments

In section 5.2, we compared the two methods, using Hoeffding's inequality and Dvoretzky-Kiefer-Wolfowitz inequality to derive the required lower bounds, for the certificates. We

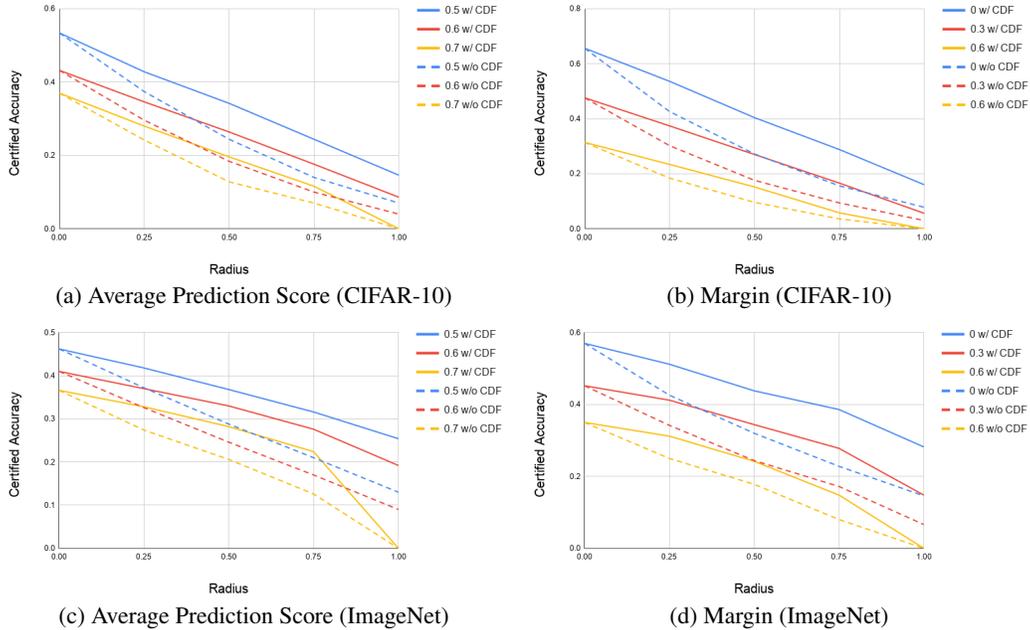


Figure 1: Certified accuracy vs. radius (CIFAR-10 & ImageNet) at different cutoffs for average confidence score with  $\sigma = 0.50$ . Solid and dashed lines represent certificates computed with and without CDF bound respectively.

repeat the same experiments in figure 1 for  $\sigma = 0.50$ . Then, in figure 2, we show that the CDF-based method (using the DKW inequality) outperforms the baseline approach regardless of how tight a lower-bound for  $e_i(x)$  is used in the baseline certificate (1). We replace  $e_i(x)$  with the empirical estimate of the expectation  $\hat{e}_i(x) = \sum_{j=1}^m f_i(x + \delta_j)/m$ , which is an upper bound on  $e_i(x)$ . And since bound (1) is an increasing function of  $e_i(x)$ , any valid lower bound  $\underline{e}_i(x)$  on the expectation cannot yield a certified accuracy better than that obtained using  $\hat{e}_i(x)$ . We compare our certificate with the best-possible baseline certificate for some of Cohen et al.’s ResNet-110 models trained on the CIFAR-10 dataset using the same value of  $\alpha$  as in section 5.2. The baseline mostly stays below the CDF-based method for both types of confidence measures under the noise levels considered.

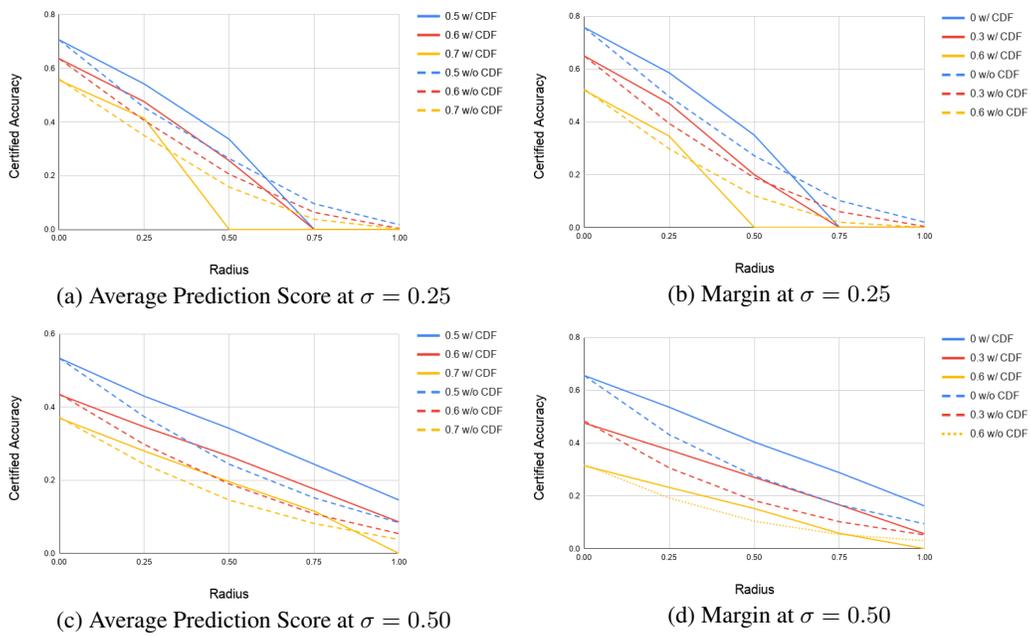


Figure 2: Certified accuracy vs. radius (CIFAR-10 only) at different cutoffs for average confidence score. Solid lines represent certificates computed with the CDF bound and dashed lines represent the best-possible baseline certificate.