A Proofs of Section 3

In all proofs in this paper, for a sequence $x = (x_0, x_1, \ldots, x_n)$, we use $x_{a:b}$ to denote its consecutive subsequence $(x_a, x_{a+1}, \ldots, x_b)$.

A.1 Proof of Theorem 3.2

Let $w_t$ be i.i.d. with zero mean and covariance matrix $W$. Suppose the controller has $k \geq 1$ predictions. Then, the optimal control policy at each step $t$ is given by:

$$u_t = -(R + B^T PB)^{-1}B^T (PAx_t + \sum_{i=0}^{k-1} (A^T - A^T PH)^i PW_{t+i}),$$

where $P$ is the solution of DARE in Equation (1). The cost under this policy is:

$$STO_k = \text{Tr}\left\{ P - \sum_{i=0}^{k-1} P(A - HPA)^i H(A^T - A^T PH)^i P \right\} W,$$

where $H = B(R + B^T PB)^{-1}B^T$.

Proof. Our proof technique closely follows that in Section 4.1 of [16]. To begin, note that the definition of $STO_k$ has a structure of repeating min’s and E’s. We use dynamic programming to compute the value iteratively. In particular, we apply backward induction to solve the optimal cost-to-go functions, from time step $T$ to the initial state. Given state $x_t$ and predictions $w_t, \ldots, w_{t+k-1}$, we define the cost-to-go function:

$$V_t(x_t; w_{t:t+k-1}) := \min_{u_t} \cdots \min_{w_{t+k}} \min_{u_{t+k+1}} \text{Tr}\left\{ P - \sum_{i=0}^{k-1} P(A - HPA)^i H(A^T - A^T PH)^i P \right\} W,$$

with $V_T(x_T; \ldots) = x_T^T Q x_T$. Note that $\text{E}_{w_{t+k}}$ has no effect for $t \geq T - k$. This function measures the expected overall control cost from a given state to the end, assuming the controller makes the optimal decision at each time.

We will show by backward induction that for every $t = 0, \ldots, T$, $V_t(x_t; w_{t:t+k-1}) = x_t^T P_t x_t + v_t^T x_t + q_t$, where $P_t, v_t, q_t$ are coefficients that may depend on $w_{t:t+k-1}$. This is clearly true for $t = T$. Suppose this is true at $t + 1$. Then,

$$V_t(x; w_{t:t+k-1}) = x^T Q x + \min_{u} \left( u^T R u + (Ax + Bu + w_t)^T P_{t+1} (Ax + Bu + w_t) \right. \left. + \text{E}_{w_{t+1}} [v_{t+1}^T (Ax + Bu + w_t)] \right) $$

$$\quad + \text{E}_{w_{t+k}} [q_{t+1}] $$

$$= x^T (Q + (Ax + w_t)^T P_{t+1} (Ax + w_t) + \text{E}_{w_{t+k}} [v_{t+1}^T (Ax + w_t)] + \text{E}_{w_{t+k}} [q_{t+1}]) $$

$$+ \min_{u} \left( u^T (R + B^T P_{t+1} B) u + u^T B^T \left( 2P_{t+1} Ax + 2P_{t+1} w_t + \text{E}_{w_{t+k}} [v_{t+1}] \right) \right).$$

The optimal $u$ is obtained by setting the derivative to be zero:

$$u^* = -(R + B^T P_{t+1} B)^{-1} B^T \left( P_{t+1} Ax + P_{t+1} w_t + \frac{1}{2} \text{E}_{w_{t+k}} [v_{t+1}] \right).$$

Let $H_t = B(R + B^T P_{t+1} B)^{-1} B^T$. Plugging $u^*$ back into $V_t$, we have

$$V_t(x_t; w_{t:t+k-1}) = x_t^T Q x + (Ax + w_t)^T P_{t+1} (Ax + w_t) + \text{E}_{w_{t+k}} [v_{t+1}^T (Ax + w_t)] + \text{E}_{w_{t+k}} [q_{t+1}].$$
Assuming i.i.d. disturbance with zero mean, the MPC policy is optimal.

Thus, the recursive formulae, which parallel [16], are given by:

\[
\begin{align*}
P_t &= Q + A^T P_{t+1} P_t A - A^T P_{t+1} H_t P_{t+1} A, \\
v_t &= (A^T - A^T P_{t+1} H_t) E_{w_{t+k}} [v_{t+1}] + 2(A^T - A^T P_{t+1} H_t) P_{t+1} w_t, \\
q_t &= w_t^T (P_{t+1} - P_{t+1} H_t P_{t+1}) w_t + w_t^T (I - P_{t+1} H_t) E_{w_{t+k}} [v_{t+1}] \\
&\quad - \frac{1}{4} E_{w_{t+k}} [v_{t+1}]^T H_t E_{w_{t+k}} [v_{t+1}] + E_{w_{t+k}} [q_{t+1}].
\end{align*}
\]

As \( T \to \infty \), \( P_t \) and \( H_t \) converge to \( P \) and \( H \) respectively, where \( P \) is the solution of discrete-time algebraic Riccati equation (DARE) \( P = Q + A^T PA - A^T PHA \), and \( H = B(R + B^T PB)^{-1} B^T \).

Note that \( v_T = 0 \) and \( q_T = 0 \). Then,

\[
\begin{align*}
v_t &= 2 \sum_{i=0}^{k-1} (A^T - A^T PH)^i P w_{t+i}, \\
q_t &= w_t^T (P - PHP) w_t + w_t^T (I - PH) E_{w_{t+k}} [v_{t+1}] - \frac{1}{4} E_{w_{t+k}} [v_{t+1}]^T H_t E_{w_{t+k}} [v_{t+1}] + E_{w_{t+k}} [q_{t+1}], \\
E_{w_{t+k}} [v_{t+1}] &= 2 \sum_{i=1}^{k-1} (A^T - A^T PH)^i P w_{t+i}.
\end{align*}
\]

Taking the expectation of \( q_t \) over all randomness, namely \( w_0, w_1, w_2, \ldots \), we have

\[
E[q_t] = \text{Tr} \{ (P - PHP) W \} - \sum_{i=1}^{k-1} \text{Tr} \{ P (A - HPA)^i H (A^T - A^T PH)^i PW \} + E[q_{t+1}]
\]

\[
= \text{Tr} \left\{ \left( P - \sum_{i=0}^{k-1} P (A - HPA)^i H (A^T - A^T PH)^i P \right) W \right\} + E[q_{t+1}],
\]

where in the first equality we use \( E[w_t] = 0 \) and the independence of the disturbances. Thus, as \( T \to \infty \), in each time step, a constant cost is incurred and the average cost \( \text{STO}_k \) is exactly this value.

\[
\text{STO}_k = \lim_{T \to \infty} \frac{1}{T} \text{STO}_k^T = \lim_{T \to \infty} \frac{1}{T} \mathbb{E} [V_0(x_0; w_{0:k-1})] = \lim_{T \to \infty} \frac{1}{T} \mathbb{E} [q_0]
\]

\[
= \lim_{T \to \infty} \frac{1}{T} \sum_{i=0}^{T-1} E[q_i] - E[q_{t+1}] = \text{Tr} \left\{ \left( P - \sum_{i=0}^{k-1} P (A - HPA)^i H (A^T - A^T PH)^i P \right) W \right\}.
\]

The explicit form of the optimal control policy is obtained by combining Equations 6 and 10. \( \blacklozenge \)

A.2 Proof of Theorem 3.3

In Algorithm 2 let \( \tilde{Q}_t = P \). Then, the MPC policy with \( k \) predictions is also given by Equation 2.

Assuming i.i.d. disturbance with zero mean, the MPC policy is optimal.
The optimal control policy with general stochastic disturbance is given by:

\[
V_t(x_t; w_{t:t+k-1}) = \min_{u_{t,t+k-1}} \sum_{i=t}^{t+k-1} (x_i^T Q x_i + u_i^T R u_i) + x_{t+k}^T P x_{t+k},
\]

where \( x_{i+1} = Ax_i + Bu_i + w_i \), given \( x_t, w_t, \ldots, w_{t+k-1} \). Define the cost-to-go function at time \( i \) given \( x_i, w_i, \ldots, w_{t+k-1} \):

\[
V_i(x_i; w_{i:t+k-1}) = \min_{u_{i,t+k-1}} \sum_{j=i}^{t+k-1} (x_j^T Q x_j + u_j^T R u_j) + x_{t+k}^T P x_{t+k}
\]

\[
= x_i^T Q x_i + \min_{u_i} (u_i^T R u_i + V_{i+1}(Ax_i + Bu_i + w_i; w_{i+1:t+k-1})).
\]

Note that \( V_{t+k}(x_{t+k}) = x_{t+k}^T P x_{t+k} \). Similar to the proof of Theorem 3.2, we can inductively show that \( V_i(x_i; w_{i:t+k-1}) = x_i^T P x_i + v_i^T x_i + q_i \) for some \( v_i \) and \( q_i \). Note that the second-degree coefficient no longer depends on the index \( i \) as in the previous proof because we start from \( P \), the solution of DARE. We then have the followings equations that parallel with Equations (6) and (8):

\[
v_t = 2 \sum_{j=0}^{t+k-i-1} F^{t+j} P w_{i+j},
\]

\[
u^*_t = - (R + B^T P B)^{-1} B^T (PAx_t + Pw_t + \frac{1}{2} v_{t+1})
\]

\[
= - (R + B^T P B)^{-1} B^T (PAx_t + \sum_{j=0}^{t+k-i-1} F^{t+j} P w_{i+j})
\]

The case \( i = t \) gives:

\[
u^*_t = - (R + B^T P B)^{-1} B^T (PAx_t + \sum_{j=0}^{k-1} F^{t+j} P w_{t+j})
\]

which is the MPC policy at time step \( t \), and is same as Equation (3). \( \square \)

**B Proofs of Section 4**

**B.1 Proof of Theorem 4.1**

The optimal control policy with general stochastic disturbance is given by:

\[
u_t = - (R + B^T P B)^{-1} B^T (PAx_t + \sum_{i=0}^{k-1} F^{t+i} P w_{t+i} + \sum_{i=k}^{\infty} F^{t+i} P \mu_{t+i|t+k-1})
\]

where \( \mu_{t+i|t+k-1} = \mathbb{E}[w_{t+i} | w_t, \ldots, w_{t+k-1}] \). Under this policy, the marginal benefit of obtaining an extra prediction decays exponentially fast in the existing number \( k \) of predictions. Formally, for \( k \geq 1 \),

\[
STO_k - STO_{k+1} = O(||F^k||^2) = O(\lambda^{2k}).
\]

**Proof.** Similar to the proof of Theorem 3.2, we assume

\[
V_i(x_i; w_{i:t+k-1}) = x_i^T P x_i + x_i^T v_i + q_i,
\]

where \( V_i \) has a similar definition as in Equation (5) but may further depend on \( w_{i}, \ldots, w_{i-1} \) because the disturbance sequence is no longer Markovian. In this case, \( P_t, v_t \) and \( q_t \) still satisfy the recursive
forms in Equation (7). However, the expected values of \( w_t \) and \( v_t \) are different since we have a more general distribution now. Let \( T - t \to \infty \), \( \mu_{t+i}^{T} = \mathbb{E}[w_{t+i} | w_{0}, \ldots, w_{t}] \) and \( F = A - HPA \). Then,

\[
\psi_{t}^{k} = 2 \sum_{i=0}^{k-1} F^{\top+i+1} P w_{t+i} + 2 \sum_{i=k}^{\infty} F^{\top+i+1} P \mu_{t+i[t+k-1]},
\]

where the superscript \( k \) denotes the number of predictions.

The optimal policy in this case has the same form as Equation (6). Plugging Equation (12) into it, we obtain the optimal policy in the theorem.

Further,

\[
\mathbb{E}[q_{t}^{k} - q_{t}^{k+1}] = \mathbb{E}\left[w_{t}^{\top}(I - PH) \left( w_{t+k} \mathbb{E}[v_{t+k+1}] - \mathbb{E}[w_{t+k+1}] \mathbb{E}[v_{t+k+1}] \right) \right] + \frac{1}{4} \mathbb{E}\left[ w_{t+k+1} \mathbb{E}[v_{t+k+1}] \mathbb{E}[v_{t+k+1}] - \mathbb{E}[w_{t+k+1}] \mathbb{E}[v_{t+k+1}] \right] + \mathbb{E}\left[ q_{t}^{k+1} - q_{t}^{k+1} \right],
\]

where the expectation \( \mathbb{E} \) is taken over all randomness. Part (13a) is zero because

\[
\mathbb{E}[v_{t+k+1}] = \mathbb{E}[w_{t+k+1} v_{t+k+1}].
\]

Part (13b) becomes

\[
\mathbb{E}\left[w_{t+k} \mathbb{E}[v_{t+k+1}] \mathbb{E}[v_{t+k+1}] - \mathbb{E}[w_{t+k+1}] \mathbb{E}[v_{t+k+1}] \right] = \mathbb{E}\left[w_{t+k} z_{k,t} H z_{k,t} \right],
\]

where

\[
z_{k,t} = F^{\top} P(w_{t+k} - \mu_{t+k[t+k-1]} + \sum_{i=k+1}^{\infty} F^{\top} P(\mu_{t+i[t+k-1]} - \mu_{t+i[t+k-1]}).
\]

Note that \( z_{k,t} = F^{\top} z_{k-1,t+1} = F^{\top} z_{0,t+k} \). Thus,

\[
STO_{k} - STO_{k+1} = \lim_{T \to \infty} \frac{1}{T} \mathbb{E}[q_{0}^{k} - q_{0}^{k+1}] = \lim_{T \to \infty} \frac{1}{T} \sum_{t=0}^{T-1} \mathbb{E}\left[z_{k,t}^{\top} H z_{k,t} \right] = \lim_{T \to \infty} \frac{1}{T} \sum_{t=0}^{T-1} \mathbb{E}\left[z_{0,t+k}^{\top} F^{\top} k H F^{\top} k z_{0,t+k} \right] = \lim_{T \to \infty} \frac{1}{T} \sum_{t=0}^{T-1} \text{Tr}\left(F^{\top} k H F^{\top} k \mathbb{E}[z_{0,t+k}^{\top} z_{0,t+k}] \right) \leq \|F^{\top} k\|^{2} \|H\| \lim_{T \to \infty} \frac{1}{T} \sum_{t=0}^{T-1} \text{Tr}\left(z_{0,t+k}^{\top} z_{0,t+k} \right)
\]

where in the last line we use the fact that if \( A \) is symmetric, then \( \text{Tr}\{AB\} \leq \lambda_{\text{max}}(A) \text{Tr}\{B\} \).

Finally we just need to show the last item \( \text{Tr}\mathbb{E}[z_{0,t+k}^{\top} z_{0,t+k}] \) is uniformly bounded for all \( t \). This is straightforward because the cross-correlation of each disturbance pair is uniformly bounded, i.e., there exists \( m > 0 \) such that for all \( t, t' \geq 1, \mathbb{E}[w_{t}^{\top} w_{t'}] \leq m \).

\[
\text{Tr}\mathbb{E}[z_{0,t}^{\top} z_{0,t}] = \sum_{i,j=0}^{\infty} \text{Tr}\mathbb{E}[P^{\top} F^{\top} P(\mu_{t+j}|t - \mu_{t+j}|t-1)(\mu_{t+i}|t - \mu_{t+i}|t-1)^{\top}]
\]
θ(\sum_{i,j=0}^{\infty} \text{Tr} \left\{ PF_i F^T J P \mathbb{E} \left[ \mu_{t+j|i} \mu_{t+i}[t] - \mu_{t+j|i} \mu_{t+i}[t-1] \right] \right\} ) \\
\leq \sum_{i,j=0}^{\infty} \| F_i \| \| F_j \| \| P \|^2 \mathbb{E} \left[ w_{t+j} w_{t+i} - w_{t+j} w_{t+i} \right] \\
\leq \sum_{i,j=0}^{\infty} c \lambda^j \| P \|^2 2m = 2 \frac{c^2}{(1-\lambda)^2} \| P \|^2 m
for some constant c from Gelfand’s formula. Thus \( \text{Tr} \mathbb{E} [z_{0,t} z_{0,t}^T] \) is bounded by a constant independent of \( t \). Thus, \( \text{STO}_k - \text{STO}_{k+1} = O(\| F_k \|^2) \).

\[ \square \]

**B.2 Proof of Theorem 4.4**

\( \text{MPCS}_k - \text{MPCS}_{k+1} = O(\| F_k \|^2) = O(\lambda^{2k}) \). Moreover, in Example 4.3, \( \text{MPCS}_k - \text{MPCS}_{k+1} = O(\| F_k \|^2) \).

**Proof.** To recursively calculate the value of \( J^\text{MPC}_k \), we define:

\[ V^\text{MPC}_k(x_t; w_{0:t+k-1}) = \sum_{i=0}^{T-1} (x_i^T Q x_i + u_i^T R u_i) + x_T^T Q x_T \]
\[ = x_t^T Q x_t + u_t^T R u_t + V_{t+1}(A x_t + B u_t + w_t; w_{0:t+k}) \]

as the cost-to-go function with MPC as the policy, i.e., \( u_t \) is the control at time step \( t \) from the MPC policy with \( k \) predictions. Similar to the previous proofs, we assume \( V^\text{MPC}_k(x) = x^T P x + x^T v_t + q_t \) (which turns out to be correct by induction) and \( T - t \to \infty \) so that \( P_t = P \). Then,

\[ V^\text{MPC}_k(x_t; w_{0:t+k-1}) = x_t^T Q x_t + u_t^T R u_t + (A x_t + B u_t + w_t)^T P (A x_t + B u_t + w_t) \]
\[ + (A x_t + B u_t + w_t)^T v_{t+1} + q_{t+1} \]
\[ = u_t^T (R + B^T P B) u_t + 2 u_t^T B^T (P A x_t + P w_t + v_{t+1}/2) \]
\[ + x_t^T Q x_t + (A x_t + w_t)^T P (A x_t + w_t) + (A x_t + w_t)^T v_{t+1} + q_{t+1}. \]

Let \( F = A - H P A \). Plugging in the formula of \( u_t \) in Theorem 3.3, we have

\[ V^\text{MPC}_k(x_t; w_{0:t+k-1}) = \left( \frac{1}{2} v_{t+1} - \sum_{i=1}^{k-1} F^{T+i} P w_{t+i} \right)^T H \left( \frac{1}{2} v_{t+1} - \sum_{i=1}^{k-1} F^{T+i} P w_{t+i} \right) \]
\[ - (P A x_t + P w_t + \frac{1}{2} v_{t+1})^T H (P A x_t + P w_t + \frac{1}{2} v_{t+1}) \]
\[ + x_t^T Q x_t + (A x_t + w_t)^T P (A x_t + w_t) + (A x_t + w_t)^T v_{t+1} + q_{t+1} \]
\[ = x_t^T (Q + A^T P A - A^T P H P A) x_t + x_t^T (F^T v_{t+1} + 2 F^T P w_t) \]
\[ + \left( \frac{1}{2} v_{t+1} - \sum_{i=1}^{k-1} F^{T+i} P w_{t+i} \right)^T H \left( \frac{1}{2} v_{t+1} - \sum_{i=1}^{k-1} F^{T+i} P w_{t+i} \right) \]
\[ - (P w_t + \frac{1}{2} v_{t+1})^T H (P w_t + \frac{1}{2} v_{t+1}) + w_t^T P w_t + w_t^T v_{t+1} + q_{t+1} \]
\[ = x_t^T P x_t + x_t^T v_t + q_t. \]

Thus,

\[ v_t = F^T v_{t+1} + 2 F^T P w_t = 2 \sum_{i=0}^{\infty} F^{T+i} P w_{t+i}. \]
Then, we can plug \( v_{t+1} \) into \( q_t \):

\[
q_t = \sum_{i=k}^\infty F^{T_i} P w_{t+i} H \left( \sum_{i=k}^\infty F^{T_i} P w_{t+i} \right) \nonumber
- \left( \sum_{i=0}^k F^{T_i} P w_{t+i} \right) H \left( \sum_{i=0}^k F^{T_i} P w_{t+i} \right) + w_t^T P w_t + 2w_t^T \left( \sum_{i=1}^\infty F^{T_i} P w_{t+i} \right).
\]

(15)

Note that Equation (15) is for MPC with \( k \) predictions. With the disturbance sequence \( \{w_i\} \) fixed, we can compare the per-step cost of MPC with \( k \) predictions and that with \( k+1 \) predictions:

\[
q_t^k - q_t^{k+1} = q_t^k - q_t^{k+1} + \left( \sum_{i=k}^\infty F^{T_i} P w_{t+i} \right) H \left( \sum_{i=k}^\infty F^{T_i} P w_{t+i} \right) \nonumber
- \left( \sum_{i=k+1}^\infty F^{T_i} P w_{t+i} \right) H \left( \sum_{i=k+1}^\infty F^{T_i} P w_{t+i} \right)
= q_t^k - q_t^{k+1} + w_t^T P F^k H F^{T_k} \left( P w_{t+k} + 2 \sum_{i=1}^\infty F^{T_i} P w_{t+i+k} \right).
\]

Thus,

\[
\mathbb{E}[q_t^k - q_t^{k+1} - (q_t^k - q_t^{k+1})] = \mathbb{E} \left[ w_t^T P F^k H F^{T_k} \left( P w_{t+k} + 2 \sum_{i=1}^\infty F^{T_i} P w_{t+i+k} \right) \right]
\]

\[
= \text{Tr} \left\{ P F^k H F^{T_k} \left( P \mathbb{E}[w_{t+k} w_t^T] + 2 \sum_{i=1}^\infty F^{T_i} P \mathbb{E}[w_{t+i+k} w_t^T] \right) \right\}
\]

\[
= \text{Tr} \left\{ P F^k H F^{T_k} Z_{k,t} \right\},
\]

where \( Z_{k,t} = P \mathbb{E}[w_{t+k} w_t^T] + 2 \sum_{i=1}^\infty F^{T_i} P \mathbb{E}[w_{t+i+k} w_t^T] \). Note that \( Z_{k,t} = Z_{k-1,t+1} \).

\[
\text{MPCS}_k - \text{MPCS}_{k+1} = \lim_{T \to \infty} \frac{1}{T} \mathbb{E}[q_0^k - q_0^{k+1}]
\]

\[
= \lim_{T \to \infty} \frac{1}{T} \sum_{t=0}^{T-1} \text{Tr} \left\{ P F^k H F^{T_k} Z_{k,t} \right\}
\]

\[
\leq \lim_{T \to \infty} \frac{1}{T} \sum_{t=0}^{T-1} \|P\| \|H\| \|F^k\|^2 \text{Tr}\{Z_{k,t}\},
\]

where in the last line we use the fact that if \( A \) is symmetric, then \( \text{Tr}\{AB\} \leq \|A\| \text{Tr}\{B\} \). Similarly to the last part in the proof of Theorem 4.1, now we just need to show the last term \( \text{Tr}\{Z_{k,t}\} \) is uniformly bounded for all \( t \). Again, this is because the cross-correlation of each disturbance pair is uniformly bounded.

\[
\text{Tr}\{Z_{k,t}\} \leq \|P\| \text{Tr}\mathbb{E}[w_{t+k} w_{t+k}^T] + 2 \sum_{i=1}^\infty \|P\| \|F^i\| \mathbb{E} \left[ \sum_j \sigma_j(w_{t+i+k} w_{t+i+k}^T) \right]
\]

\[
\leq \|P\| m + 2 \sum_{i=1}^\infty c \lambda^i \|P\| m \leq \|P\| m + 2c \lambda \|P\| m \]

where \( c \) is some constant, and in the first line, we use the fact that \( \text{Tr}\{AB\} \leq \|A\| \sum_j \sigma_j(B) \) with \( \sigma_j(\cdot) \) denoting the \( j \)-th singular value. Thus, \( \text{Tr}\{Z_{k,t}\} \) is uniformly bounded. Therefore, \( \text{MPCS}_k - \text{MPCS}_{k+1} = O(\|F^k\|^2) \). □
### B.3 Proof of Theorem 4.6

\(Reg^S(\text{MPC}_k) = \text{MPCS}^T_k - \text{STOT}^T_k = O(\|F\|^2T + 1) = O(\lambda^{2k}T + 1),\) where the second term results from the difference between finite/infinite horizons.

**Proof.** To calculate the dynamic regret, we cannot simply let \(T - t \to \infty\) as we did before Equation (4) in the proof of Theorem 4.4 and instead need to handle the expressions in a more delicate manner. In particular, we need to rigorously analyze the impact of finite horizon. Let \(\Delta_t = P_t - P.

\[
V_t^{\text{MPC}_k}(x_t; w_{0:t+k-1}) = u_t^T (R + B^T P_t 1) u_t + 2u_t^T B^T (P_{t+1} Ax_t + P_{t+1} w_t + v_{t+1}/2) + x_t^T Q x_t + (Ax_t + w_t)^T P_{t+1} (Ax_t + w_t) + (Ax_t + w_t)^T v_{t+1} + q_{t+1} = u_t^T (R + B^T P) u_t + 2u_t^T B^T (P Ax_t + P w_t + v_{t+1}/2) + x_t^T Q x_t + (Ax_t + w_t)^T P (Ax_t + w_t) + (Ax_t + w_t)^T v_{t+1} + q_{t+1} + u_t^T B^T \Delta_{t+1} u_t + 2u_t^T B^T \Delta_{t+1} (Ax_t + w_t) + (Ax_t + w_t)^T \Delta_{t+1} (Ax_t + w_t).
\]

Plugging in the MPC policy as in Theorem 3.3, we have:

\[
V_t^{\text{MPC}_k}(x_t; w_{0:t+k-1}) = x_t^T (Q + A^T P A - A^T PHPA) x_t + x_t^T (F^T v_{t+1} + 2F^T P w_t) + \left( \frac{1}{2}v_{t+1} - \sum_{i=1}^{k-1} F^T P w_{t+i} \right)^T H \left( \frac{1}{2}v_{t+1} - \sum_{i=1}^{k-1} F^T P w_{t+i} \right) - \left( P w_t + \frac{1}{2}v_{t+1} \right)^T H \left( P w_t + \frac{1}{2}v_{t+1} \right) + w_t^T P w_t + w_t^T v_{t+1} + q_{t+1} + \left( F x_t + w_t - \sum_{i=0}^{k-1} F^T P w_{t+i} \right)^T \Delta_{t+1} \left( F x_t + w_t - \sum_{i=0}^{k-1} F^T P w_{t+i} \right) = x_t^T (Q + A^T P A - A^T PHPA + F^T \Delta_{t+1} F) x_t + x_t^T \left( F^T v_{t+1} + 2F^T P w_t + 2F^T \Delta_{t+1} \left( w_t - \sum_{i=0}^{k-1} F^T P w_{t+i} \right) \right) + \left( \frac{1}{2}v_{t+1} - \sum_{i=1}^{k-1} F^T P w_{t+i} \right)^T H \left( \frac{1}{2}v_{t+1} - \sum_{i=1}^{k-1} F^T P w_{t+i} \right) - \left( P w_t + \frac{1}{2}v_{t+1} \right)^T H \left( P w_t + \frac{1}{2}v_{t+1} \right) + w_t^T P w_t + w_t^T v_{t+1} + q_{t+1} + \left( w_t - \sum_{i=0}^{k-1} F^T P w_{t+i} \right)^T \Delta_{t+1} \left( w_t - \sum_{i=0}^{k-1} F^T P w_{t+i} \right)
\]

Comparing this with the induction hypothesis \(V_t^{\text{MPC}_k} = x_t^T (P + \Delta_t) x_t + x_t^T v_t + q_t\), we obtain the recursive formulae for \(\Delta_t, v_t, q_t\).

\[
\Delta_t = F^T \Delta_{t+1} F = F^{T-t} \Delta_T F^{T-t} = F^{T-t} (Q_f - P) F^{T-t}.
\]

This implies that \(P_t\) converges to \(P\) exponentially fast, i.e., \(\|\Delta_t\| = O(\|F^{T-t}\|^2) = O(\lambda^{2(T-t)})\).

\[
v_t = F^T v_{t+1} + 2F^T P w_t + 2F^T \Delta_{t+1} \left( w_t - \sum_{i=0}^{k-1} F^T P w_{t+i} \right) = 2 \sum_{j=0}^{T-t-1} \left( F^{T+j+1} P w_{t+j} + F^{T+j+1} \Delta_{t+j+1} \left( w_{t+j} - \sum_{i=0}^{k-1} F^T P w_{t+j+i} \right) \right)
\]
We have a formula for
\[
= \left( \left( \sum_{i=0}^{T-1} F_{t+i}^T P_{w_{t+i}} + 2 \sum_{j=0}^{T-1} F_{t+j+1}^T \Delta_{t+j+1} \left( w_{t+j} - \sum_{i=0}^{k-1} F_{t+j+i}^T P_{w_{t+j+i}} \right) \right) \right).
\]

Denote the second term by \(2d_t\). We have
\[
d_t = \sum_{j=0}^{T-t-1} F_{t+j+1}^T \Delta_{t+j+1} \left( w_{t+j} - \sum_{i=0}^{k-1} F_{t+j+i}^T P_{w_{t+j+i}} \right)
= \sum_{j=0}^{T-t-1} O(\lambda^j \lambda^2(T-t-j)) = O(\lambda^{T-t}).
\]

Finally, we have a formula for \(q_t\) that parallels Equation (15):
\[
q_t = q_{t+1} + \left( d_{t+1} + \sum_{i=k}^{T-t-1} F_{t+i}^T P_{w_{t+i}} \right)^\top H \left( d_{t+1} + \sum_{i=k}^{T-t-1} F_{t+i}^T P_{w_{t+i}} \right)
- \left( d_{t+1} + \sum_{i=1}^{T-t-1} F_{t+i}^T P_{w_{t+i}} \right)^\top H \left( d_{t+1} + \sum_{i=1}^{T-t-1} F_{t+i}^T P_{w_{t+i}} \right)
+ w_t^T P w_t + 2w_t^T \left( d_{t+1} + \sum_{i=1}^{T-t-1} F_{t+i}^T P_{w_{t+i}} \right).
\]

Taking the difference between \(k\) and \(k+1\) predictions, we have
\[
\begin{align*}
d_t^k - d_t^{k+1} & = \left( w_{t+k}^T P F_k + (d_{t+1}^k - d_{t+1}^{k+1}) \right) H \left( d_{t+1}^k + d_{t+1}^{k+1} + F_{t+k}^T P_{w_{t+k}} + 2 \sum_{i=1}^{T-t-k-1} F_{t+i+k}^T P_{w_{t+i+k}} \right) \\
& = (w_{t+k}^T P F_k + O(\lambda^{T-k}||F_k||)) H \left( O(\lambda^{T-k}) + F_{t+k}^T P_{w_{t+k}} + 2 \sum_{i=1}^{T-t-k-1} F_{t+i+k}^T P_{w_{t+i+k}} \right) \\
& = (w_{t+k}^T P F_k + O(\lambda^{T-k}||F_k||)) \left( O(\lambda^{T-k}) + F_{t+k}^T P_{w_{t+k}} + 2 \sum_{i=1}^{T-t-k-1} F_{t+i+k}^T P_{w_{t+i+k}} \right),
\end{align*}
\]

and thus
\[
\begin{align*}
\mathbb{E}[q_t^k - q_t^{k+1} - (q_{t+1}^k - q_{t+1}^{k+1})] & = O(||F_k|| (\lambda^{T-t} + ||F_k||)). \\
\mathbb{E}[q_0^k - q_0^T] & = \sum_{t=0}^{T-1} \mathbb{E}[q_t^k - q_t^{k+1} - (q_{t+1}^k - q_{t+1}^{k+1})] \\
& = \sum_{t=0}^{T-1} O(||F_k|| (\lambda^{T-t} + ||F_k||)) \\
& = O(||F_k|| (\lambda^{T} + ||F_k||)). \\
\mathbb{E}[v_0^k - v_0^T] & = 2(d_0^k - d_0^T) = O(\lambda^{T+k}||F_k||). \\
\mathbb{E}[J_{\text{MP}} - J_{\text{MP}C}] & = \mathbb{E}[V_0^k (x_0) - V_0^T (x_0)] \\
& = \mathbb{E}[x_0^T (v_0^k - v_0^T) + (q_0^k + q_0^T)].
\end{align*}
\]
By definition, $J^{\text{MPC}_T}$ is the cost of MPC policy given all future disturbances before making any decisions. It almost equals to $\min_u J$, the optimal policy given all future disturbances, except that during optimization, MPC assumes the final-step cost to be $x_T^T P x_T$ instead of $x_T^T Q_f x_T$. This will incur at most constant extra cost, i.e.,

$$J^{\text{MPC}_T} - \min_u J = O(P - Q_f) = O(1).$$

By Equations (19) and (20),

$$\text{Reg}^S(\text{MPC}_k) = \mathbb{E} J^{\text{MPC}_k} - \mathbb{E} \min_u J = O(\|F^k\|^2 T + \|F^k\| + 1) = O(\|F^k\|^2 T + 1).$$

**B.4 Proof of Theorem 4.7**

The optimal dynamic regret $\text{Reg}^S_k = \text{STO}_T^k - \text{STO}_T = O(\|F^k\|^2 T + 1) = O(\lambda^{2k} T + 1)$ and there exist $A, B, Q, R, Q_f, x_0$, and $W$ such that $\text{Reg}^S_k = \Theta(\|F^k\|^2 (T - k))$. 

**Proof.** The first part follows from Theorem 4.6 and the fact that $\text{Reg}^S_k \leq \text{Reg}^S(\text{MPC}_k)$. The second part is shown by Example 4.3, i.e., suppose $n = d = 1$ and the disturbance are i.i.d. and zero-mean. Additionally, let $Q_f = P$ and $x_0 = 0$. In this case, MPC has not only the same policy but also the same cost as the optimal control policy. Also, $P_t = P$ for all $t$. To calculate the total cost, we follow the approach used in the proof of Theorem 3.2. Since $T$ is finite now, we have a similar (to Equation (8)) but different form of $v_t$:

$$v_t = 2 \sum_{i=0}^{\min\{k-1, T-t-1\}} F_{t+i+1}^T P w_{t+i}.$$ 

Thus,

$$\mathbb{E}[q_t] = \text{Tr}\left\{ \left( P - \sum_{i=0}^{\min\{k-1, T-t-1\}} Pf_i H F_{t+i}^T \right) W \right\} + \mathbb{E}[q_{t+1}].$$

$$\mathbb{E}[q_0] = \text{Tr}\left\{ \sum_{i=0}^{T-1} \left( P - \sum_{i=0}^{\min\{k-1, T-t-1\}} Pf_i H F_{t+i}^T \right) W \right\}.$$ 

Let $q^k_t$ denote $q_t$ in the scenario of $k$ predictions.

$$\text{Reg}^S_k = \mathbb{E}[q^k_0 - q^k_T] = \text{Tr}\left\{ \sum_{i=0}^{T-k-1} \sum_{i=0}^{T-t-1} Pf_i H F_{t+i}^T P W \right\} \geq (T-k) \text{Tr}\left\{ Pf_k H F_{k}^T P W \right\} = \Omega(\|F^k\|^2 (T - k)).$$

On the other hand,

$$\text{Reg}^S_k = \mathbb{E}[q^k_0 - q^k_T] \leq (T-k) \text{Tr}\left\{ \sum_{i=k}^{\infty} Pf_i H F_{t+i}^T P W \right\} = O(\|F^k\|^2 (T - k)).$$

Therefore, $\text{Reg}^S_k = \Theta(\|F^k\|^2 (T - k)).$ 

**C Proofs of Section 5**

**C.1 Proof of Theorem 5.1**

For $k \geq 1$, $\text{ADV}_k - \text{ADV}_{k+1} = O(\|F^k\|^2) = O(\lambda^{2k})$. 

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Proof. This proof is based on Theorem 5.3. It turns out that the behavior of the MPC policy and its cost is easier to analyze than the optimal one, especially in the adversarial setting.

\[ \text{ADV}_k - \text{ADV}_{k+1} \leq \text{ADV}_k - \text{ADV}_\infty \leq \text{MPCA}_k - \text{ADV}_\infty = \sum_{i=k}^{\infty} \text{MPCA}_i - \text{MPCA}_{i+1}. \]

By Theorem 5.3

\[ \text{MPCA}_i - \text{MPCA}_{i+1} \leq O\left(\|F^i\|^2\right) \leq O\left(\|F^k\|^2\|F^{i-k}\|^2\right) \leq O\left(\|F^k\|^2\lambda^{2(i-k)}\right). \]

Thus,

\[ \text{ADV}_k - \text{ADV}_{k+1} \leq O\left(\|F^k\|^2 \sum_{i=k}^{\infty} \lambda^{2(i-k)}\right) = O(\|F^k\|^2). \]

\[ \square \]

C.2 Proof of Example 5.2

Let \( A = B = Q = R = 1 \) and \( \Omega = [-1, 1] \). In this case, one prediction is enough to leverage the full power of prediction. Formally, we have \( \text{ADV}_1 = \text{ADV}_\infty = 1 \). In other words, for all \( k \geq 1 \), \( \text{ADV}_k = 1 \). The optimal control policy (as \( T \to \infty \)) is a piecewise function:

\[ u^*(x, w) = \begin{cases} 
-(x + w), & -1 \leq x + w \leq 1 \\
-(x + w) + \frac{3 - \sqrt{2}}{2}(x + w - 1), & x + w > 1 \\
-(x + w) + \frac{3 - \sqrt{2}}{2}(x + w + 1), & x + w < -1
\end{cases}. \]

The proof leverages two different cost-to-go functions for the min player and the sup player.

Proof. We will show \( \text{ADV}_1 = 1 \) and \( \text{ADV}_\infty = 1 \) separately. The system dynamics is given by \( x_{t+1} = x_t + u_t + w_t \) with \( w_t \in [-1, 1] \) and

\[ \text{ADV}_1^T = \min_{u_0} \max_{u_1} \cdot \max_{u_T} \min_{u_{T-1}} \sum_{t=0}^{T-1} (u_t^2 + u_{t+1}^2) + x_T^2. \]

We will calculate the results of each min and max by dynamical programming. In particular, we will define two cost-to-go functions for the min player and the max player respectively. Let \( z_t = x_t + w_t \). Then, \( z_t \) can be regarded as the disturbed state. This is natural since the controller has one prediction and decides \( u_t \) after knowing \( w_t \). Thus, the system dynamics can be split into two stages: \( z_t = x_t + w_t \) and \( x_{t+1} = z_t + u_t \). Let

\[ f_t(z_t) = \min_{u_t} \max_{u_{t+1}} \cdot \min_{u_{t+1}} \cdot \max_{u_{t+1}} \sum_{i=t}^{T-1} (u_i^2 + x_{i+1}^2) = \min_{u_t} \left( u_t^2 + (z_t + u_t)^2 + g_{t+1}(z_t + u_t) \right) \]

\[ g_t(x_t) = \max_{u_t} \cdot \min_{u_{t+1}} \cdot \max_{u_{t+1}} \sum_{i=t}^{T-1} (u_i^2 + x_{i+1}^2) \]

\[ = \max_{u_t} f_t(x_t + u_t). \]

For \( t = T - 1 \), we have

\[ f_{T-1}(z) = \min_u u^2 + (z + u)^2 = \frac{z^2}{2}, \]

\[ g_{T-1}(x) = \max_w \frac{(x + w)^2}{2} = \frac{(|x| + 1)^2}{2}. \]

We will prove by backward induction that \( g_t(x) = a_t x^2 + 2b_t |x| + c_t \) where \( a_t, b_t, c_t \) are some coefficients with \( 0 < b_t < 1 \). Assuming this is true at \( t \), we will show this is true at \( t - 1 \).

\[ f_{t-1}(z) = \min_u (u^2 + (z + u)^2 + g_t(z + u)) \]
According to Equations (8) and (9) with $k$ (induction hypothesis) and the adversary — who wants to maximize 

\[ \text{The optimal control policy is obtained by plugging the above values back into Equation (21):} \]

For $c^T$, we have 

\[ \text{Solving the Riccati equation, we have} \]

\[ P = \frac{1+\sqrt{5}}{2}, \quad H = F = \frac{3-\sqrt{5}}{2}. \]

When $w_t = 1$ for all $t$, 

\[ \text{STO}_\infty = 1. \]
C.3 Proof of Theorem 5.3

\[ \text{MPCA}_k - \text{MPCA}_{k+1} = O(\|F^k\|^2) = O(\lambda^{2k}). \]

**Proof.** Note that Equation (16) in the proof of Theorem 4.4 does not rely on the type of disturbance, i.e., Equation (16) holds for adversarial disturbance as well. Let \( r = \sup_{w \in \Omega} \|w\|^2. \)

\[
q_t^k - q_{t+1}^k = w_t^\top P F^k H^\top \left( P w_{t+k} + 2 \sum_{i=1}^\infty F_i^\top P w_{t+i+k} \right)
\leq \|w_{t+k}\| P \|H\| \|F^k\|^2 \left( \|P\| \|w_{t+k}\| + 2 \sum_{i=1}^\infty F_i \|P\| \|w_{t+i+k}\| \right)
\leq \|F^k\|^2 \left( 1 + 2 \sum_{i=1}^\infty \|F_i\| \right) \|H\| \|P\|^2 r^2
\leq \|F^k\|^2 \left( 1 + 2 \frac{c \lambda}{1 - \lambda} \right) \|H\| \|P\|^2 r^2
\]

for some constant \( c. \)

\[
\text{MPCA}_k - \text{MPCA}_{k+1} = \lim_{T \to \infty} \frac{1}{T} \max_{w} (q_0^k - q_{0}^{k+1})
\leq \lim_{T \to \infty} \frac{1}{T} \max_{w} (q_0^k - q_{0}^{k+1})
\leq \lim_{T \to \infty} \frac{1}{T} \sum_{t=0}^{T-1} \max_{w} (q_t^k - q_{t+1}^k - (q_{t+1}^k - q_{t+1}^{k+1}))
\leq \|F^k\|^2 \left( 1 + 2 \frac{c \lambda}{1 - \lambda} \right) \|H\| \|P\|^2 r^2 = O(\|F^k\|^2).
\]

C.4 Proof of Theorem 5.5

\[ \text{Reg}^A(\text{MPC}_k) = O(\|F^k\|^2 T + 1) = O(\lambda^{2k} T + 1). \]

**Proof.** We follow the notations in the proof of Theorem 4.6 Equation (18) does not rely on the type of disturbance, so it holds for adversarial disturbance as well. By Equation (18) and the fact that \( w_t \) is bounded, we have

\[
q_t^k - q_{t+1}^k - (q_{t+1}^k - q_{t+1}^{k+1}) = O(\|F^k\|(\lambda^{T-t} + \|F^k\|)),
\]

where the constant in the Big-Oh notation does not depend on the disturbance sequence \( w. \) Thus,

\[
\max_{w} (q_0^k - q_{0}^T) \leq \sum_{t=0}^{T-1} \max_{w} (q_t^k - q_{t+1}^k - (q_{t+1}^k - q_{t+1}^{k+1})) = O(\|F^k\|^2 T + \|F^k\|).
\]

By Equation (17) and the boundedness of \( w_t, \)

\[
\max_{w} (v_0^T - v_0^T) = 2 \max_{w} (d_0^T - d_0^T) = O(\lambda^{T+k} \|F^k\|).
\]

\[
\max_{w} (\text{MPC}_k - \text{MPC}_T) = \max_{w} (V_0^T (x_0) - V_0^T (x_0)) \leq \max_{w} (x_0 (v_0^T - v_0^T)) + \max_{w} (q_0^k - q_0^T) = O(\|F^k\|^2 T + \|F^k\|).
\]

As Equation (20), \( J_{\text{MPC}_T} - \min_{u} J = O(1). \) Thus,

\[
\text{Reg}^A(\text{MPC}_k) = \max_{u} (\text{MPC}_k - \min_{u} J) \leq \max_{u} (\text{MPC}_k - \text{MPC}_T) + \max_{u} (\text{MPC}_T - \min_{u} J) = O(\|F^k\|^2 T + \|F^k\|^2 T + 1).
\]

\( \square \)
C.5 Proof of Theorem 5.6

\( \text{Reg}_{k}^{A^*} = O(\|F^k\|^2 T + 1) = O(\lambda^{2k} T + 1) \). Moreover, there exist \( A, B, Q, R, Q_f, x_0, \) and \( \Omega \) such that \( \text{Reg}_{k}^{A^*} = \Omega(\|F^k\|^2 (T - k)) \).

**Proof.** The first part of the theorem follows from Theorem 5.5 and the fact that \( \text{Reg}_{k}^{A^*} \leq \text{Reg}_{k}^{A^*}(\text{MPC}_k) \).

We reduce the second part of this theorem to the second part of Theorem 4.7. Since the proof of Theorem 4.7 works for any fixed distribution of \( w_t \) (with finite second moment), we can restrict that distribution to have bounded support. Denote this bounded support by \( \Omega \). Then, we have

\[
\text{Reg}_{k}^{A^*} = \sup_{u_0, \ldots, u_{k-1}} \min \sup_{w_0} \min \sup_{w_{k-1}} \min \sup_{w_{T-k-1}} \min \sup_{w_{T-k-1}} \min (J(u, w) - \min_{u'_0, \ldots, u'_{T-1}} J(u', w)) \\
\geq \mathbb{E} \min \mathbb{E} \cdots \min \mathbb{E} \min (J(u, w) - \min_{u'_0, \ldots, u'_{T-1}} J(u', w)) \\
= \text{Reg}_{k}^{\Omega^*} = \Theta(\|F^k\|^2 (T - k)).
\]

\( \square \)