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# Single-Agent Policy Tree Search With Guarantees: Supplementary Material

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\*This work was carried out while L. H. S. Lelis was at the University of Alberta, Canada.

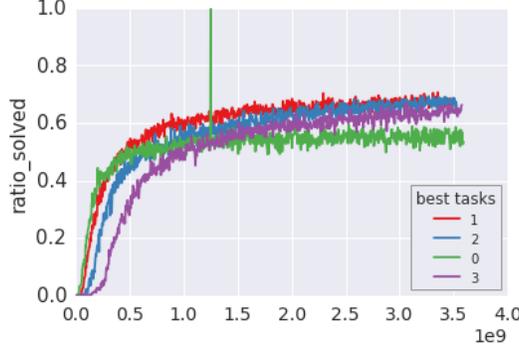


Figure 4: Learning curves of A3C for the 4 chosen learning rates (4e-4, 2e-4, 1e-4, 5e-5) on the Sokoban level generator.

## A Network architecture and learning protocol

The network takes as input a 10x10x4 grid where the last dimension is for a binary encoding of the different attributes (wall, man, goal, box), which is passed through 2 convolutional layers ( $4 \times 4$  with 64 channels, followed by  $3 \times 3$  with 64 channels as well), followed by a fully connected layer of 512 ReLU units. The output layer provides logits for the 4 actions (up, down, left, right). Training is performed using A3C [Mnih et al., 2016] with a reward function giving a reward of -0.1 per step, +1 per box on a goal and -1 for the converse action, and +10 for solving the level (all boxes on goals), with a discount factor of 0.99; the optimizer used is RMSProp [Tieleman and Hinton, 2012] (no momentum, epsilon 0.1, decay 0.99), with entropy regularization of 0.005. During training, at each episode, the learner performs a single trajectory of length 100 (like multiTS(1, 100)), receives the corresponding rewards, then moves on to the next episode. A single level is (very likely) never seen twice during training. Similarly, it is very unlikely that a level of the 1000 test levels was seen during training. We take the best performing network, which solves around 65% of the levels when sampling a single sequence of actions. The network is trained for 3.5e9 steps (node expansions), which can seem to be a lot, however notice that this is equivalent to fully searching a *single* level of Sokoban (without state cuts) uniformly with 4 actions up to depth 16 (given that solutions are usually of depth more than 30). The learning process was repeated for 4 learning rates (4e-4, 2e-4, 1e-4, 5e-5) (see Fig. 4).

## B Another universal restarting strategy for Las Vegas programs

We use the sequence<sup>5</sup> of runtimes  $f(n) := A6519(n)$ :

$$1 \ 2 \ 1 \ 4 \ 1 \ 2 \ 1 \ 8 \ 1 \ 2 \ 1 \ 4 \ 1 \ 2 \ 1 \ 16 \ 1 \ 2 \ 1 \dots$$

$$\text{For all } n \in \mathbb{N}_1 : f(n) := \begin{cases} 1 & \text{if } n \text{ is odd,} \\ 2f(n/2) & \text{o.w.} \end{cases}$$

It has the ‘fractal’ property that  $f(k2^n) = 2^n f(k)$  (since  $f(k2^n) = 2f(k2^{n-1}) = \dots = 2^n f(k2^0)$ ), for  $k \in \mathbb{N}_1$  and  $n \in \mathbb{N}_0$ , and it follows that  $f(2^n) = 2^n$  and  $f(k2^n) \geq 2^n$ .

At iteration  $n$ , the Las Vegas program is run for  $f(n)$  steps. For all  $t > 0$ , if  $f(n) \geq t$ , then it has a probability at least  $q(t)$  of halting, otherwise it does not halt and is forcibly stopped after  $f(n)$  computations steps. Let  $\hat{t} := 2^{\lceil \log_2 t \rceil}$  be the smallest power of 2 greater than or equal to  $t$ . Then Lemma 8 below tells us that for  $c < \hat{t}$  we have that  $f(k\hat{t} + c) = f(c) \leq \hat{t}/2 < t$ , that is, between two consecutive factors of  $\hat{t}$ ,  $f(n) < t$ .

Let  $p_{\text{halt}}(n)$  denote the probability that the algorithm halts exactly at the  $n$ th run, and take  $1 \leq c < \hat{t}$  and  $k \geq 0$ , then the expected number of computation steps  $L$  (sum of the lengths of the runs) before

<sup>5</sup><https://oeis.org/A006519>.

halting is given by:

$$L_{\text{univ}}(p) := \sum_{n=1}^{\infty} [t p_{\text{halt}}(n) + (1 - p_{\text{halt}}(n))f(n)] \underbrace{\prod_{j=1}^{n-1} (1 - p_{\text{halt}}(j))}_{\text{probability of not halting before run } n}.$$

where  $p_{\text{halt}}(n) = 0$  when  $f(n) < t$ , and  $p_{\text{halt}}(n) = q(t)$  otherwise.

We restate Theorem 5 more precisely:

**Theorem 7.** For all distributions  $p$  over halting times, the expected runtime of the universal restarting strategy based on A6519 is bounded by

$$L_{\text{univ}}(p) \leq \min_t t + \frac{t}{q(t)} \left( \log_2 \frac{t}{q(t)} + 6.1 \right),$$

where  $q$  is the cumulative distribution of  $p$ .

*Proof of Theorems 5 and 7.* At step  $n$ , if  $k$  is the number of past runs where  $f(m) \geq \hat{t}$  (with  $m < n$ ), then  $\prod_{j=1}^{n-1} (1 - p_{\text{halt}}(j)) = (1 - q(t))^k$  then with  $1 \leq c < \hat{t}$  and  $\gamma := 1 - q(t)$ :

$$\begin{aligned} L_{\text{univ}}(p) &= \sum_{n=0}^{\infty} \begin{cases} \gamma^k f(n) & \text{if } n = k\hat{t} + c \quad (\text{i.e., } f(n) < t) \\ \gamma^k pt + \gamma^{k+1} f(n) & \text{if } n = k\hat{t} + \hat{t}, \end{cases} \\ &= \sum_{n=0}^{\infty} \begin{cases} \gamma^k f(c) & \text{if } n = k\hat{t} + c \\ \gamma^k pt + \gamma^{k+1} \hat{t} f(k+1) & \text{if } n = k\hat{t} + \hat{t}. \end{cases} \end{aligned}$$

where we used  $f((k+1)\hat{t}) = \hat{t}f(k+1)$  (remembering that  $\hat{t}$  is a power of 2) and Lemma 8 for  $f(k\hat{t} + c) = f(c)$ . Since  $f(n) = f(c) < t$  when  $n = k\hat{t} + c$ , we can decompose  $L_{\text{univ}}(p)$  into the steps where  $f(n) < t$  and the rest:

$$\begin{aligned} L_{\text{univ}}(p) &= L^< + L^{\geq} \\ L^< &:= \sum_{k=0}^{\infty} \gamma^k \sum_{c=1}^{\hat{t}-1} f(c) = \frac{1}{1-\gamma} \sum_{c=1}^{\hat{t}-1} f(c) = \frac{\hat{t}}{2q(t)} \log_2 \hat{t} \quad (\text{Lemma 9}) \\ L^{\geq} &:= \sum_{k=0}^{\infty} \gamma^k (1-\gamma)t + \gamma^{k+1} \hat{t} f(k+1) = t + \hat{t} \sum_{k=1}^{\infty} \gamma^k f(k) \\ &\leq t + \frac{\hat{t}}{q(\hat{t})} \left( \frac{1}{e} + \frac{1}{\ln 2} + \frac{1}{2} \log_2 \ln 16 + \frac{1}{2} \log_2 \frac{1}{q(\hat{t})} \right) \end{aligned}$$

where we used Lemma 13 on the last line with  $\gamma = 1 - q(t)$ . Finally, since  $\hat{t} = 2^{\lceil \log_2 t \rceil} \leq 2t$  and  $q(\hat{t}) \geq q(t)$  and  $\lceil \log_2 t \rceil \leq \log_2 t + 1$ :

$$\begin{aligned} L &\leq t + \frac{t}{q(t)} \left( \log_2 t + 1 + \frac{2}{e} + \frac{2}{\ln 2} + \log_2 \ln 16 + \log_2 \frac{1}{q(t)} \right) \\ &\leq t + \frac{t}{q(t)} \left( \log_2 \frac{t}{q(t)} + 6.1 \right) \end{aligned}$$

which proves the result.  $\square$

**Lemma 8.** For  $f = \text{A6519}$ , with  $k \in \mathbb{N}_0, n \in \mathbb{N}_0, a \in \mathbb{N}_1, b \in \mathbb{N}_0$  and  $a2^b < 2^n$ , and with  $a$  odd, then

$$f(k2^n + a2^b) = f(a2^b) = 2^b.$$

*Proof.* Since  $a$  is odd, then so is  $k2^{n-b} + a$ , and so  $f(k2^n + a2^b) = f(2^b(k2^{n-b} + a)) = 2^b f(k2^{n-b} + a) = 2^b$ .  $\square$

Hence, for all numbers between two adjacent factors of  $2^n$ ,  $f(k2^n + c) = f(c) \leq 2^{n-1}$ .

**Lemma 9.** For  $n \in \mathbb{N}_1$  and  $f = A6519$ ,

$$\sum_{c=1}^{2^n-1} f(c) = n2^{n-1}.$$

*Proof.* If  $n \geq 1$  and using Lemma 8 again at  $2^{n-1}$ :

$$\begin{aligned} \sum_{c=1}^{2^n-1} f(c) &= \sum_{c=1}^{2^{n-1}-1} f(c) + f(2^{n-1}) + \sum_{c=2^{n-1}+1}^{2^n-1} f(c) \\ &= 2^{n-1} + 2 \sum_{c=1}^{2^{n-1}-1} f(c) \\ &= \dots = 2^0 2^{n-1} + 2^1 2^{n-2} + 2^2 2^{n-3} + \dots + 2^{n-1} 2^0 + 2^n \sum_{c=1}^{2^0-1} f(c) \\ &= n2^{n-1}. \end{aligned}$$

□

**Lemma 10.** Let  $f = A6519$ , then for  $k \in \mathbb{N}_1, n \in \mathbb{N}_0, c \in \mathbb{N}_0$ :

$$f(k) = 2^n \Leftrightarrow k = (2c + 1)2^n.$$

*Proof.* Since any number  $k$  can be uniquely written in the form  $k = (2c + 1)2^a$ , and  $f((2c + 1)2^a) = 2^a f(2c + 1) = 2^a$  with  $a \in \mathbb{N}_0$ , then  $f(k) = 2^n \Leftrightarrow a = n$ . □

**Lemma 11.** For  $\gamma \in [0, 1)$ ,

$$\sum_{n=0}^{\infty} 2^n \gamma^{2^n} \leq \frac{1}{\ln \frac{1}{\gamma}} \left( \frac{1}{e} + \frac{\gamma}{\ln 2} \right).$$

*Proof.* Let  $h(x) := 2^x \gamma^{2^x}$  for  $x \in \mathbb{R}$ , then  $h'(x) = \ln(2) 2^x \gamma^{2^x} (2^x \ln \gamma + 1)$  where  $h'(x_0) = 0$  for the unique  $x_0$  such that  $2^{x_0} = \frac{1}{\ln \frac{1}{\gamma}}$  and since  $\ln \gamma < 0$ , we have that  $h'(x)$  is positive for  $x < x_0$  and negative for  $x > x_0$ . Thus  $h$  is unimodal, and since furthermore  $h(x)$  is positive the sum can be upper bounded by the integral of the continuous function plus its maximum:

$$\begin{aligned} \sum_{n=0}^{\infty} h(n) &\leq \int_0^{\infty} h(x) dx + \max_x h(x), \\ \max_x h(x) &= h(x_0) = \frac{1}{\ln \frac{1}{\gamma}} \frac{1}{e}, \\ \int_0^{\infty} 2^x \gamma^{2^x} dx &= \frac{1}{\ln 2} \int_0^{\infty} 2^x \ln 2 \gamma^{2^x} dx = \frac{1}{\ln 2} \int_1^{\infty} \gamma^y dy = \frac{\gamma}{\ln 2 \ln \frac{1}{\gamma}}, \end{aligned}$$

where we used integration by substitution. Adding the two terms finishes the proof. □

**Lemma 12.** For  $\gamma \in [0, 1)$  and  $a \geq 1$ :

$$\sum_{n=0}^{\infty} \gamma^{2^n} \leq \gamma \left\lceil \log_2 \frac{1}{\log_2 \frac{1}{\gamma}} \right\rceil + 1 \leq \log_2 \frac{1}{\ln \frac{1}{\gamma}} + \log_2 \ln 16.$$

*Proof.* Let  $N = \min \left\{ n \in \mathbb{N}_0 : \gamma^{2^N} \leq \frac{1}{2} \right\} = \left\lceil \log_2 \frac{1}{\log_2 \frac{1}{\gamma}} \right\rceil$ , then

$$\begin{aligned}
\sum_{n=0}^{\infty} \gamma^{2^n} &= \sum_{n=0}^{N-1} \gamma^{2^n} + \sum_{n=N}^{\infty} \gamma^{2^n} \\
&\leq N\gamma + \sum_{n=0}^{\infty} \left( \gamma^{2^N} \right)^{2^n} \leq N\gamma + \sum_{n=0}^{\infty} 2^{-2^n} \leq N\gamma + 1 \\
&\leq \left\lceil \log_2 \frac{1}{\log_2 \frac{1}{\gamma}} \right\rceil + 1 \\
&\leq \log_2 \frac{1}{\log_2 \frac{1}{\gamma}} + 2.
\end{aligned}$$

Extracting  $\log_2 \ln 2$  finishes the proof.  $\square$

**Lemma 13.** Let  $f = \text{A6519}$  and  $\gamma \in [0, 1)$ . Then

$$\sum_{k=1}^{\infty} \gamma^k f(k) \leq \frac{1}{1-\gamma} \left( \frac{1}{e} + \frac{1}{\ln 2} + \frac{1}{2} \log_2 \ln 16 + \frac{1}{2} \log_2 \frac{1}{1-\gamma} \right).$$

*Proof.* Since  $f(n)$  is a power of 2 for all  $n \in \mathbb{N}_1$ , we regroup the runs by powers of 2:

$$\begin{aligned}
\sum_{k=1}^{\infty} \gamma^k f(k) &= \sum_{n=0}^{\infty} 2^n \sum_{k=1}^{\infty} \gamma^k \llbracket f(k) = 2^n \rrbracket \\
&= \sum_{n=0}^{\infty} 2^n \sum_{c=0}^{\infty} \gamma^{(2c+1)2^n} \quad (\text{Lemma 10}) \\
&= \sum_{n=0}^{\infty} 2^n \gamma^{2^n} \sum_{c=0}^{\infty} \left( \gamma^{2^{n+1}} \right)^c = \sum_{n=0}^{\infty} 2^n \gamma^{2^n} \frac{1}{1-\gamma^{2^{n+1}}} \\
&\leq \sum_{n=0}^{\infty} 2^n \gamma^{2^n} \left( 1 + \frac{\gamma}{2^{n+1}(1-\gamma)} \right) \quad (\text{Lemma 14}) \\
&= \sum_{n=0}^{\infty} 2^n \gamma^{2^n} + \frac{1}{2} \frac{\gamma}{1-\gamma} \sum_{n=0}^{\infty} \gamma^{2^n} \\
&\leq \frac{1}{1-\gamma} \left( \frac{1}{e} + \frac{\gamma}{\ln 2} + \frac{\gamma}{2} + \frac{\gamma^2}{2} \left( \log_2 \ln 4 + \log_2 \frac{1}{1-\gamma} \right) \right) \\
&\leq \frac{1}{1-\gamma} \left( \frac{1}{e} + \frac{1}{\ln 2} + \frac{1}{2} \log_2 \ln 16 + \frac{1}{2} \log_2 \frac{1}{1-\gamma} \right)
\end{aligned}$$

where we used Lemma 11 and Lemma 12 on the second to last line together with  $\ln \frac{1}{\gamma} \geq 1 - \gamma$ .  $\square$

**Lemma 14.** For  $\gamma \in [0, 1)$  and  $a \geq 1$ :

$$\frac{1}{1-\gamma^a} \leq 1 + \frac{1}{a} \frac{\gamma}{1-\gamma}.$$

*Proof.* For  $\epsilon > 0$  and  $a \geq 1$ , it can be shown that  $(1 + \epsilon)^a \geq 1 + a\epsilon$ . Then, taking  $\gamma := \frac{1}{1+\epsilon}$ :

$$\begin{aligned}(1 + \epsilon)^a \geq 1 + a\epsilon &\Leftrightarrow (1 + \epsilon)^a - 1 \geq a((1 + \epsilon) - 1) \\ &\Leftrightarrow \frac{1}{(1 + \epsilon)^a - 1} \leq \frac{1}{a((1 + \epsilon) - 1)} \\ &\Leftrightarrow \frac{1}{\gamma^{-a} - 1} \leq \frac{1}{a(\gamma^{-1} - 1)} \\ &\Leftrightarrow \frac{\gamma^a}{1 - \gamma^a} \leq \frac{\gamma}{a(1 - \gamma)} \\ &\Leftrightarrow \frac{1}{1 - \gamma^a} \leq 1 + \frac{1}{a} \frac{\gamma}{1 - \gamma},\end{aligned}$$

which proves the result. □