Supplementary Material for Dual Space Gradient Descent for Online Learning

Trung Le, Tu Dinh Nguyen, Vu Nguyen, Dinh Phung
Centre for Pattern Recognition and Data Analytics
Deakin University, Australia
{trung.l, tu.nguyen, v.nguyen, dinh.phung}@deakin.edu.au

1 Suitability of Loss Functions

In this section, we present the suitability of the loss functions for Hinge, smooth Hinge, and Logistic for classification and ℓ_1, and ε-insensitive for regression. We prove that these losses satisfy the condition: there exists a positive constant A such that $|\nabla_o l(y,o)| \leq A$, $\forall y,\, o$. For each loss, we show its two forms used in the paper w.r.t o and $\mathbf{w}$.

Hinge loss

$$\begin{aligned}
l(y,o) &= \max(0, 1-yo)\\
l(\mathbf{w},\boldsymbol{x},y) &= \max\left(0, 1-y\mathbf{w}^\top\Phi(\boldsymbol{x})\right)\\
\nabla_o l(y,o) &= -\mathbb{I}_{yo\leq 1}y\\
|\nabla_o l(y,o)| &= |\mathbb{I}_{yo\leq 1}| \leq 1 = A
\end{aligned}$$

Logistic loss

$$\begin{aligned}
l(y,o) &= \log\left(1+e^{-yo}\right)\\
l(\mathbf{w},\boldsymbol{x},y) &= \log\left(1+e^{-y\mathbf{w}^\top\Phi(\boldsymbol{x})}\right)\\
\nabla_o l(y,o) &= \frac{-ye^{-yo}}{e^{-yo}+1}\\
|\nabla_o l(y,o)| &= \left|\frac{e^{-yo}}{e^{-yo}+1}\right| < 1 = A
\end{aligned}$$

Smooth Hinge loss [4]

$$\begin{aligned}
l(y,o) &= \begin{cases} 0 & \text{if } yo > 1\\ 1-yo-\frac{\tau}{2} & \text{if } yo < 1-\tau\\ \frac{1}{2\tau}(1-yo)^2 & \text{otherwise}\end{cases}\\
l(\mathbf{w},\boldsymbol{x},y) &= \begin{cases} 0 & \text{if } y\mathbf{w}^\top\Phi(\boldsymbol{x}) > 1\\ 1-y\mathbf{w}^\top\Phi(\boldsymbol{x})-\frac{\tau}{2} & \text{if } y\mathbf{w}^\top\Phi(\boldsymbol{x}) < 1-\tau\\ \frac{1}{2\tau}\left(1-y\mathbf{w}^\top\Phi(\boldsymbol{x})\right)^2 & \text{otherwise}\end{cases}\\
\nabla_o l(y,o) &= -\mathbb{I}_{\{yo<1-\tau\}}y + \tau^{-1}\mathbb{I}_{1-\tau\leq yo\leq 1}(yo-1)\,y\\
|\nabla_o l(y,o)| &= \left|\mathbb{I}_{\{yo<1-\tau\}}\right| + \left|\tau^{-1}\mathbb{I}_{1-\tau\leq yo\leq 1}(yo-1)\right|\\
&\leq \left|\mathbb{I}_{\{yo<1-\tau\}}\right| + \tau^{-1}\tau\left|\mathbb{I}_{1-\tau\leq yo\leq 1}\right| \leq 1 = A
\end{aligned}$$

30th Conference on Neural Information Processing Systems (NIPS 2016), Barcelona, Spain
Acknowledgment: This work is partially supported by the Australian Research Council under the Discovery Project DP160109394.

$\boldsymbol{\ell}_1$ loss

$$
\begin{aligned}
l\left(y, o\right) &= \left|y - o\right| \\
l\left(\mathbf{w}, \boldsymbol{x}, y\right) &= \left|y - \mathbf{w}^{\top}\Phi\left(\boldsymbol{x}\right)\right| \\
\nabla_{o} l\left(y, o\right) &= \textbf{sign}\left(o - y\right) \\
\left|\nabla_{o} l\left(y, o\right)\right| &\leq 1 = A
\end{aligned}
$$

ε-insensitive loss

$$
\begin{aligned}
l\left(y, o\right) &= \max\left(0, \left|y - o\right| - \varepsilon\right) \\
l\left(\mathbf{w}, \boldsymbol{x}, y\right) &= \max\left(0, \left|y - \mathbf{w}^{\top}\Phi\left(\boldsymbol{x}\right)\right| - \varepsilon\right) \\
\nabla_{o} l\left(y, o\right) &= \mathbb{I}_{\left|y-o\right| \geq \varepsilon}\textbf{sign}\left(o - y\right) \\
\left|\nabla_{o} l\left(y, o\right)\right| &\leq 1 = A
\end{aligned}
$$

We note that $\mathbb{I}_A$ denotes the indicator function which renders 1 if A is true and 0 otherwise.

2 Proofs

Lemma 1. *After the iteration t, we have the following representations*

$$\hat{\mathbf{w}}_t = \sum_{j=1}^{t} \alpha_j \left(1 - \beta_j\right) \Phi\left(\boldsymbol{x}_j\right) \tag{1}$$

$$\tilde{\mathbf{w}}_t = \sum_{j=1}^{t} \alpha_j \beta_j \boldsymbol{z}\left(\boldsymbol{x}_j\right) \tag{2}$$

$$\mathbf{w}_t = \sum_{j=1}^{t} \alpha_j \Phi\left(\boldsymbol{x}_j\right) \tag{3}$$

where $\alpha_j = -\eta_t \nabla_o l\left(y_j, f_j^h\left(\boldsymbol{x}_j\right)\right),\ \forall j = 1, \ldots, t$ and $\eta_t = \frac{1}{\lambda t}$.

Proof. Since if $\beta_j = 1$, we perform the budget maintenance procedure and move the current vector to the random-feature space, we have the representations in Eqs. (1,2,3). In addition at the iteration j, $\Phi\left(\boldsymbol{x}_j\right)$ arrives with the initial coefficient $\alpha_j = -\eta_j \nabla_o l\left(y_j, f_j^h\left(\boldsymbol{x}_j\right)\right)$. After the iteration $t > j$, this coefficient becomes

$$\alpha_j = -\frac{t-1}{t}\frac{t-2}{t-1}...\frac{j}{j+1}\frac{1}{\lambda j}\nabla_o l\left(y_i, f_j^h\left(\boldsymbol{x}_j\right)\right) = -\eta_t \nabla_o l\left(y_j, f_j^h\left(\boldsymbol{x}_j\right)\right)$$

□

Theorem 2. *With a probability at least $1 - 2^8\left(\frac{\sigma_\mu A d_{\mathcal{X}}}{\lambda\varepsilon}\right)\exp\left(-\frac{D\lambda^2\varepsilon^2}{4(M+2)A^2}\right)$ where $d_{\mathcal{X}}$ specifies the diameter of the compact set $\mathcal{X}$, we have*

i) $\left|f_t\left(\boldsymbol{x}\right) - f_t^h\left(\boldsymbol{x}\right)\right| \leq \varepsilon$ for all $t > 0$ and $\boldsymbol{x} \in \mathcal{X}$.

ii) $\mathbb{E}\left[\left|f_t\left(\boldsymbol{x}\right) - f_t^h\left(\boldsymbol{x}\right)\right|\right] \leq A^{-1}\lambda\varepsilon\sum_{j=1}^{t}\mathbb{E}\left[\alpha_j^2\right]^{1/2}\mu_j^{1/2}$ where $\mu_j = p\left(\beta_j = 1\right)$.

Let us define a random map $z : \mathbb{R}^d \to \mathbb{R}^{2D}$ where $z\left(x\right) = \frac{1}{D^{1/2}}\left[\cos\left(\omega_i^{\mathsf{T}}x\right), \sin\left(\omega_i^{\mathsf{T}}x\right)\right]_{i=1}^{D}$ and $\omega_1, ..., \omega_D \overset{\text{i.i.d}}{\sim} \mathcal{N}\left(0, \sigma^{-2}I\right)$ for every $x \in \mathbb{R}^d$. We would like to restate Claim 1 in [3].

Let $\mathcal{M}$ be a compact subset of $\mathbb{R}^d$ with diameter $diam\left(\mathcal{M}\right)$. Then, for the random mapping $\boldsymbol{z}\left(.\right)$, we have

$$\mathbb{P}\left(\underset{x,x^{'}\in\mathcal{M}}{sup}\left|K\left(x, x^{'}\right) - \boldsymbol{z}\left(x\right)^{\mathsf{T}}\boldsymbol{z}\left(x^{'}\right)\right| < \varepsilon\right) \geq 1 - 2^8\left(\frac{\sigma diam\left(\mathcal{M}\right)}{\varepsilon}\right)exp\left(\frac{-D\varepsilon^2}{4\left(d+2\right)}\right)$$

where $K\left(x, x^{'}\right) = e^{-\frac{\left\|x-x^{'}\right\|^2}{2\sigma^2}}$.

Proof. We denote

$$\omega = (\omega_1, ..., \omega_D) \sim p_\omega(\omega) = \prod_{i=1}^{D} \mathcal{N}\left(\omega_i | 0, \sigma^{-2} I\right)$$

$$\tilde{K}\left(x, x^{'}\right) = z(x)^\intercal z\left(x^{'}\right) = D^{-1} \sum_{i=1}^{D} \left(cos\left(\omega_i^\intercal x\right) cos\left(\omega_i^\intercal x^{'}\right) + sin\left(\omega_i^\intercal x\right) sin\left(\omega_i^\intercal x^{'}\right)\right)$$

We further denote

$$g(\omega) = \sup_{x, x^{'} \in \mathcal{M}} \left| K\left(x, x^{'}\right) - \tilde{K}\left(x, x^{'}\right) \right|$$

$$G_\varepsilon = \left\{\omega : g(\omega) < A^{-1} \lambda \varepsilon\right\}$$

It is certain that $\mathbb{P}_\omega(G_\varepsilon) \geq 1 - \theta$ where $\theta = 2^8 \left(\frac{\sigma A \text{diam}(\mathcal{M})}{\lambda\varepsilon}\right) exp\left(\frac{-D\lambda^2\varepsilon^2}{4(d+2)A^2}\right)$ and for every $\omega \in G_\varepsilon$ and $x, x^{'} \in \mathcal{M}$ we have

$$\left| K\left(x, x^{'}\right) - \tilde{K}\left(x, x^{'}\right) \right| < A^{-1}\lambda\varepsilon$$

We now turn back to Theorem 2. It appears that

$$\left| f_t(x) - f_t^h(x) \right| \leq \sum_{j=1}^{t} \beta_j \left|\alpha_j\right| \left| K(x_j, x) - \tilde{K}(x_j, x) \right|$$

Therefore, for every $\omega \in G_\varepsilon$ we have

$$\left| f_t(x) - f_t^h(x) \right| \leq A^{-1}\lambda\varepsilon \sum_{j=1}^{t} \beta_j \left|\alpha_j\right|$$

Let us denote $s = (x_1, y_1), ..., (x_t, y_t)$. Taking expectation of the above inequality w.r.t s, we gain for all $\omega \in G_\varepsilon$

$$\begin{aligned} \mathbb{E}_s\left[\left| f_t(x) - f_t^h(x) \right|\right] &\leq A^{-1}\lambda\varepsilon \sum_{j=1}^{t} \mathbb{E}_s\left[\beta_j^2\right]^{1/2} \mathbb{E}_s\left[\alpha_j^2\right]^{1/2} \\ &\leq A^{-1}\lambda\varepsilon \sum_{j=1}^{t} \mu_j \mathbb{E}_s\left[\alpha_j^2\right]^{1/2} \end{aligned}$$

It means that

$$\mathbb{P}_\omega\left(\mathbb{E}_s\left[\left| f_t(x) - f_t^h(x) \right|\right] \leq A^{-1}\lambda\varepsilon \sum_{j=1}^{t} \mu_j \mathbb{E}_s\left[\alpha_j^2\right]^{1/2}\right) \geq \mathbb{P}_\omega(G_\varepsilon) \geq 1 - \theta$$

□

Lemma 3. *The following statement holds for all* t

$$\|\mathbf{w}_t\| \leq \frac{A}{\lambda}$$

Proof. Using Lemma 1, we have

$$\mathbf{w}_t = \sum_{j=1}^{t} \alpha_j \Phi\left(\boldsymbol{x}_j\right)$$

where $\alpha_j = -\eta_t \nabla_o l\left(y_j, f_j^h\left(\boldsymbol{x}_j\right)\right)$.

It implies that

$$\|\mathbf{w}_t\| \leq \sum_{j=1}^{t} |\alpha_j| \left\|\Phi\left(\boldsymbol{x}_j\right)\right\| \leq \sum_{j=1}^{t} |\alpha_j| \leq \sum_{j=1}^{t} \frac{A}{\lambda t} = \frac{A}{\lambda}$$

□

Lemma 4. *The following statement holds for all t*

$$\|g_t\| \leq G = 2A$$

where we define $g_t = \lambda \mathbf{w}_t + \nabla_{\mathbf{w}} l\left(\mathbf{w}_t, \boldsymbol{x}_t, y_t\right) = \lambda \mathbf{w}_t + \nabla_o l\left(y_t, f_t\left(\boldsymbol{x}_t\right)\right) \Phi\left(\boldsymbol{x}_t\right)$.

Proof. We derive as

$$\|g_t\| \leq \lambda \|\mathbf{w}_t\| + \left\|\nabla_o l\left(y_t, f_t\left(\boldsymbol{x}_t\right)\right) \Phi\left(\boldsymbol{x}_t\right)\right\| \leq \lambda \frac{A}{\lambda} + A = 2A$$

□

Lemma 5. *The following statement holds for all t*

$$\mathbb{E}\left[\left\|\mathbf{w}_t - \mathbf{w}^\star\right\|^2\right] \leq W^2$$

where $W = \frac{2A\left(1+\sqrt{5}\right)}{\lambda}$.

Proof. Recall that $g_t = \lambda \mathbf{w}_t + \nabla_{\mathbf{w}} l\left(\mathbf{w}_t, \boldsymbol{x}_t, y_t\right) = \lambda \mathbf{w}_t + \nabla_o l\left(y_t, f_t\left(\boldsymbol{x}_t\right)\right) \Phi\left(\boldsymbol{x}_t\right)$. It is obvious that g_t satisfies

$$\mathbb{E}_{\left(\boldsymbol{x}_t, y_t\right)}\left[g_t \mid \mathbf{w}_t\right] = \mathcal{J}^{'}\left(\mathbf{w}_t\right)$$

We have the following if we denote $\delta g_t = g_t - g_t^h$

$$\begin{aligned}
\left\|\mathbf{w}_{t+1} - \mathbf{w}^\star\right\|^2 &= \left\|\mathbf{w}_t - \eta_t g_t^h - \mathbf{w}^\star\right\| = \left\|\mathbf{w}_t - \eta_t g_t - \mathbf{w}^\star + \eta_t \delta g_t\right\|^2 \\
= \left\|\mathbf{w}_t - \mathbf{w}^\star\right\|^2 - 2\eta_t g_t^\top \left(\mathbf{w}_t - \mathbf{w}^\star\right) &+ \eta_t^2 \left\|g_t\right\|^2 - 2\eta_t^2 g_t^\top \delta g_t + \eta_t^2 \left\|\delta g_t\right\|^2 + 2\eta_t \left(\mathbf{w}_t - \mathbf{w}^\star\right)^\top \delta g_t
\end{aligned}$$

It appears that

$$\begin{aligned}
\delta g_t &= \left[\nabla_o l\left(y_t, f_t\left(\boldsymbol{x}_t\right)\right) - \nabla_o l\left(y_t, f_t^h\left(\boldsymbol{x}_t\right)\right)\right] \Phi\left(\boldsymbol{x}_t\right) \\
\left\|\delta g_t\right\| &= \left|\nabla_o l\left(y_t, f_t\left(\boldsymbol{x}_t\right)\right) - \nabla_o l\left(y_t, f_t^h\left(\boldsymbol{x}_t\right)\right)\right| \leq 2A
\end{aligned}$$

Hence, we obtain

$$\begin{aligned}
\left\|\mathbf{w}_{t+1} - \mathbf{w}^\star\right\|^2 \leq \left\|\mathbf{w}_t - \mathbf{w}^\star\right\|^2 &- 2\eta_t g_t^\top \left(\mathbf{w}_t - \mathbf{w}^\star\right) + \eta_t^2 G^2 + 4\eta_t^2 G A + 4\eta_t^2 A^2 \\
&+ 2\eta_t \left\|\mathbf{w}_t - \mathbf{w}^\star\right\| \left\|\delta g_t\right\|
\end{aligned}$$

Taking conditional expectation w.r.t $\mathbf{w}_t$ on both sides of the above inequality, we gain

$$\begin{aligned}
\mathbb{E}\left[\left\|\mathbf{w}_{t+1} - \mathbf{w}^\star\right\|^2\right] &\leq \mathbb{E}\left[\left\|\mathbf{w}_t - \mathbf{w}^\star\right\|^2\right] - 2\eta_t \nabla_{\mathbf{w}} \mathcal{J}\left(\mathbf{w}_t\right)^\top \left(\mathbf{w}_t - \mathbf{w}^\star\right) + \eta_t^2 G^2 + 4\eta_t^2 G A \\
&\quad + 4\eta_t^2 A^2 + 2\eta_t \mathbb{E}\left[\left\|\mathbf{w}_t - \mathbf{w}^\star\right\| \left\|\delta g_t\right\|\right] \\
&\leq \mathbb{E}\left[\left\|\mathbf{w}_t - \mathbf{w}^\star\right\|^2\right] + 16A^2\eta_t^2 + 2\eta_t \mathbb{E}\left[\left\|\mathbf{w}_t - \mathbf{w}^\star\right\| \left\|\delta g_t\right\|\right] - \frac{1}{t}\left\|\mathbf{w}_t - \mathbf{w}^\star\right\|
\end{aligned}$$

Here we note that we have used

$$\nabla_{\mathbf{w}} \mathcal{J}\left(\mathbf{w}_t\right)^\top \left(\mathbf{w}_t - \mathbf{w}^\star\right) \geq \mathcal{J}\left(\mathbf{w}_t\right) - \mathcal{J}\left(\mathbf{w}^\star\right) + \frac{\lambda}{2}\left\|\mathbf{w}_t - \mathbf{w}^\star\right\|^2 \geq \frac{\lambda}{2}\left\|\mathbf{w}_t - \mathbf{w}^\star\right\|^2$$

Taking expectation on both sides again, we obtain

$$\begin{aligned}\mathbb{E}\left[\|\mathbf{w}_{t+1}-\mathbf{w}^\star\|^2\right] &\le \frac{t-1}{t}\mathbb{E}\left[\|\mathbf{w}_t-\mathbf{w}^\star\|^2\right]+\frac{16A^2}{\lambda^2t^2}+\frac{4A\mathbb{E}\left[\|\mathbf{w}_t-\mathbf{w}^\star\|^2\right]^{1/2}}{\lambda t}\\ &\le \frac{t-1}{t}\mathbb{E}\left[\|\mathbf{w}_t-\mathbf{w}^\star\|^2\right]+\frac{16A^2}{\lambda^2t}+\frac{4A\mathbb{E}\left[\|\mathbf{w}_t-\mathbf{w}^\star\|^2\right]^{1/2}}{\lambda t}\end{aligned}$$

Choose $W=\frac{2A\left(1+\sqrt{5}\right)}{\lambda}$, we have if $\mathbb{E}\left[\|\mathbf{w}_t-\mathbf{w}^\star\|^2\right]\le W^2$ then $\mathbb{E}\left[\|\mathbf{w}_{t+1}-\mathbf{w}^\star\|^2\right]\le W^2$. □

Theorem 6. *The following statement guarantees for all T*

$$\mathbb{E}\left[\mathcal{J}\left(\overline{\mathbf{w}}_T\right)-\mathcal{J}\left(\mathbf{w}^\star\right)\right]\le\mathbb{E}\left[\frac{1}{T}\sum_{t=1}^{T}\mathcal{J}\left(\mathbf{w}_t\right)-\mathcal{J}\left(\mathbf{w}^\star\right)\right]\le\frac{8A^2\left(\log\left(T\right)+1\right)}{\lambda T}+\frac{1}{T}W\sum_{t=1}^{T}\mathbb{E}\left[M_t^2\right]^{1/2}$$

where $\overline{\mathbf{w}}_T=\frac{1}{T}\sum_{t=1}^{T}\mathbf{w}_t$, $M_t=\nabla_o l\left(y_t,f_t\left(\boldsymbol{x}_t\right)\right)-\nabla_o l\left(y_t,f_t^h\left(\boldsymbol{x}_t\right)\right)$.

Proof. Recall that $g_t=\lambda\mathbf{w}_t+\nabla_{\mathbf{w}}l\left(\mathbf{w}_t,\boldsymbol{x}_t,y_t\right)=\lambda\mathbf{w}_t+\nabla_o l\left(y_t,f_t\left(\boldsymbol{x}_t\right)\right)\Phi\left(\boldsymbol{x}_t\right)$. It is obvious that g_t satisfies

$$\mathbb{E}_{\left(\boldsymbol{x}_t,y_t\right)}\left[g_t|\mathbf{w}_t\right]=\nabla_{\mathbf{w}}\mathcal{J}\left(\mathbf{w}_t\right)$$

We have the following if we denote $\delta g_t=g_t-g_t^h$

$$\begin{aligned}\|\mathbf{w}_{t+1}-\mathbf{w}^\star\|^2&=\left\|\mathbf{w}_t-\eta_tg_t^h-\mathbf{w}^\star\right\|=\|\mathbf{w}_t-\eta_tg_t-\mathbf{w}^\star+\eta_t\delta g_t\|^2\\ =\|\mathbf{w}_t-\mathbf{w}^\star\|^2-2\eta_tg_t^\top\left(\mathbf{w}_t-\mathbf{w}^\star\right)&+\eta_t^2\|g_t\|^2-2\eta_t^2g_t^\top\delta g_t+\eta_t^2\|\delta g_t\|^2+2\eta_t\left(\mathbf{w}_t-\mathbf{w}^\star\right)^\top\delta g_t\end{aligned}$$

It appears that

$$\begin{aligned}\delta g_t&=\left[\nabla_o l\left(y_t,f_t\left(\boldsymbol{x}_t\right)\right)-\nabla_o l\left(y_t,f_t^h\left(\boldsymbol{x}_t\right)\right)\right]\Phi\left(x_t\right)\\ \|\delta g_t\|&=\left|\nabla_o l\left(y_t,f_t\left(\boldsymbol{x}_t\right)\right)-\nabla_o l\left(y_t,f_t^h\left(\boldsymbol{x}_t\right)\right)\right|\le 2A\end{aligned}$$

Hence, we obtain

$$\begin{aligned}\|\mathbf{w}_{t+1}-\mathbf{w}^\star\|^2\le\|\mathbf{w}_t-\mathbf{w}^\star\|^2&-2\eta_tg_t^\top\left(\mathbf{w}_t-\mathbf{w}^\star\right)+\eta_t^2G^2+4\eta_t^2GA+4\eta_t^2A^2\\ &+2\eta_t\|\mathbf{w}_t-\mathbf{w}^\star\|\|\delta g_t\|\end{aligned}$$

$$g_t^\intercal\left(\mathbf{w}_t-\mathbf{w}^\star\right)\le\frac{\|\mathbf{w}_t-\mathbf{w}^\star\|^2-\|\mathbf{w}_{t+1}-\mathbf{w}^\star\|^2}{2\eta_t}+8A^2\eta_t+\|\mathbf{w}_t-\mathbf{w}^\star\|\|\delta g_t\|$$

Taking conditional expectation w.r.t $\mathbf{w}_t$ on both sides, we gain

$$\nabla_{\mathbf{w}}\mathcal{J}\left(\mathbf{w}_t\right)^\top\left(\mathbf{w}_t-\mathbf{w}^\star\right)\le\mathbb{E}\left[\frac{\|\mathbf{w}_t-\mathbf{w}^\star\|^2}{2\eta_t}\right]-\mathbb{E}\left[\frac{\|\mathbf{w}_{t+1}-\mathbf{w}^\star\|^2}{2\eta_t}\right]+8A^2\eta_t+\mathbb{E}\left[\|\mathbf{w}_t-\mathbf{w}^\star\|\|\delta g_t\|\right]$$

$$\begin{aligned}\mathcal{J}\left(\mathbf{w}_t\right)-\mathcal{J}\left(\mathbf{w}^\star\right)+\frac{\lambda}{2}\|\mathbf{w}_t-\mathbf{w}^\star\|^2\le\mathbb{E}\left[\frac{\|\mathbf{w}_t-\mathbf{w}^\star\|^2}{2\eta_t}\right]&-\mathbb{E}\left[\frac{\|\mathbf{w}_{t+1}-\mathbf{w}^\star\|^2}{2\eta_t}\right]\\ +8A^2\eta_t+\mathbb{E}&\left[\|\mathbf{w}_t-\mathbf{w}^\star\|\|\delta g_t\|\right]\end{aligned}$$

Taking expectation on both sides once again, we achieve

$$\begin{aligned}\mathbb{E}\left[\mathcal{J}\left(\mathbf{w}_t\right)-\mathcal{J}\left(\mathbf{w}^\star\right)\right]&\le\frac{\lambda}{2}\left(t-1\right)\mathbb{E}\left[\|\mathbf{w}_t-\mathbf{w}^\star\|^2\right]-\frac{\lambda}{2}t\mathbb{E}\left[\|\mathbf{w}_{t+1}-\mathbf{w}^\star\|^2\right]\\ &+8A^2\eta_t+\mathbb{E}\left[\|\mathbf{w}_t-\mathbf{w}^\star\|\|\delta g_t\|\right]\\ \mathbb{E}\left[\mathcal{J}\left(\mathbf{w}_t\right)-\mathcal{J}\left(\mathbf{w}^\star\right)\right]&\le\frac{\lambda}{2}\left(t-1\right)\mathbb{E}\left[\|\mathbf{w}_t-\mathbf{w}^\star\|^2\right]-\frac{\lambda}{2}t\mathbb{E}\left[\|\mathbf{w}_{t+1}-\mathbf{w}^\star\|^2\right]\\ &+8A^2\eta_t+\mathbb{E}\left[\|\mathbf{w}_t-\mathbf{w}^\star\|^2\right]^{1/2}\mathbb{E}\left[\|\delta g_t\|^2\right]^{1/2}\end{aligned}$$

Taking sum the above inequality when $t = 1, ..., T$, we obtain

$$\mathbb{E}\left[\frac{1}{T}\sum_{t=1}^{T}\mathcal{J}\left(\mathbf{w}_t\right)-\mathcal{J}\left(\mathbf{w}^{\star}\right)\right] \leq \frac{8A^2}{\lambda}\sum_{t=1}^{T}\frac{1}{t}+\frac{1}{T}W\sum_{t=1}^{T}\mathbb{E}\left[M_t^2\right]^{1/2}$$
$$\leq \frac{8A^2\left(\log T+1\right)}{\lambda T}+\frac{1}{T}W\sum_{t=1}^{T}\mathbb{E}\left[M_t^2\right]^{1/2}$$

Here we note that

$$\left\|\delta g_t\right\| = \left\|\left[\nabla_o l\left(y_t, f_t\left(\boldsymbol{x}_t\right)\right)-\nabla_o l\left(y_t, f_t^h\left(\boldsymbol{x}_t\right)\right)\right]\Phi\left(\boldsymbol{x}_t\right)\right\| = \left|M_t\right|$$

The last conclusion comes from the convexity of the function $\mathcal{J}\left(.\right)$. □

Theorem 7. *Assume that $l\left(y, o\right)$ is a γ-strongly smooth loss function. With a probability at least $1-\theta$, the following statements hold*

i) $\mathbb{E}\left[\mathcal{J}\left(\overline{\mathbf{w}}_T\right)-\mathcal{J}\left(\mathbf{w}^{\star}\right)\right] \leq \mathbb{E}\left[\frac{1}{T}\sum_{t=1}^{T}\mathcal{J}\left(\mathbf{w}_t\right)-\mathcal{J}\left(\mathbf{w}^{\star}\right)\right] \leq \frac{8A^2\left(\log T+1\right)}{\lambda T} + \frac{1}{T}W\gamma\varepsilon\sum_{t=1}^{T}\left(\frac{\sum_{i=1}^{t}\mu_i}{t}\right)^{1/2}$

ii) $\mathbb{E}\left[\mathcal{J}\left(\overline{\mathbf{w}}_T\right)-\mathcal{J}\left(\mathbf{w}^{\star}\right)\right] \leq \mathbb{E}\left[\frac{1}{T}\sum_{t=1}^{T}\mathcal{J}\left(\mathbf{w}_t\right)-\mathcal{J}\left(\mathbf{w}^{\star}\right)\right] \leq \frac{8A^2\left(\log T+1\right)}{\lambda T}+W\gamma\varepsilon$

where $\theta = 2^8\left(\frac{\sigma_\mu A d_{\mathcal{X}}}{\lambda\varepsilon}\right)\exp\left(-\frac{D\lambda^2\varepsilon^2}{4(M+2)A^2}\right)$.

Proof. From the smoothness of the loss function, we have

$$\left|\nabla_o l\left(y_t, f_t\left(\boldsymbol{x}_t\right)\right)-\nabla_o l\left(y_t, f_t^h\left(\boldsymbol{x}_t\right)\right)\right| \leq \gamma\left|f_t\left(\boldsymbol{x}_t\right)-f_t^h\left(\boldsymbol{x}_t\right)\right|$$

Referring to Lemma 2, with a probability at least $1-2^8\left(\frac{\sigma_\mu A d_{\mathcal{X}}}{\lambda\varepsilon}\right)\exp\left(-\frac{D\lambda^2\varepsilon^2}{4(M+2)A^2}\right) = 1-\theta$ we have

$$\left|M_t\right| \leq \gamma A^{-1}\lambda\varepsilon\sum_{j=1}^{t}\left|\alpha_j\right|\beta_i \leq \gamma A^{-1}\lambda\varepsilon\sum_{j=1}^{t}\frac{A}{\lambda t}\beta_j = \frac{\gamma\varepsilon}{t}\sum_{j=1}^{t}\beta_j$$
$$M_t^2 \leq \frac{\gamma^2\varepsilon^2}{t^2}\left(\sum_{j=1}^{t}\beta_j\right)^2 \leq \frac{\gamma^2\varepsilon^2}{t}\sum_{j=1}^{t}\beta_j^2 = \frac{\gamma^2\varepsilon^2}{t}\sum_{j=1}^{t}\beta_j \quad (\text{since } \beta_i = 0 \text{ or } 1)$$
$$\mathbb{E}\left[M_t^2\right] \leq \frac{\gamma^2\varepsilon^2}{t}\left(\sum_{j=1}^{t}\mu_j\right)$$

and $\left|M_t\right| \leq \gamma\varepsilon$. Therefore, with a probability at least $1-\theta$ we achieve

$$\mathbb{E}\left[\mathcal{J}\left(\overline{\mathbf{w}}_T\right)-\mathcal{J}\left(\mathbf{w}^{\star}\right)\right] \leq \mathbb{E}\left[\frac{1}{T}\sum_{t=1}^{T}\mathcal{J}\left(\mathbf{w}_t\right)-\mathcal{J}\left(\mathbf{w}^{\star}\right)\right]$$
$$\leq \frac{8A^2\left(\log T+1\right)}{\lambda T}+\frac{1}{T}W\gamma\varepsilon\sum_{t=1}^{T}\frac{\left(\sum_{j=1}^{t}\mu_j\right)^{1/2}}{t^{1/2}}$$
$$\leq \frac{8A^2\left(\log T+1\right)}{\lambda T}+\frac{1}{T}W\gamma\varepsilon\sum_{t=1}^{T}\left(\frac{\sum_{j=1}^{t}\mu_j}{t}\right)^{1/2}$$

and

$$
\begin{aligned}
\mathbb{E}\left[\mathcal{J}\left(\overline{\mathbf{w}}_T\right)-\mathcal{J}\left(\mathbf{w}^\star\right)\right] &\leq \mathbb{E}\left[\frac{1}{T}\sum_{t=1}^{T}\mathcal{J}\left(\mathbf{w}_t\right)-\mathcal{J}\left(\mathbf{w}^\star\right)\right] \\
&\leq \frac{8A^2\left(\log T+1\right)}{\lambda T}+\frac{1}{T}W\sum_{t=1}^{T}\gamma\varepsilon \\
&\leq \frac{8A^2\left(\log T+1\right)}{\lambda T}+W\gamma\varepsilon
\end{aligned}
$$

□

3 Computational Complexities of DualSGD and FOGD

We compare the computational complexities of our proposed DualSGD and Fourier Online Gradient Descent (FOGD) [2]. Recall that M and D denote the dimensions of input space and feature space, and B the budget size. There are four operators: (i) random feature mapping; (ii) kernel function; (iii) sorting coefficients of support vectors and (iv) prediction. The random feature mapping first projects the input data vector to random feature space with $\mathcal{O}\left(MD\right)$ computational complexity, and then compute *sin*, *cos* on the random feature dimension with $\mathcal{O}\left(2D*2^{\log n}n\log^2 n\right)$ where n is the number of bits accuracy [1]. The kernel function, sorting coefficients and prediction operate in $\mathcal{O}\left(MB\right)$, $\mathcal{O}\left(B\log B\right)$ and $\mathcal{O}\left(D\right)$ complexity, respectively. The FOGD performs random feature mapping and prediction whilst the DualSGD performs all four operators.

Let D_1 and D_2 denote the number of random features of FOGD and DualSGD. The computational complexities of FOGD and DualSGD reads

$$
\begin{aligned}
\mathcal{O}_{\text{FOGD}} &= \mathcal{O}\left(MD_1+2D_1*2^{\log n}n\log^2 n+D_1\right)=U\left(MD_1+2D_1*2^{\log n}n\log^2 n+D_1\right) \\
\mathcal{O}_{\text{DualSGD}} &= \mathcal{O}\left(MD_2+2D_2*2^{\log n}n\log^2 n+D_2+MB+B\log B\right) \\
&= V\left(MD_2+2D_2*2^{\log n}n\log^2 n+D_2+MB+B\log B\right)
\end{aligned}
$$

where U, V are the number of iterations.

Taking the subtraction of $\mathcal{O}_{\text{FOGD}}$ and $\mathcal{O}_{\text{DualSGD}}$, we obtain:

$$
\begin{aligned}
\hat{\mathcal{O}} &= \mathcal{O}_{\text{FOGD}}-\mathcal{O}_{\text{DualSGD}} \\
&= M\left(UD_1-VD_2-B\right)+\left(UD_1-VD_2\right)\left(2*2^{\log n}n\log^2 n+1\right)-B\log B
\end{aligned}
$$

According Fig. 1 in the introduction section, $D_1 \gg D_2$ and $D_1 \gg B$, thus $D_1 - D_2 \gg B$. In addition, we assume that $U = V$ and normally use double-precision floating-point with $n = 64$ (bits) for storing and computing real number, thus $2*2^{\log n}n\log^2 n+1 > \log B$. Finally, we can see that $\mathcal{O} \gg 0$, thus the computational complexity of DualSGD, in practice, is significantly lower than that of FOGD.

References

[1] R. P Brent and P. Zimmermann. *Modern computer arithmetic*, volume 18. Cambridge University Press, 2010.

[2] J. Lu, S. C.H. Hoi, J. Wang, P. Zhao, and Z.-Y. Liu. Large scale online kernel learning. *J. Mach. Learn. Res.*, 2015.

[3] A. Rahimi and B. Recht. Random features for large-scale kernel machines. In *Advances in Neural Infomration Processing Systems*, 2007.

[4] S. Shalev-Shwartz and T. Zhang. Stochastic dual coordinate ascent methods for regularized loss. *Journal of Machine Learning Research*, 14(1):567–599, 2013.