
Coin Betting and Parameter-Free Online Learning

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Abstract

In the recent years, a number of parameter-free algorithms have been developed for online linear optimization over Hilbert spaces and for learning with expert advice. These algorithms achieve optimal regret bounds that depend on the unknown competitors, without having to tune the learning rates with oracle choices.

We present a new intuitive framework to design parameter-free algorithms for *both* online linear optimization over Hilbert spaces and for learning with expert advice, based on reductions to betting on outcomes of adversarial coins. We instantiate it using a betting algorithm based on the Krichevsky-Trofimov estimator. The resulting algorithms are simple, with no parameters to be tuned, and they improve or match previous results in terms of regret guarantee and per-round complexity.

1 Introduction

We consider the Online Linear Optimization (OLO) [4, 25] setting. In each round t , an algorithm chooses a point w_t from a convex *decision set* K and then receives a reward vector g_t . The algorithm’s goal is to keep its *regret* small, defined as the difference between its cumulative reward and the cumulative reward of a fixed strategy $u \in K$, that is

$$\text{Regret}_T(u) = \sum_{t=1}^T \langle g_t, u \rangle - \sum_{t=1}^T \langle g_t, w_t \rangle .$$

We focus on two particular decision sets, the N -dimensional probability simplex $\Delta_N = \{x \in \mathbb{R}^N : x \geq 0, \|x\|_1 = 1\}$ and a Hilbert space \mathcal{H} . OLO over Δ_N is referred to as the problem of Learning with Expert Advice (LEA). We assume bounds on the norms of the reward vectors: For OLO over \mathcal{H} , we assume that $\|g_t\| \leq 1$, and for LEA we assume that $g_t \in [0, 1]^N$.

OLO is a basic building block of many machine learning problems. For example, Online Convex Optimization (OCO), the problem analogous to OLO where $\langle g_t, u \rangle$ is generalized to an arbitrary convex function $\ell_t(u)$, is solved through a reduction to OLO [25]. LEA [17, 27, 5] provides a way of combining classifiers and it is at the heart of boosting [12]. Batch and stochastic convex optimization can also be solved through a reduction to OLO [25].

To achieve optimal regret, most of the existing online algorithms require the user to set the learning rate (step size) η to an unknown/oracle value. For example, to obtain the optimal bound for Online Gradient Descent (OGD), the learning rate has to be set with the knowledge of the norm of the competitor u , $\|u\|$; second entry in Table 1. Likewise, the optimal learning rate for Hedge depends on the KL divergence between the prior weighting π and the unknown competitor u , $D(u\|\pi)$; seventh entry in Table 1. Recently, new parameter-free algorithms have been proposed, both for LEA [6, 8, 18, 19, 15, 11] and for OLO/OCO over Hilbert spaces [26, 23, 21, 22, 24]. These algorithms adapt to the number of experts and to the norm of the optimal predictor, respectively, without the need to tune parameters. However, their *design and underlying intuition* is still a challenge. Foster et al. [11] proposed a unified framework, but it is not constructive. Furthermore, all existing algorithms for LEA either have sub-optimal regret bound (e.g. extra $\mathcal{O}(\log \log T)$ factor) or sub-optimal running time (e.g. requiring solving a numerical problem in every round, or with extra factors); see Table 1.

Algorithm	Worst-case regret guarantee	Per-round time complexity	Adaptive	Unified analysis
OGD, $\eta = \frac{1}{\sqrt{T}}$ [25]	$\mathcal{O}((1 + \ u\ ^2)\sqrt{T}), \forall u \in \mathcal{H}$	$\mathcal{O}(1)$		
OGD, $\eta = \frac{U}{\sqrt{T}}$ [25]	$U\sqrt{T}$ for any $u \in \mathcal{H}$ s.t. $\ u\ \leq U$	$\mathcal{O}(1)$		
[23]	$\mathcal{O}(\ u\ \ln(1 + \ u\ T)\sqrt{T}), \forall u \in \mathcal{H}$	$\mathcal{O}(1)$	✓	
[22, 24]	$\mathcal{O}(\ u\ \sqrt{T \ln(1 + \ u\ T)}), \forall u \in \mathcal{H}$	$\mathcal{O}(1)$	✓	
This paper, Sec. 7.1	$\mathcal{O}(\ u\ \sqrt{T \ln(1 + \ u\ T)}), \forall u \in \mathcal{H}$	$\mathcal{O}(1)$	✓	✓
Hedge, $\eta = \sqrt{\frac{\ln N}{T}}, \pi_i = \frac{1}{N}$ [12]	$\mathcal{O}(\sqrt{T \ln N}), \forall u \in \Delta_N$	$\mathcal{O}(N)$		
Hedge, $\eta = \frac{U}{\sqrt{T}}$ [12]	$\mathcal{O}(U\sqrt{T})$ for any $u \in \Delta_N$ s.t. $\sqrt{D(u \pi)} \leq U$	$\mathcal{O}(N)$		
[6]	$\mathcal{O}(\sqrt{T(1 + D(u \pi))} + \ln^2 N), \forall u \in \Delta_N$	$\mathcal{O}(N K)^1$	✓	
[8]	$\mathcal{O}(\sqrt{T(1 + D(u \pi))}), \forall u \in \Delta_N$	$\mathcal{O}(N K)^1$	✓	
[8, 19, 15] ²	$\mathcal{O}(\sqrt{T(\ln \ln T + D(u \pi))}), \forall u \in \Delta_N$	$\mathcal{O}(N)$	✓	
[11]	$\mathcal{O}(\sqrt{T(1 + D(u \pi))}), \forall u \in \Delta_N$	$\mathcal{O}(N \ln \max_{u \in \Delta_N} D(u \pi))^3$	✓	✓
This paper, Sec. 7.2	$\mathcal{O}(\sqrt{T(1 + D(u \pi))}), \forall u \in \Delta_N$	$\mathcal{O}(N)$	✓	✓

Table 1: Algorithms for OLO over Hilbert space and LEA.

Contributions. We show that a more fundamental notion subsumes *both* OLO and LEA parameter-free algorithms. We prove that the ability to maximize the wealth in bets on the outcomes of coin flips *implies* OLO and LEA parameter-free algorithms. We develop a novel potential-based framework for betting algorithms. It gives intuition to previous constructions and, instantiated with the Krichevsky-Trofimov estimator, provides new and elegant algorithms for OLO and LEA. The new algorithms also have optimal worst-case guarantees on regret and time complexity; see Table 1.

2 Preliminaries

We begin by providing some definitions. The Kullback-Leibler (KL) divergence between two discrete distributions p and q is $D(p||q) = \sum_i p_i \ln(p_i/q_i)$. If p, q are real numbers in $[0, 1]$, we denote by $D(p||q) = p \ln(p/q) + (1-p) \ln((1-p)/(1-q))$ the KL divergence between two Bernoulli distributions with parameters p and q . We denote by \mathcal{H} a Hilbert space, by $\langle \cdot, \cdot \rangle$ its inner product, and by $\|\cdot\|$ the induced norm. We denote by $\|\cdot\|_1$ the 1-norm in \mathbb{R}^N . A function $F : I \rightarrow \mathbb{R}_+$ is called *logarithmically convex* iff $f(x) = \ln(F(x))$ is convex. Let $f : V \rightarrow \mathbb{R} \cup \{\pm\infty\}$, the Fenchel conjugate of f is $f^* : V^* \rightarrow \mathbb{R} \cup \{\pm\infty\}$ defined on the dual vector space V^* by $f^*(\theta) = \sup_{x \in V} \langle \theta, x \rangle - f(x)$. A function $f : V \rightarrow \mathbb{R} \cup \{+\infty\}$ is said to be *proper* if there exists $x \in V$ such that $f(x)$ is finite. If f is a proper lower semi-continuous convex function then f^* is also proper lower semi-continuous convex and $f^{**} = f$.

Coin Betting. We consider a gambler making repeated bets on the outcomes of adversarial coin flips. The gambler starts with an initial endowment $\epsilon > 0$. In each round t , he bets on the outcome of a coin flip $g_t \in \{-1, 1\}$, where $+1$ denotes heads and -1 denotes tails. We do not make any assumption on how g_t is generated, that is, it can be chosen by an adversary.

The gambler can bet any amount on either heads or tails. However, he is not allowed to borrow any additional money. If he loses, he loses the betted amount; if he wins, he gets the betted amount back and, in addition to that, he gets the same amount as a reward. We encode the gambler's bet in round t by a single number w_t . The sign of w_t encodes whether he is betting on heads or tails. The absolute value encodes the betted amount. We define Wealth_t as the gambler's wealth at the end of round t and Reward_t as the gambler's net reward (the difference of wealth and initial endowment), that is

$$\text{Wealth}_t = \epsilon + \sum_{i=1}^t w_i g_i \quad \text{and} \quad \text{Reward}_t = \text{Wealth}_t - \epsilon. \quad (1)$$

In the following, we will also refer to a bet with β_t , where β_t is such that

$$w_t = \beta_t \text{Wealth}_{t-1}. \quad (2)$$

The absolute value of β_t is the *fraction* of the current wealth to bet, and sign of β_t encodes whether he is betting on heads or tails. The constraint that the gambler cannot borrow money implies that $\beta_t \in [-1, 1]$. We also generalize the problem slightly by allowing the outcome of the coin flip g_t to be any real number in the interval $[-1, 1]$; wealth and reward in (1) remain exactly the same.

¹These algorithms require to solve a numerical problem at each step. The number K is the number of steps needed to reach the required precision. Neither the precision nor K are calculated in these papers.

²The proof in [15] can be modified to prove a KL bound, see <http://blog.wouterkoolen.info>.

³A variant of the algorithm in [11] can be implemented with the stated time complexity [10].

3 Warm-Up: From Betting to One-Dimensional Online Linear Optimization

In this section, we sketch how to reduce one-dimensional OLO to betting on a coin. The reasoning for generic Hilbert spaces (Section 5) and for LEA (Section 6) will be similar. We will show that the betting view provides a natural way for the analysis and design of online learning algorithms, where the only design choice is the potential function of the betting algorithm (Section 4). A specific example of coin betting potential and the resulting algorithms are in Section 7.

As a warm-up, let us consider an algorithm for OLO over one-dimensional Hilbert space \mathbb{R} . Let $\{w_t\}_{t=1}^\infty$ be its sequence of predictions on a sequence of rewards $\{g_t\}_{t=1}^\infty$, $g_t \in [-1, 1]$. The total reward of the algorithm after t rounds is $\text{Reward}_t = \sum_{i=1}^t g_i w_i$. Also, even if in OLO there is no concept of “wealth”, define the wealth of the OLO algorithm as $\text{Wealth}_t = \epsilon + \text{Reward}_t$, as in (1).

We now restrict our attention to algorithms whose predictions w_t are of the form of a bet, that is $w_t = \beta_t \text{Wealth}_{t-1}$, where $\beta_t \in [-1, 1]$. We will see that the restriction on β_t does not prevent us from obtaining parameter-free algorithms with optimal bounds.

Given the above, it is immediate to see that any coin betting algorithm that, on a sequence of coin flips $\{g_t\}_{t=1}^\infty$, $g_t \in [-1, 1]$, bets the amounts w_t can be used as an OLO algorithm in a one-dimensional Hilbert space \mathbb{R} . But, what would be the regret of such OLO algorithms?

Assume that the betting algorithm at hand guarantees that its wealth is at least $F(\sum_{t=1}^T g_t)$ starting from an endowment ϵ , for a given potential function F , then

$$\text{Reward}_T = \sum_{t=1}^T g_t w_t = \text{Wealth}_T - \epsilon \geq F\left(\sum_{t=1}^T g_t\right) - \epsilon. \quad (3)$$

Intuitively, if the reward is big we can expect the regret to be small. Indeed, the following lemma converts the lower bound on the reward to an upper bound on the regret.

Lemma 1 (Reward-Regret relationship [22]). *Let V, V^* be a pair of dual vector spaces. Let $F : V \rightarrow \mathbb{R} \cup \{+\infty\}$ be a proper convex lower semi-continuous function and let $F^* : V^* \rightarrow \mathbb{R} \cup \{+\infty\}$ be its Fenchel conjugate. Let $w_1, w_2, \dots, w_T \in V$ and $g_1, g_2, \dots, g_T \in V^*$. Let $\epsilon \in \mathbb{R}$. Then,*

$$\underbrace{\sum_{t=1}^T \langle g_t, w_t \rangle}_{\text{Reward}_T} \geq F\left(\sum_{t=1}^T g_t\right) - \epsilon \quad \text{if and only if} \quad \forall u \in V^*, \quad \underbrace{\sum_{t=1}^T \langle g_t, u - w_t \rangle}_{\text{Regret}_T(u)} \leq F^*(u) + \epsilon.$$

Applying the lemma, we get a regret upper bound: $\text{Regret}_T(u) \leq F^*(u) + \epsilon$ for all $u \in \mathcal{H}$.

To summarize, if we have a betting algorithm that guarantees a minimum wealth of $F(\sum_{t=1}^T g_t)$, it can be used to design and analyze a one-dimensional OLO algorithm. The faster the growth of the wealth, the smaller the regret will be. Moreover, the lemma also shows that trying to design an algorithm that is adaptive to u is *equivalent* to designing an algorithm that is adaptive to $\sum_{t=1}^T g_t$. Also, most importantly, *methods that guarantee optimal wealth for the betting scenario are already known*, see, e.g., [4, Chapter 9]. We can just re-use them to get optimal online algorithms!

4 Designing a Betting Algorithm: Coin Betting Potentials

For sequential betting on i.i.d. coin flips, an optimal strategy has been proposed by Kelly [14]. The strategy assumes that the coin flips $\{g_t\}_{t=1}^\infty$, $g_t \in \{+1, -1\}$, are generated i.i.d. with known probability of heads. If $p \in [0, 1]$ is the probability of heads, the Kelly bet is to bet $\beta_t = 2p - 1$ at each round. He showed that, in the long run, this strategy will provide more wealth than betting any other fixed fraction of the current wealth [14].

For adversarial coins, Kelly betting does not make sense. With perfect knowledge of the future, the gambler could always bet everything on the right outcome. Hence, after T rounds from an initial endowment ϵ , the maximum wealth he would get is $\epsilon 2^T$. Instead, assume he bets the same fraction β of its wealth at each round. Let $\text{Wealth}_t(\beta)$ the wealth of such strategy after t rounds. As observed in [21], the optimal fixed fraction to bet is $\beta^* = (\sum_{t=1}^T g_t)/T$ and it gives the wealth

$$\text{Wealth}_T(\beta^*) = \epsilon \exp\left(T \cdot D\left(\frac{1}{2} + \frac{\sum_{t=1}^T g_t}{2T} \parallel \frac{1}{2}\right)\right) \geq \epsilon \exp\left(\frac{(\sum_{t=1}^T g_t)^2}{2T}\right), \quad (4)$$

where the inequality follows from Pinsker’s inequality [9, Lemma 11.6.1].

However, even without knowledge of the future, it is possible to go very close to the wealth in (4). This problem was studied by Krichevsky and Trofimov [16], who proposed that after seeing the coin flips g_1, g_2, \dots, g_{t-1} the empirical estimate $k_t = \frac{1/2 + \sum_{i=1}^{t-1} \mathbf{1}[g_i = +1]}{t}$ should be used instead of p . Their estimate is commonly called *KT estimator*.¹ The KT estimator results in the betting

$$\beta_t = 2k_t - 1 = \frac{\sum_{i=1}^{t-1} g_i}{t} \quad (5)$$

which we call *adaptive Kelly betting based on the KT estimator*. It looks like an online and slightly biased version of the oracle choice of β^* . This strategy guarantees²

$$\text{Wealth}_T \geq \frac{\text{Wealth}_T(\beta^*)}{2\sqrt{T}} = \frac{\epsilon}{2\sqrt{T}} \exp \left(T \cdot D \left(\frac{1}{2} + \frac{\sum_{t=1}^T g_t}{2T} \parallel \frac{1}{2} \right) \right).$$

This guarantee is optimal up to constant factors [4] and mirrors the guarantee of the Kelly bet.

Here, we propose a new set of definitions that allows to generalize the strategy of adaptive Kelly betting based on the KT estimator. For these strategies it will be possible to prove that, for any $g_1, g_2, \dots, g_t \in [-1, 1]$,

$$\text{Wealth}_t \geq F_t \left(\sum_{i=1}^t g_i \right), \quad (6)$$

where $F_t(x)$ is a certain function. We call such functions *potentials*. The betting strategy will be determined uniquely by the potential (see (c) in the Definition 2), and we restrict our attention to potentials for which (6) holds. These constraints are specified in the definition below.

Definition 2 (Coin Betting Potential). *Let $\epsilon > 0$. Let $\{F_t\}_{t=0}^\infty$ be a sequence of functions $F_t : (-a_t, a_t) \rightarrow \mathbb{R}_+$ where $a_t > t$. The sequence $\{F_t\}_{t=0}^\infty$ is called a **sequence of coin betting potentials for initial endowment ϵ** , if it satisfies the following three conditions:*

- (a) $F_0(0) = \epsilon$.
- (b) For every $t \geq 0$, $F_t(x)$ is even, logarithmically convex, strictly increasing on $[0, a_t)$, and $\lim_{x \rightarrow a_t} F_t(x) = +\infty$.
- (c) For every $t \geq 1$, every $x \in [-(t-1), (t-1)]$ and every $g \in [-1, 1]$, $(1 + g\beta_t) F_{t-1}(x) \geq F_t(x + g)$, where

$$\beta_t = \frac{F_t(x+1) - F_t(x-1)}{F_t(x+1) + F_t(x-1)}. \quad (7)$$

The sequence $\{F_t\}_{t=0}^\infty$ is called a **sequence of excellent coin betting potentials for initial endowment ϵ** if it satisfies conditions (a)–(c) and the condition (d) below.

- (d) For every $t \geq 0$, F_t is twice-differentiable and satisfies $x \cdot F_t''(x) \geq F_t'(x)$ for every $x \in [0, a_t)$.

Let’s give some intuition on this definition. First, let’s show by induction on t that (b) and (c) of the definition together with (2) give a betting strategy that satisfies (6). The base case $t = 0$ is trivial. At time $t \geq 1$, bet $w_t = \beta_t \text{Wealth}_{t-1}$ where β_t is defined in (7), then

$$\begin{aligned} \text{Wealth}_t &= \text{Wealth}_{t-1} + w_t g_t = (1 + g_t \beta_t) \text{Wealth}_{t-1} \\ &\geq (1 + g_t \beta_t) F_{t-1} \left(\sum_{i=1}^{t-1} g_i \right) \geq F_t \left(\sum_{i=1}^{t-1} g_i + g_t \right) = F_t \left(\sum_{i=1}^t g_i \right). \end{aligned}$$

The formula for the potential-based strategy (7) might seem strange. However, it is derived—see Theorem 8 in Appendix B—by minimizing the worst-case value of the right-hand side of the inequality used w.r.t. to g_t in the induction proof above: $F_{t-1}(x) \geq \frac{F_t(x+g_t)}{1+g_t\beta_t}$.

The last point, (d), is a technical condition that allows us to seamlessly reduce OLO over a Hilbert space to the one-dimensional problem, characterizing the worst case direction for the reward vectors.

¹Compared to the maximum likelihood estimate $\frac{\sum_{i=1}^{t-1} \mathbf{1}[g_i = +1]}{t-1}$, KT estimator shrinks slightly towards $1/2$.

²See Appendix A for a proof. For lack of space, all the appendices are in the supplementary material.

Regarding the design of coin betting potentials, we expect any potential that approximates the best possible wealth in (4) to be a good candidate. In fact, $F_t(x) = \epsilon \exp(x^2/(2t)) / \sqrt{t}$, essentially the potential used in the parameter-free algorithms in [22, 24] for OLO and in [6, 18, 19] for LEA, approximates (4) and it is an excellent coin betting potential—see Theorem 9 in Appendix B. Hence, our framework provides intuition to previous constructions and in Section 7 we show new examples of coin betting potentials.

In the next two sections, we presents the reductions to effortlessly solve *both* the generic OLO case and LEA with a betting potential.

5 From Coin Betting to OLO over Hilbert Space

In this section, generalizing the one-dimensional construction in Section 3, we show how to use a sequence of excellent coin betting potentials $\{F_t\}_{t=0}^\infty$ to construct an algorithm for OLO over a Hilbert space and how to prove a regret bound for it.

We define reward and wealth analogously to the one-dimensional case: $\text{Reward}_t = \sum_{i=1}^t \langle g_i, w_i \rangle$ and $\text{Wealth}_t = \epsilon + \text{Reward}_t$. Given a sequence of coin betting potentials $\{F_t\}_{t=0}^\infty$, using (7) we define the fraction

$$\beta_t = \frac{F_t(\|\sum_{i=1}^{t-1} g_i\| + 1) - F_t(\|\sum_{i=1}^{t-1} g_i\| - 1)}{F_t(\|\sum_{i=1}^{t-1} g_i\| + 1) + F_t(\|\sum_{i=1}^{t-1} g_i\| - 1)}. \quad (8)$$

The prediction of the OLO algorithm is defined similarly to the one-dimensional case, but now we also need a direction in the Hilbert space:

$$w_t = \beta_t \text{Wealth}_{t-1} \frac{\sum_{i=1}^{t-1} g_i}{\|\sum_{i=1}^{t-1} g_i\|} = \beta_t \frac{\sum_{i=1}^{t-1} g_i}{\|\sum_{i=1}^{t-1} g_i\|} \left(\epsilon + \sum_{i=1}^{t-1} \langle g_i, w_i \rangle \right). \quad (9)$$

If $\sum_{i=1}^{t-1} g_i$ is the zero vector, we define w_t to be the zero vector as well. For this prediction strategy we can prove the following regret guarantee, proved in Appendix C. The proof reduces the general Hilbert case to the 1-d case, thanks to (d) in Definition 2, then it follows the reasoning of Section 3.

Theorem 3 (Regret Bound for OLO in Hilbert Spaces). *Let $\{F_t\}_{t=0}^\infty$ be a sequence of excellent coin betting potentials. Let $\{g_t\}_{t=1}^\infty$ be any sequence of reward vectors in a Hilbert space \mathcal{H} such that $\|g_t\| \leq 1$ for all t . Then, the algorithm that makes prediction w_t defined by (9) and (8) satisfies*

$$\forall T \geq 0 \quad \forall u \in \mathcal{H} \quad \text{Regret}_T(u) \leq F_T^*(\|u\|) + \epsilon.$$

6 From Coin Betting to Learning with Expert Advice

In this section, we show how to use the algorithm for OLO over one-dimensional Hilbert space \mathbb{R} from Section 3—which is itself based on a coin betting strategy—to construct an algorithm for LEA.

Let $N \geq 2$ be the number of experts and Δ_N be the N -dimensional probability simplex. Let $\pi = (\pi_1, \pi_2, \dots, \pi_N) \in \Delta_N$ be any *prior* distribution. Let A be an algorithm for OLO over the one-dimensional Hilbert space \mathbb{R} , based on a sequence of the coin betting potentials $\{F_t\}_{t=0}^\infty$ with initial endowment³ 1. We instantiate N copies of A .

Consider any round t . Let $w_{t,i} \in \mathbb{R}$ be the prediction of the i -th copy of A . The LEA algorithm computes $\hat{p}_t = (\hat{p}_{t,1}, \hat{p}_{t,2}, \dots, \hat{p}_{t,N}) \in \mathbb{R}_{0,+}^N$ as

$$\hat{p}_{t,i} = \pi_i \cdot [w_{t,i}]_+, \quad (10)$$

where $[x]_+ = \max\{0, x\}$ is the positive part of x . Then, the LEA algorithm predicts $p_t = (p_{t,1}, p_{t,2}, \dots, p_{t,N}) \in \Delta^N$ as

$$p_t = \frac{\hat{p}_t}{\|\hat{p}_t\|_1}. \quad (11)$$

If $\|\hat{p}_t\|_1 = 0$, the algorithm predicts the prior π . Then, the algorithm receives the reward vector $g_t = (g_{t,1}, g_{t,2}, \dots, g_{t,N}) \in [0, 1]^N$. Finally, it feeds the reward to each copy of A . The reward for

³Any initial endowment $\epsilon > 0$ can be rescaled to 1. Instead of $F_t(x)$ we would use $F_t(x)/\epsilon$. The w_t would become w_t/ϵ , but p_t is invariant to scaling of w_t . Hence, the LEA algorithm is the same regardless of ϵ .

the i -th copy of A is $\tilde{g}_{t,i} \in [-1, 1]$ defined as

$$\tilde{g}_{t,i} = \begin{cases} g_{t,i} - \langle g_t, p_t \rangle & \text{if } w_{t,i} > 0, \\ [g_{t,i} - \langle g_t, p_t \rangle]_+ & \text{if } w_{t,i} \leq 0. \end{cases} \quad (12)$$

The construction above defines a LEA algorithm defined by the predictions p_t , based on the algorithm A . We can prove the following regret bound for it.

Theorem 4 (Regret Bound for Experts). *Let A be an algorithm for OLO over the one-dimensional Hilbert space \mathbb{R} , based on the coin betting potentials $\{F_t\}_{t=0}^\infty$ for an initial endowment of 1. Let f_t^{-1} be the inverse of $f_t(x) = \ln(F_t(x))$ restricted to $[0, \infty)$. Then, the regret of the LEA algorithm with prior $\pi \in \Delta_N$ that predicts at each round with p_t in (11) satisfies*

$$\forall T \geq 0 \quad \forall u \in \Delta_N \quad \text{Regret}_T(u) \leq f_T^{-1}(D(u||\pi)).$$

The proof, in Appendix D, is based on the fact that (10)–(12) guarantee that $\sum_{i=1}^N \pi_i \tilde{g}_{t,i} w_{t,i} \leq 0$ and on a variation of the change of measure lemma used in the PAC-Bayes literature, e.g. [20].

7 Applications of the Krichevsky-Trofimov Estimator to OLO and LEA

In the previous sections, we have shown that a coin betting potential with a guaranteed rapid growth of the wealth will give good regret guarantees for OLO and LEA. Here, we show that the KT estimator has associated an excellent coin betting potential, which we call *KT potential*. Then, the optimal wealth guarantee of the KT potentials will translate to optimal parameter-free regret bounds.

The sequence of excellent coin betting potentials for an initial endowment ϵ corresponding to the adaptive Kelly betting strategy β_t defined by (5) based on the KT estimator are

$$F_t(x) = \epsilon \frac{2^t \cdot \Gamma\left(\frac{t+1}{2} + \frac{x}{2}\right) \cdot \Gamma\left(\frac{t+1}{2} - \frac{x}{2}\right)}{\pi \cdot t!} \quad t \geq 0, \quad x \in (-t-1, t+1), \quad (13)$$

where $\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt$ is Euler's gamma function—see Theorem 13 in Appendix E. This potential was used to prove regret bounds for online prediction with the logarithmic loss [16][4, Chapter 9.7]. Theorem 13 also shows that the KT betting strategy β_t as defined by (5) satisfies (7).

This potential has the nice property that it satisfies the inequality in (c) of Definition 2 with equality when $g_t \in \{-1, 1\}$, i.e. $F_t(x + g_t) = (1 + g_t \beta_t) F_{t-1}(x)$.

We also generalize the KT potentials to δ -shifted KT potentials, where $\delta \geq 0$, defined as

$$F_t(x) = \frac{2^t \cdot \Gamma(\delta+1) \cdot \Gamma\left(\frac{t+\delta+1}{2} + \frac{x}{2}\right) \cdot \Gamma\left(\frac{t+\delta+1}{2} - \frac{x}{2}\right)}{\Gamma\left(\frac{\delta+1}{2}\right)^2 \cdot \Gamma(t+\delta+1)}.$$

The reason for its name is that, up to a multiplicative constant, F_t is equal to the KT potential shifted in time by δ . Theorem 13 also proves that the δ -shifted KT potentials are excellent coin betting potentials with initial endowment 1, and the corresponding betting fraction is $\beta_t = \frac{\sum_{j=1}^{t-1} g_j}{\delta+t}$.

7.1 OLO in Hilbert Space

We apply the KT potential for the construction of an OLO algorithm over a Hilbert space \mathcal{H} . We will use (9), and we just need to calculate β_t . According to Theorem 13 in Appendix E, the formula for β_t simplifies to $\beta_t = \frac{\|\sum_{i=1}^{t-1} g_i\|}{t}$ so that $w_t = \frac{1}{t} \left(\epsilon + \sum_{i=1}^{t-1} \langle g_i, w_i \rangle \right) \sum_{i=1}^{t-1} g_i$.

The resulting algorithm is stated as Algorithm 1. We derive a regret bound for it as a very simple corollary of Theorem 3 to the KT potential (13). The only technical part of the proof, in Appendix F, is an upper bound on F_t^* since it cannot be expressed as an elementary function.

Corollary 5 (Regret Bound for Algorithm 1). *Let $\epsilon > 0$. Let $\{g_t\}_{t=1}^\infty$ be any sequence of reward vectors in a Hilbert space \mathcal{H} such that $\|g_t\| \leq 1$. Then Algorithm 1 satisfies*

$$\forall T \geq 0 \quad \forall u \in \mathcal{H} \quad \text{Regret}_T(u) \leq \|u\| \sqrt{T \ln \left(1 + \frac{24T^2 \|u\|^2}{\epsilon^2} \right)} + \epsilon \left(1 - \frac{1}{e\sqrt{\pi T}} \right).$$

Algorithm 1 Algorithm for OLO over Hilbert space \mathcal{H} based on KT potential

Require: Initial endowment $\epsilon > 0$

- 1: **for** $t = 1, 2, \dots$ **do**
 - 2: Predict with $w_t \leftarrow \frac{1}{t} \left(\epsilon + \sum_{i=1}^{t-1} \langle g_i, w_i \rangle \right) \sum_{i=1}^{t-1} g_i$
 - 3: Receive reward vector $g_t \in \mathcal{H}$ such that $\|g_t\| \leq 1$
 - 4: **end for**
-

Algorithm 2 Algorithm for Learning with Expert Advice based on δ -shifted KT potential

Require: Number of experts N , prior distribution $\pi \in \Delta_N$, number of rounds T

- 1: **for** $t = 1, 2, \dots, T$ **do**
 - 2: For each $i \in [N]$, set $w_{t,i} \leftarrow \frac{\sum_{j=1}^{t-1} \tilde{g}_{j,i}}{t+T/2} \left(1 + \sum_{j=1}^{t-1} \tilde{g}_{j,i} w_{j,i} \right)$
 - 3: For each $i \in [N]$, set $\hat{p}_{t,i} \leftarrow \pi_i [w_{t,i}]_+$
 - 4: Predict with $p_t \leftarrow \begin{cases} \hat{p}_t / \|\hat{p}_t\|_1 & \text{if } \|\hat{p}_t\|_1 > 0 \\ \pi & \text{if } \|\hat{p}_t\|_1 = 0 \end{cases}$
 - 5: Receive reward vector $g_t \in [0, 1]^N$
 - 6: For each $i \in [N]$, set $\tilde{g}_{t,i} \leftarrow \begin{cases} g_{t,i} - \langle g_t, p_t \rangle & \text{if } w_{t,i} > 0 \\ [g_{t,i} - \langle g_t, p_t \rangle]_+ & \text{if } w_{t,i} \leq 0 \end{cases}$
 - 7: **end for**
-

It is worth noting the elegance and extreme simplicity of Algorithm 1 and contrast it with the algorithms in [26, 22–24]. Also, the regret bound is optimal [26, 23]. The parameter ϵ can be safely set to any constant, e.g. 1. Its role is equivalent to the initial guess used in doubling tricks [25].

7.2 Learning with Expert Advice

We will now construct an algorithm for LEA based on the δ -shifted KT potential. We set δ to $T/2$, requiring the algorithm to know the number of rounds T in advance; we will fix this later with the standard doubling trick.

To use the construction in Section 6, we need an OLO algorithm for the 1-d Hilbert space \mathbb{R} . Using the δ -shifted KT potentials, the algorithm predicts for any sequence $\{\tilde{g}_t\}_{t=1}^\infty$ of reward

$$w_t = \beta_t \text{Wealth}_{t-1} = \beta_t \left(1 + \sum_{j=1}^{t-1} \tilde{g}_j w_j \right) = \frac{\sum_{i=1}^{t-1} \tilde{g}_i}{T/2 + t} \left(1 + \sum_{j=1}^{t-1} \tilde{g}_j w_j \right).$$

Then, following the construction in Section 6, we arrive at the final algorithm, Algorithm 2. We can derive a regret bound for Algorithm 2 by applying Theorem 4 to the δ -shifted KT potential.

Corollary 6 (Regret Bound for Algorithm 2). *Let $N \geq 2$ and $T \geq 0$ be integers. Let $\pi \in \Delta_N$ be a prior. Then Algorithm 2 with input N, π, T for any rewards vectors $g_1, g_2, \dots, g_T \in [0, 1]^N$ satisfies*

$$\forall u \in \Delta_N \quad \text{Regret}_T(u) \leq \sqrt{3T(3 + D(u\|\pi))}.$$

Hence, the Algorithm 2 has *both* the best known guarantee on worst-case regret and per-round time complexity, see Table 1. Also, it has the advantage of being very simple.

The proof of the corollary is in the Appendix F. The only technical part of the proof is an upper bound on $f_t^{-1}(x)$, which we conveniently do by lower bounding $F_t(x)$.

The reason for using the shifted potential comes from the analysis of $f_t^{-1}(x)$. The unshifted algorithm would have a $O(\sqrt{T(\log T + D(u\|\pi))})$ regret bound; the shifting improves the bound to $O(\sqrt{T(1 + D(u\|\pi))})$. By changing $T/2$ in Algorithm 2 to another constant fraction of T , it is possible to trade-off between the two constants 3 present in the square root in the regret upper bound.

The requirement of knowing the number of rounds T in advance can be lifted by the standard doubling trick [25, Section 2.3.1], obtaining an anytime guarantee with a bigger leading constant,

$$\forall T \geq 0 \quad \forall u \in \Delta_N \quad \text{Regret}_T(u) \leq \frac{\sqrt{2}}{\sqrt{2}-1} \sqrt{3T(3 + D(u\|\pi))}.$$

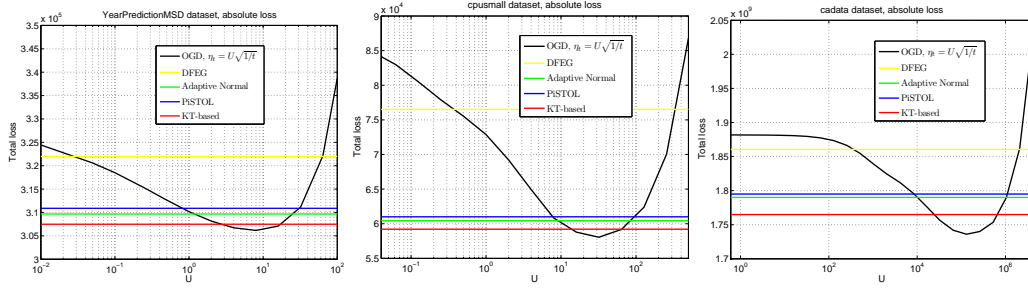


Figure 1: Total loss versus learning rate parameter of OGD (in log scale), compared with parameter-free algorithms DFEG [23], Adaptive Normal [22], PiSTOL [24] and the KT-based Algorithm 1.

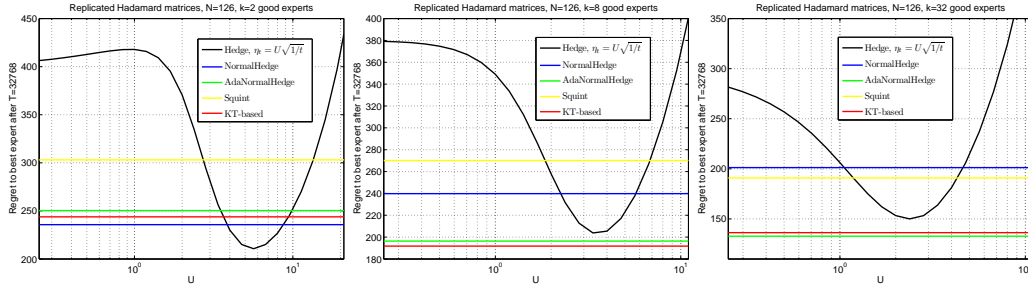


Figure 2: Regrets to the best expert after $T = 32768$ rounds, versus learning rate parameter of Hedge (in log scale). The “good” experts are $\epsilon = 0.025$ better than the others. The competitor algorithms are NormalHedge [6], AdaNormalHedge [19], Squint [15], and the KT-based Algorithm 2. $\pi_i = 1/N$ for all algorithms.

8 Discussion of the Results

We have presented a new interpretation of parameter-free algorithms as coin betting algorithms. This interpretation, far from being just a mathematical gimmick, reveals the *common* hidden structure of previous parameter-free algorithms for both OLO and LEA and also allows the design of new algorithms. For example, we show that the characteristic of parameter-freeness is just a consequence of having an algorithm that guarantees the maximum reward possible. The reductions in Sections 5 and 6 are also novel and they are in a certain sense optimal. In fact, the obtained Algorithms 1 and 2 achieve the optimal worst case upper bounds on the regret, see [26, 23] and [4] respectively.

We have also run an empirical evaluation to show that the theoretical difference between classic online learning algorithms and parameter-free ones is real and not just theoretical. In Figure 1, we have used three regression datasets⁴, and solved the OCO problem through OLO. In all the three cases, we have used the absolute loss and normalized the input vectors to have L2 norm equal to 1. From the empirical results, it is clear that the optimal learning rate is completely data-dependent, yet *parameter-free algorithms have performance very close to the unknown optimal tuning of the learning rate*. Moreover, the KT-based Algorithm 1 seems to dominate all the other similar algorithms.

For LEA, we have used the synthetic setting in [6]. The dataset is composed of Hadamard matrices of size 64, where the row with constant values is removed, the rows are duplicated to 126 inverting their signs, 0.025 is subtracted to k rows, and the matrix is replicated in order to generate $T = 32768$ samples. For more details, see [6]. Here, the KT-based algorithm is the one in Algorithm 2, where the term $T/2$ is removed, so that the final regret bound has an additional $\ln T$ term. Again, we see that the parameter-free algorithms have a performance close or *even better* than Hedge with an oracle tuning of the learning rate, with no clear winners among the parameter-free algorithms.

Notice that since the adaptive Kelly strategy based on KT estimator is very close to optimal, the only possible improvement is to have a data-dependent bound, for example like the ones in [24, 15, 19]. In future work, we will extend our definitions and reductions to the data-dependent case.

⁴Datasets available at <https://www.csie.ntu.edu.tw/~cjlin/libsvmtools/datasets/>.

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A From Log Loss to Wealth

Guarantees for betting or sequential investment algorithm are often expressed as upper bounds on the regret with respect to the log loss. Here, for the sake of completeness, we show how to convert such a guarantee to a lower bound on the wealth of the corresponding betting algorithm.

We consider the problem of predicting a binary outcome. The algorithm predicts at each round probability $p_t \in [0, 1]$. The adversary generates a sequences of outcomes $x_t \in \{0, 1\}$ and the algorithm's loss is

$$\ell(p_t, x_t) = -x_t \ln p_t - (1 - x_t) \ln(1 - p_t) .$$

We define the regret with respect to a fixed probability vector β as

$$\text{Regret}_T^{\text{logloss}} = \sum_{t=1}^T \ell(p_t, x_t) - \min_{\beta \in [0,1]} \sum_{t=1}^T \ell(\beta, x_t) .$$

Lemma 7. *Assume that an algorithm that predicts p_t guarantees $\text{Regret}_T^{\text{logloss}} \leq R_T$. Then, the coin betting strategy with endowment ϵ and $\beta_t = 2p_t - 1$ guarantees*

$$\text{Wealth}_T \geq \epsilon \exp \left(T \cdot D \left(\frac{1}{2} + \frac{\sum_{t=1}^T g_t}{2T} \middle\| \frac{1}{2} \right) - R_T \right)$$

against any sequence of outcomes $g_t \in [-1, +1]$.

Proof. Define $x_t = \frac{1+g_t}{2}$. We have

$$\begin{aligned} \ln \text{Wealth}_T &= \ln(\text{Wealth}_{t-1} + w_t g_t) \\ &= \ln(\text{Wealth}_{t-1}(1 + g_t \beta_t)) \\ &= \ln \epsilon \prod_{t=1}^T (1 + g_t \beta_t) \\ &= \ln \epsilon + \sum_{t=1}^T \ln(1 + g_t \beta_t) \\ &\geq \ln \epsilon + \sum_{t=1}^T \left(\frac{1+g_t}{2} \right) \ln(1 + \beta_t) + \left(\frac{1-g_t}{2} \right) \ln(1 - \beta_t) \\ &= \ln \epsilon + \sum_{t=1}^T \left(\frac{1+g_t}{2} \right) \ln(2p_t) + \left(\frac{1-g_t}{2} \right) \ln(2(1-p_t)) \\ &= \ln \epsilon + T \ln(2) + \sum_{t=1}^T \left(\frac{1+g_t}{2} \right) \ln(p_t) + \left(\frac{1-g_t}{2} \right) \ln(1-p_t) \\ &= \ln \epsilon + T \ln(2) - \sum_{t=1}^T \ell(p_t, x_t) \\ &= \ln \epsilon + T \ln(2) - \text{Regret}_T^{\text{logloss}} - \min_{\beta \in [0,1]} \sum_{t=1}^T \ell(\beta, x_t) \\ &\geq \ln \epsilon + T \ln(2) - R_T - \min_{\beta \in [0,1]} \sum_{t=1}^T \ell(\beta, x_t) , \end{aligned}$$

where the first inequality is due to the concavity of \ln and the second one is due to the assumption of the regret.

It is easy to see that the $\beta^* = \arg \min_{\beta \in [0,1]} \sum_{t=1}^T \ell(\beta, x_t) = \frac{\sum_{t=1}^T x_t}{T}$. Hence, we have

$$\min_{\beta \in [0,1]} \sum_{t=1}^T \ell(\beta, x_t) = T (-\beta^* \ln \beta^* - (1 - \beta^*) \ln(1 - \beta^*)) .$$

Also, we have that for any $\beta \in [0, 1]$

$$-\beta \ln \beta - (1 - \beta) \ln(1 - \beta) = -D\left(\beta \parallel \frac{1}{2}\right) + \ln 2.$$

Putting all together, we have the stated lemma. \square

The lower bound on the wealth of the adaptive Kelly betting based on the KT estimator is obtained simply by the stated Lemma and reminding that the log loss regret of the KT estimator is upper bounded by $\frac{1}{2} \ln T + \ln 2$.

B Optimal Betting Fraction

Theorem 8 (Optimal Betting Fraction). *Let $x \in \mathbb{R}$. Let $F : [x - 1, x + 1] \rightarrow \mathbb{R}$ be a logarithmically convex function. Then,*

$$\arg \min_{\beta \in (-1, 1)} \max_{g \in [-1, 1]} \frac{F(x + g)}{1 + \beta g} = \frac{F(x + 1) - F(x - 1)}{F(x + 1) + F(x - 1)}.$$

Moreover, $\beta^* = \frac{F(x+1) - F(x-1)}{F(x+1) + F(x-1)}$ satisfies

$$\ln(F(x + 1)) - \ln(1 + \beta^*) = \ln(F(x - 1)) - \ln(1 - \beta^*).$$

Proof. We define the functions $h, f : [-1, 1] \times (-1, 1) \rightarrow \mathbb{R}$ as

$$h(g, \beta) = \frac{F(x + g)}{1 + \beta g} \quad \text{and} \quad f(g, \beta) = \ln(h(g, \beta)) = \ln(F(x + g)) - \ln(1 + \beta g).$$

Clearly, $\arg \min_{\beta \in (-1, 1)} \max_{g \in [-1, 1]} h(g, \beta) = \arg \min_{\beta \in (-1, 1)} \max_{g \in [-1, 1]} f(g, \beta)$ and we can work with f instead of h . The function h is logarithmically convex in g and thus f is convex in g . Therefore,

$$\forall \beta \in (-1, 1) \quad \max_{g \in [-1, 1]} f(g, \beta) = \max \{f(+1, \beta), f(-1, \beta)\}.$$

Let $\phi(\beta) = \max \{f(+1, \beta), f(-1, \beta)\}$. We seek to find the $\arg \min_{\beta \in (-1, 1)} \phi(\beta)$. Since $f(+1, \beta)$ is decreasing in β and $f(-1, \beta)$ is increasing in β , the minimum of $\phi(\beta)$ is at a point β^* such that $f(+1, \beta^*) = f(-1, \beta^*)$. In other words, β^* satisfies

$$\ln(F(x + 1)) - \ln(1 + \beta^*) = \ln(F(x - 1)) - \ln(1 - \beta^*).$$

The only solution of this equation is

$$\beta^* = \frac{F(x + 1) - F(x - 1)}{F(x + 1) + F(x - 1)}.$$

\square

Theorem 9. *The functions $F_t(x) = \epsilon \exp(\frac{x^2}{2t} - \frac{1}{2} \sum_{i=1}^t \frac{1}{i})$ are excellent coin betting potentials.*

Proof. The first and second properties of Definition 2 are trivially true. For the third property, we first use Theorem 8 to have

$$\ln(1 + \beta_t g) - \ln F_t(x + g) \geq \ln(1 + \beta_t) - \ln F_t(x + 1) = \ln \frac{2}{F_t(x + 1) + F_t(x - 1)},$$

where the definition of β_t is from (7). Hence, we have

$$\begin{aligned}
\ln(1 + \beta_t g) - \ln F_t(x + g) + \ln F_{t-1}(x) &\geq \ln \frac{2}{F_t(x+1) + F_t(x-1)} + \ln F_{t-1}(x) \\
&= -\frac{x^2 + 1}{2t} + \frac{1}{2} \sum_{i=1}^t \frac{1}{i} - \ln \cosh \frac{x}{t} + \frac{x^2}{2(t-1)} - \frac{1}{2} \sum_{i=1}^{t-1} \frac{1}{i} \\
&= -\frac{x^2}{2t} - \ln \cosh \frac{x}{t} + \frac{x^2}{2(t-1)} \\
&\geq -\frac{x^2}{2t} - \frac{x^2}{2t^2} + \frac{x^2}{2(t-1)} \\
&\geq -\frac{x^2}{2t} - \frac{x^2}{2t(t-1)} + \frac{x^2}{2(t-1)} = 0,
\end{aligned}$$

where in the second inequality we have used the elementary inequality $\ln \cosh x \leq \frac{x^2}{2}$.

The fourth property of Definition 2 is also true because $F_t(x)$ is of the form $h(x^2)$ with $h(\cdot)$ convex [22]. \square

C Proof of Lemma 11

First we state the following Lemma from [22] and reported here with our notation for completeness.

Lemma 10 (Extremes). *Let $h : (-a, a) \rightarrow \mathbb{R}$ be an even twice-differentiable function that satisfies $x \cdot h''(x) \geq h'(x)$ for all $x \in [0, a)$. Let $c : [0, \infty) \times [0, \infty) \rightarrow \mathbb{R}$ be an arbitrary function. Then, if vectors $u, v \in \mathcal{H}$ satisfy $\|u\| + \|v\| < a$, then*

$$\begin{aligned}
c(\|u\|, \|v\|) \cdot \langle u, v \rangle - h(\|u + v\|) &\geq \min \{ c(\|u\|, \|v\|) \cdot \|u\| \cdot \|v\| - h(\|v\| + \|v\|), \\
&\quad -c(\|u\|, \|v\|) \cdot \|u\| \cdot \|v\| - h(\|u\| - \|v\|) \} . \quad (14)
\end{aligned}$$

Proof. If u or v is zero, the inequality (14) clearly holds. From now on we assume that u, v are non-zero. Let α be the cosine of the angle of between u and v . More formally,

$$\alpha = \frac{\langle u, v \rangle}{\|u\| \cdot \|v\|} .$$

With this notation, the left-hand side of (14) is

$$f(\alpha) = c(\|u\|, \|v\|) \cdot \alpha \|u\| \cdot \|v\| - h(\sqrt{\|u\|^2 + \|v\|^2 + 2\alpha \|u\| \cdot \|v\|}) .$$

Since h is even, the inequality (14) is equivalent to

$$\forall \alpha \in [-1, 1] \quad f(\alpha) \geq \min \{ f(+1), f(-1) \} .$$

The last inequality is clearly true if $f : [-1, 1] \rightarrow \mathbb{R}$ is concave. We now check that f is indeed concave, which we prove by showing that the second derivative is non-positive. The first derivative of f is

$$f'(\alpha) = c(\|u\|, \|v\|) \cdot \|u\| \cdot \|v\| - \frac{h'(\sqrt{\|u\|^2 + \|v\|^2 + 2\alpha \|u\| \cdot \|v\|}) \cdot \|u\| \cdot \|v\|}{\sqrt{\|u\|^2 + \|v\|^2 + 2\alpha \|u\| \cdot \|v\|}} .$$

The second derivative of f is

$$\begin{aligned}
f''(\alpha) &= -\frac{\|u\|^2 \cdot \|v\|^2}{\|u\|^2 + \|v\|^2 + 2\alpha \|u\| \cdot \|v\|} \\
&\quad \cdot \left(h''(\sqrt{\|u\|^2 + \|v\|^2 + 2\alpha \|u\| \cdot \|v\|}) - \frac{h'(\sqrt{\|u\|^2 + \|v\|^2 + 2\alpha \|u\| \cdot \|v\|})}{\sqrt{\|u\|^2 + \|v\|^2 + 2\alpha \|u\| \cdot \|v\|}} \right) .
\end{aligned}$$

If we consider $x = \sqrt{\|u\|^2 + \|v\|^2 + 2\alpha \|u\| \cdot \|v\|}$, the assumption $x \cdot h''(x) \geq h'(x)$ implies that $f''(\alpha)$ is non-positive. This finishes the proof of the inequality (14). \square

We also need the following technical Lemma whose proof relies mainly on property (d) of Definition 2.

Lemma 11. *Let $\{F_t\}_{t=0}^\infty$ be a sequence of excellent coin betting potentials. Let g_1, g_2, \dots, g_t be vectors in a Hilbert space \mathcal{H} such that $\|g_1\|, \|g_2\|, \dots, \|g_t\| \leq 1$. Let β_t be defined by (8) and let $x = \sum_{i=1}^{t-1} g_i$. Then,*

$$\left(1 + \beta_t \frac{\langle g_t, x \rangle}{\|x\|}\right) F_{t-1}(\|x\|) \geq F_t(\|x + g_t\|) .$$

Proof. Since $F_t(x)$ is an excellent coin betting potential, it satisfies $xF_t''(x) \geq F_t'(x)$. Hence,

$$\begin{aligned} & \left(1 + \beta_t \frac{\langle g_t, x \rangle}{\|x\|}\right) F_{t-1}(\|x\|) - F_t(\|x + g_t\|) \\ &= F_{t-1}(\|x\|) + \beta_t \frac{\langle g_t, x \rangle}{\|x\|} F_{t-1}(\|x\|) - F_t(\|x + g_t\|) \\ &\geq F_{t-1}(\|x\|) + \min_{r \in \{-1, 1\}} \beta_t r \|g_t\| F_{t-1}(\|x\|) - F_t(\|x\| + r \|g_t\|) \\ &= \min_{r \in \{-1, 1\}} (1 + \beta_t r \|g_t\|) F_{t-1}(\|x\|) - F_t(\|x\| + r \|g_t\|) \\ &\geq 0 . \end{aligned}$$

If $x \neq 0$, the first inequality comes from Lemma 10 with $c(z, \cdot) = \frac{F_{t-1}(z+1) - F_{t-1}(z-1)}{F_{t-1}(z+1) + F_{t-1}(z-1)} F_{t-1}(z)/z$ and $h(z) = F_t(z)$, $u = g_t$, $v = x$. If $x = 0$ then, according to (8), $\beta_t = 0$ and the first inequality trivially holds. The second inequality follows from the property (c) of a coin betting potential. \square

Proof of Theorem 3. First, by induction on t we show that

$$\text{Wealth}_t \geq F_t \left(\left\| \sum_{t=1}^T g_t \right\| \right) . \quad (15)$$

The base case $t = 0$ is trivial, since both sides of the inequality are equal to ϵ . For $t \geq 1$, if we let $x = \sum_{i=1}^{t-1} g_i$, we have

$$\begin{aligned} \text{Wealth}_t &= \langle g_t, w_t \rangle + \text{Wealth}_{t-1} = \left(1 + \beta_t \frac{\langle g_t, x \rangle}{\|x\|}\right) \text{Wealth}_{t-1} \\ &\geq \left(1 + \beta_t \frac{\langle g_t, x \rangle}{\|x\|}\right) F_{t-1}(\|x\|) \stackrel{(*)}{\geq} F_t(\|x + g_t\|) = F_t \left(\left\| \sum_{i=1}^t g_i \right\| \right) . \end{aligned}$$

The inequality marked with $(*)$ follows from Lemma 11.

This establishes (15), from which we immediately have a reward lower bound

$$\text{Reward}_T = \sum_{t=1}^T \langle g_t, w_t \rangle = \text{Wealth}_T - \epsilon \geq F_T \left(\left\| \sum_{t=1}^T g_t \right\| \right) - \epsilon . \quad (16)$$

We apply Lemma 1 to the function $F(x) = F_T(\|x\|) - \epsilon$ and we are almost done. The only remaining property we need is that if F is an even function then the Fenchel conjugate of $F(\|\cdot\|)$ is $F^*(\|\cdot\|)$; see Bauschke and Combettes [3, Example 13.7]. \square

D Proof of Theorem 4

Proof. We first prove that $\sum_{i=1}^N \pi_i \tilde{g}_{t,i} w_{t,i} \leq 0$. Indeed,

$$\begin{aligned} \sum_{i=1}^N \pi_i \tilde{g}_{t,i} w_{t,i} &= \sum_{i: \pi_i w_{t,i} > 0} \pi_i [w_{t,i}]_+ (g_{t,i} - \langle g_t, p_t \rangle) + \sum_{i: \pi_i w_{t,i} \leq 0} \pi_i w_{t,i} [g_{t,i} - \langle g_t, p_t \rangle]_+ \\ &= \|\hat{p}_t\|_1 \sum_{i=1}^N p_{t,i} (g_{t,i} - \langle g_t, p_t \rangle) + \sum_{i: \pi_i w_{t,i} \leq 0} \pi_i w_{t,i} [g_{t,i} - \langle g_t, p_t \rangle]_+ \\ &= 0 + \sum_{i: \pi_i w_{t,i} \leq 0} \pi_i w_{t,i} [g_{t,i} - \langle g_t, p_t \rangle]_+ \leq 0. \end{aligned}$$

The first equality follows from definition of $g_{t,i}$. To see the second equality, consider two cases: If $\pi_i w_{t,i} \leq 0$ for all i then $\|\hat{p}_t\|_1 = 0$ and therefore both $\|\hat{p}_t\|_1 \sum_{i=1}^N p_{t,i} (g_{t,i} - \langle g_t, p_t \rangle)$ and $\sum_{i: \pi_i w_{t,i} > 0} \pi_i [w_{t,i}]_+ (g_{t,i} - \langle g_t, p_t \rangle)$ are trivially zero. If $\|\hat{p}_t\|_1 > 0$ then $\pi_i [w_{t,i}]_+ = \hat{p}_{t,i} = \|\hat{p}_t\|_1 p_{t,i}$ for all i .

From the assumption on A , we have, for any sequence $\{\tilde{g}_t\}_{t=1}^\infty$ such that $\tilde{g}_t \in [-1, 1]$, satisfies

$$\text{Wealth}_t = 1 + \sum_{i=1}^t \tilde{g}_i w_i \geq F_t \left(\sum_{i=1}^t \tilde{g}_i \right). \quad (17)$$

Inequality $\sum_{i=1}^N \pi_i \tilde{g}_{t,i} w_{t,i} \leq 0$ and (17) imply

$$\sum_{i=1}^N \pi_i F_T \left(\sum_{t=1}^T \tilde{g}_{t,i} \right) \leq 1 + \sum_{i=1}^N \pi_i \sum_{t=1}^T \tilde{g}_{t,i} w_{t,i} \leq 1. \quad (18)$$

Now, let $\tilde{G}_{T,i} = \sum_{t=1}^T \tilde{g}_{t,i}$. For any competitor $u \in \Delta_N$,

$$\begin{aligned} \text{Regret}_T(u) &= \sum_{t=1}^T \langle g_t, u - p_t \rangle = \sum_{t=1}^T \sum_{i=1}^N u_i (g_{t,i} - \langle g_t, p_t \rangle) \\ &\leq \sum_{t=1}^T \sum_{i=1}^N u_i \tilde{g}_{t,i} \quad (\text{by definition of } \tilde{g}_{t,i}) \\ &\leq \sum_{i=1}^N u_i \left| \tilde{G}_{T,i} \right| \quad (\text{since } u_i \geq 0, i = 1, \dots, N) \\ &= \sum_{i=1}^N u_i f_T^{-1} \left(\ln[F_T(\tilde{G}_{T,i})] \right) \quad (\text{since } F_T(x) = \exp(f_T(x)) \text{ is even}) \\ &\leq f_T^{-1} \left(\sum_{i=1}^N u_i \ln[F_T(\tilde{G}_{T,i})] \right) \quad (\text{by concavity of } f_T^{-1}) \\ &= f_T^{-1} \left(\sum_{i=1}^N u_i \left\{ \ln \left[\frac{u_i}{\pi_i} \right] + \ln \left[\frac{\pi_i}{u_i} F_T(\tilde{G}_{T,i}) \right] \right\} \right) = f_T^{-1} \left(D(u \parallel \pi) + \sum_{i=1}^N u_i \ln \left[\frac{\pi_i}{u_i} F_T(\tilde{G}_{T,i}) \right] \right) \\ &\leq f_T^{-1} \left(D(u \parallel \pi) + \ln \left(\sum_{i=1}^N \pi_i F_T(\tilde{G}_{T,i}) \right) \right) \quad (\text{by concavity of } \ln(\cdot)) \\ &\leq f_T^{-1} (D(u \parallel \pi)) \quad (\text{by (18)}). \end{aligned} \quad \square$$

E Properties of Krichevsky-Trofimov Potential

Lemma 12 (Analytic Properties of KT potential). *Let $a > 0$. The function $F : (-a, a) \rightarrow \mathbb{R}_+$,*

$$F(x) = \Gamma(a+x)\Gamma(a-x)$$

is even, logarithmically convex, strictly increasing on $[0, a)$, satisfies

$$\lim_{x \nearrow a} F(x) = \lim_{x \searrow -a} F(x) = +\infty$$

and

$$\forall x \in [0, a) \quad x \cdot F''(x) \geq F'(x) . \quad (19)$$

Proof. $F(x)$ is obviously even. $\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt$ is defined for any real number $z > 0$. Hence, F is defined on the interval $(-a, a)$. According to Bohr-Mollerup theorem [1, Theorem 2.1], $\Gamma(x)$ is logarithmically convex on $(0, \infty)$. Hence, $F(x)$ is also logarithmically convex, since $\ln(F(x)) = \ln(\Gamma(a+x)) + \ln(\Gamma(a-x))$ is a sum of convex functions.

It is well known that $\lim_{z \searrow 0} \Gamma(z) = +\infty$. Thus,

$$\lim_{x \nearrow a} F(x) = \lim_{x \nearrow a} \Gamma(a+x) \Gamma(a-x) = \Gamma(2a) \lim_{x \nearrow a} \Gamma(a-x) = \Gamma(2a) \lim_{z \searrow 0} \Gamma(z) = +\infty ,$$

since Γ is continuous and not zero at $2a$. Because $F(x)$ is even, we also have $\lim_{x \searrow -a} F(x) = +\infty$.

To show that $F(x)$ is increasing and that it satisfies (19), we write $f(x) = \ln(F(x))$ as a Mclaurin series. The derivatives of $\ln(\Gamma(z))$ are the so called polygamma functions

$$\psi^{(n)}(z) = \frac{d^{n+1}}{dz^{n+1}} \ln(\Gamma(z)) \quad \text{for } z > 0 \text{ and } n = 0, 1, 2, \dots$$

Polygamma functions have the well-known integral representation

$$\psi^{(n)}(z) = (-1)^{n+1} \int_0^\infty \frac{t^n e^{-zt}}{1 - e^{-t}} dt \quad \text{for } z > 0 \text{ and } n = 1, 2, \dots$$

Using polygamma functions, we can write the Mclaurin series for $f(x) = \ln(F(x))$ as

$$f(x) = \ln(F(x)) = \ln(\Gamma(a+x)) + \ln(\Gamma(a-x)) = 2 \ln(\Gamma(a)) + 2 \sum_{\substack{n \geq 2 \\ n \text{ even}}} \frac{\psi^{(n-1)}(a) x^n}{n!} .$$

The series converges for $x \in (-a, a)$, since for even $n \geq 2$, $\psi^{(n-1)}(a)$ is positive and can be upper bounded as

$$\begin{aligned} \psi^{(n-1)}(a) &= \int_0^\infty \frac{t^{n-1} e^{-at}}{1 - e^{-t}} dt \\ &= \int_0^1 \frac{t^{n-1} e^{-at}}{1 - e^{-t}} dt + \int_1^\infty \frac{t^{n-1} e^{-zt}}{1 - e^{-t}} dt \\ &\leq \int_0^1 \frac{t^{n-1} e^{-at}}{t(1 - 1/e)} dt + \int_1^\infty t^{n-1} e^{-at} dt \\ &\leq \frac{1}{1 - 1/e} \int_0^\infty t^{n-2} e^{-at} dt + \int_0^\infty t^{n-1} e^{-at} dt \\ &= \frac{1}{1 - 1/e} a^{1-n} \Gamma(n-1) + a^{-n} \Gamma(n) \\ &\leq \frac{1}{1 - 1/e} a^{-n} (a+1)(n-1)! . \end{aligned}$$

From the Mclaurin expansion we see that $f(x)$ is increasing on $[0, a)$ since all the coefficients are positive (except for zero order term).

Finally, to prove (19), note that for any $x \in (-a, a)$,

$$f(x) = c_0 + \sum_{n=2}^\infty c_n x^n$$

where c_2, c_3, \dots are non-negative coefficients. Thus

$$f'(x) = \sum_{n=2}^\infty n c_n x^{n-1} \quad \text{and} \quad f''(x) = \sum_{n=2}^\infty n(n-1) c_n x^{n-2} .$$

and hence $x \cdot f''(x) \geq f'(x)$ for $x \in [0, a)$. Since $F(x) = \exp(f(x))$,

$$F'(x) = f'(x) \cdot F(x) \quad \text{and} \quad F''(x) = [f''(x) + (f'(x))^2] \cdot F(x).$$

Therefore, for $x \in [0, a)$,

$$x \cdot F''(x) = x [f''(x) + (f'(x))^2] F(x) \geq [f'(x) + x(f'(x))^2] F(x) \geq f'(x)F(x) = F'(x).$$

This proves (19). \square

Theorem 13 (KT potential). *Let $\delta \geq 0$ and $\epsilon > 0$. The sequence of functions $\{F_t\}_{t=0}^\infty$, $F_t : (-t - \delta - 1, t + \delta + 1) \rightarrow \mathbb{R}_+$ defined by*

$$F_t(x) = \epsilon \frac{2^t \cdot \Gamma(\delta + 1) \Gamma(\frac{t+\delta+1}{2} + \frac{x}{2}) \Gamma(\frac{t+\delta+1}{2} - \frac{x}{2})}{\Gamma(\frac{\delta+1}{2})^2 \Gamma(t + \delta + 1)}.$$

is a sequence of excellent coin betting potentials for initial endowment ϵ . Furthermore, for any $x \in (-t - \delta - 1, t + \delta + 1)$,

$$\frac{F_t(x+1) - F_t(x-1)}{F_t(x+1) + F_t(x-1)} = \frac{x}{t + \delta}. \quad (20)$$

Proof. Property (b) and (d) of the definition follow from Lemma 12. Property (a) follows by simple substitution for $t = 0$ and $x = 0$.

Before verifying property (c), we prove (20). We use an algebraic property of the gamma function that states that $\Gamma(1+z) = z\Gamma(z)$ for any positive z . Equation (20) follows from

$$\begin{aligned} \frac{F_t(x+1) - F_t(x-1)}{F_t(x+1) + F_t(x-1)} &= \frac{\Gamma(\frac{t+\delta+2}{2} + \frac{x}{2}) \Gamma(\frac{t+\delta}{2} - \frac{x}{2}) - \Gamma(\frac{t+\delta}{2} + \frac{x}{2}) \Gamma(\frac{t+\delta+2}{2} - \frac{x}{2})}{\Gamma(\frac{t+\delta+2}{2} + \frac{x}{2}) \Gamma(\frac{t+\delta}{2} - \frac{x}{2}) + \Gamma(\frac{t+\delta}{2} + \frac{x}{2}) \Gamma(\frac{t+\delta+2}{2} - \frac{x}{2})} \\ &= \frac{(\frac{t+\delta}{2} + \frac{x}{2}) \Gamma(\frac{t+\delta}{2} + \frac{x}{2}) \Gamma(\frac{t+\delta}{2} - \frac{x}{2}) - (\frac{t+\delta}{2} - \frac{x}{2}) \Gamma(\frac{t+\delta}{2} + \frac{x}{2}) \Gamma(\frac{t+\delta}{2} - \frac{x}{2})}{(\frac{t+\delta}{2} + \frac{x}{2}) \Gamma(\frac{t+\delta}{2} + \frac{x}{2}) \Gamma(\frac{t+\delta}{2} - \frac{x}{2}) + (\frac{t+\delta}{2} - \frac{x}{2}) \Gamma(\frac{t+\delta}{2} + \frac{x}{2}) \Gamma(\frac{t+\delta}{2} - \frac{x}{2})} \\ &= \frac{(\frac{t+\delta}{2} + \frac{x}{2}) - (\frac{t+\delta}{2} - \frac{x}{2})}{(\frac{t+\delta}{2} + \frac{x}{2}) + (\frac{t+\delta}{2} - \frac{x}{2})} \\ &= \frac{x}{t + \delta}. \end{aligned}$$

Let $\phi(g) = \frac{F_t(x+g)}{F_{t-1}(x)}$. To verify property (c) of the definition, we need to show that $\phi(g) \leq 1 + g \frac{x}{t+\delta}$ for any $x \in [-t+1, t-1]$ and any $g \in [-1, 1]$. We can write $\phi(g)$ as

$$\begin{aligned} \phi(g) &= \frac{F_t(x+g)}{F_{t-1}(x)} \\ &= \frac{2 \Gamma(\frac{t+\delta+1}{2} + \frac{x+g}{2}) \Gamma(\frac{t+\delta+1}{2} - \frac{x+g}{2}) \Gamma(t+\delta)}{\Gamma(\frac{t+\delta}{2} + \frac{x}{2}) \Gamma(\frac{t+\delta}{2} - \frac{x}{2}) \Gamma(t+\delta+1)} \\ &= \frac{2}{t+\delta} \cdot \frac{\Gamma(\frac{t+\delta+1}{2} + \frac{x+g}{2}) \Gamma(\frac{t+\delta+1}{2} - \frac{x+g}{2})}{\Gamma(\frac{t+\delta}{2} + \frac{x}{2}) \Gamma(\frac{t+\delta}{2} - \frac{x}{2})}. \end{aligned}$$

For $g = +1$, using the formula $\Gamma(1+z) = z\Gamma(z)$, we have

$$\phi(+1) = \frac{2}{t+\delta} \cdot \frac{\Gamma(\frac{t+\delta}{2} + \frac{x}{2} + 1) \Gamma(\frac{t+\delta}{2} - \frac{x}{2})}{\Gamma(\frac{t+\delta}{2} + \frac{x}{2}) \Gamma(\frac{t+\delta}{2} - \frac{x}{2})} = \frac{2}{t+\delta} \left(\frac{t+\delta}{2} + \frac{x}{2} \right) = 1 + \frac{x}{t+\delta}.$$

Similarly, for $g = -1$, using the formula $\Gamma(1+z) = z\Gamma(z)$, we have

$$\phi(-1) = \frac{2}{t+\delta} \cdot \frac{\Gamma(\frac{t+\delta}{2} + \frac{x}{2}) \Gamma(\frac{t+\delta}{2} - \frac{x}{2} + 1)}{\Gamma(\frac{t+\delta}{2} + \frac{x}{2}) \Gamma(\frac{t+\delta}{2} - \frac{x}{2})} = \frac{2}{t+\delta} \left(\frac{t+\delta}{2} - \frac{x}{2} \right) = 1 - \frac{x}{t+\delta}.$$

We can write any $g \in [-1, 1]$ as a convex combination of -1 and $+1$, i.e., $g = \lambda \cdot (-1) + (1 - \lambda) \cdot (+1)$ for some $\lambda \in [0, 1]$. Since $\phi(g)$ is (logarithmically) convex,

$$\begin{aligned}\phi(g) &= \phi(\lambda \cdot (-1) + (1 - \lambda) \cdot (+1)) \\ &\leq \lambda \phi(-1) + (1 - \lambda) \phi(+1) \\ &= \lambda \left(1 + \frac{x}{t + \delta}\right) + (1 - \lambda) \left(1 - \frac{x}{t + \delta}\right) \\ &= 1 + g \frac{x}{t + \delta} .\end{aligned}\quad \square$$

F Proofs of Corollaries 5 and 6

We state some technical lemmas that will be used in the following proofs. We start with a lower bound on the Krichevsky-Trofimov (KT) potential. It is a generalization of the lower bound proved for integers in Willems et al. [29] to real numbers.

Lemma 14 (Lower Bound on KT Potential). *If $c \geq 1$ and a, b are non-negative reals such that $a + b = c$ then*

$$\ln \left(\frac{\Gamma(a + 1/2) \cdot \Gamma(b + 1/2)}{\pi \cdot \Gamma(c + 1)} \right) \geq -\ln(e\sqrt{\pi}) - \frac{1}{2} \ln(c) + \ln \left(\left(\frac{a}{c}\right)^a \left(\frac{b}{c}\right)^b \right) .$$

Proof. From [28][p. 263 Ex. 45], we have

$$\frac{\Gamma(a + 1/2) \Gamma(b + 1/2)}{\Gamma(a + b + 1)} \geq \sqrt{2\pi} \frac{(a + 1/2)^a (b + 1/2)^b}{(a + b + 1)^{a+b+1/2}} .$$

It remains to show that

$$\sqrt{2\pi} \frac{(a + 1/2)^a (b + 1/2)^b}{(a + b + 1)^{a+b+1/2}} > \frac{\sqrt{\pi}}{e} \frac{1}{\sqrt{a+b}} \left(\frac{a}{a+b}\right)^a \left(\frac{b}{a+b}\right)^b ,$$

which is equivalent to

$$\frac{(1 + \frac{1}{2a})^a (1 + \frac{1}{2b})^b}{(1 + \frac{1}{a+b})^{a+b+1/2}} > \frac{1}{e\sqrt{2}} .$$

From the inequality $1 \leq (1 + 1/x)^x < e$ valid for any $x \geq 0$, it follows that $1 \leq (1 + \frac{1}{2a})^a < \sqrt{e}$ and $1 \leq (1 + \frac{1}{2b})^b < \sqrt{e}$ and $1 \leq (1 + 1/(a+b))^{a+b} < e$. Hence,

$$\frac{(1 + \frac{1}{2a})^a (1 + \frac{1}{2b})^b}{(1 + \frac{1}{a+b})^{a+b+1/2}} > \frac{1}{e\sqrt{1 + \frac{1}{a+b}}} \geq \frac{1}{e\sqrt{2}} .\quad \square$$

Lemma 15. *Let $\delta \geq 0$. Then*

$$\frac{\Gamma(\delta + 1)}{2^\delta \Gamma(\frac{\delta+1}{2})^2} \geq \frac{\sqrt{\delta + 1}}{\pi} .$$

Proof. We will prove the equivalent statement that

$$\ln \frac{\Gamma(\delta + 1) \pi}{2^\delta \Gamma(\frac{\delta+1}{2})^2 \sqrt{\delta + 1}} \geq 0 .$$

The inequality holds with equality in $\delta = 0$, so it is enough to prove that the derivative of the left-hand side is positive for $\delta > 0$. The derivative of the left-hand side is equal to

$$\Psi(\delta + 1) - \frac{1}{2(\delta + 1)} - \ln(2) - \Psi\left(\frac{\delta + 1}{2}\right) ,$$

where $\Psi(x)$ is the digamma function.

We will use the upper [7] and lower bound [2] to the digamma function, which state that for any $x > 0$,

$$\begin{aligned}\Psi(x) &< \ln(x) - \frac{1}{2x} - \frac{1}{12x^2} + \frac{1}{120x^4} \\ \Psi(x+1) &> \ln\left(x + \frac{1}{2}\right).\end{aligned}$$

Using these bounds we have

$$\begin{aligned}&\Psi(\delta+1) - \frac{1}{2(\delta+1)} - \ln(2) - \Psi\left(\frac{\delta+1}{2}\right) \\&\geq \ln\left(\delta + \frac{1}{2}\right) - \frac{1}{2(\delta+1)} - \ln(2) - \ln\left(\frac{\delta+1}{2}\right) + \frac{1}{\delta+1} + \frac{1}{3(\delta+1)^2} - \frac{2}{15(\delta+1)^4} \\&= \ln\left(1 - \frac{1}{2(\delta+1)}\right) + \frac{1}{2(\delta+1)} + \frac{1}{3(\delta+1)^2} - \frac{2}{15(\delta+1)^4} \\&\geq -\frac{(4\ln(2)-2)}{4(\delta+1)^2} + \frac{1}{3(\delta+1)^2} - \frac{2}{15(\delta+1)^4} \\&= \frac{[15(1/2 - \ln(2))] + 5](\delta+1)^2 - 2}{15(\delta+1)^4} \\&\geq \frac{[15(1/2 - \ln(2))] + 5] - 2}{15(\delta+1)^4} \geq 0\end{aligned}$$

where in the second inequality we used the elementary inequality $\ln(1-x) \geq -x - (4\ln(2)-2)x^2$ valid for $x \in [0, 5]$. \square

Lemma 16 (Lower Bound on Shifted KT Potential). *Let $T \geq 1$, $\delta \geq 0$, and $x \in [-T, T]$. Then*

$$\frac{2^T \cdot \Gamma(\delta+1) \Gamma\left(\frac{T+\delta+1}{2} + \frac{x}{2}\right) \cdot \Gamma\left(\frac{T+\delta+1}{2} - \frac{x}{2}\right)}{\Gamma\left(\frac{\delta+1}{2}\right)^2 \Gamma(T+\delta+1)} \geq \exp\left(\frac{x^2}{2(T+\delta)} + \frac{1}{2} \ln\left(\frac{1+\delta}{T+\delta}\right) - \ln(e\sqrt{\pi})\right).$$

Proof. Using Lemma 14, we have

$$\begin{aligned}&\ln \frac{2^T \cdot \Gamma(\delta+1) \Gamma\left(\frac{T+\delta+1}{2} + \frac{x}{2}\right) \cdot \Gamma\left(\frac{T+\delta+1}{2} - \frac{x}{2}\right)}{\Gamma\left(\frac{\delta+1}{2}\right)^2 \Gamma(T+\delta+1)} \\&\geq \ln \frac{2^{T+\delta} \sqrt{\delta+1} \cdot \Gamma\left(\frac{T+\delta+1}{2} + \frac{x}{2}\right) \cdot \Gamma\left(\frac{T+\delta+1}{2} - \frac{x}{2}\right)}{\pi \Gamma(T+\delta+1)} \\&\geq -\ln(e\sqrt{\pi}) + \frac{1}{2} \ln\left(\frac{1+\delta}{T+\delta}\right) + \ln\left(\left(1 + \frac{x}{T+\delta}\right)^{\frac{T+\delta+x}{2}} \left(1 + \frac{x}{T+\delta}\right)^{\frac{T+\delta-x}{2}}\right) \\&= -\ln(e\sqrt{\pi}) + \frac{1}{2} \ln\left(\frac{1+\delta}{T+\delta}\right) + (T+\delta) D\left(\frac{1}{2} + \frac{x}{2(T+\delta)} \parallel \frac{1}{2}\right) \\&\geq -\ln(e\sqrt{\pi}) + \frac{1}{2} \ln\left(\frac{1+\delta}{T+\delta}\right) + \frac{x^2}{2(T+\delta)},\end{aligned}$$

where in the first inequality we used Lemma 15, in the second one Lemma 14, and in third one the known lower bound to the divergence $D\left(\frac{1}{2} + \frac{x}{2} \parallel \frac{1}{2}\right) \geq \frac{x^2}{2}$. Exponentiating and overapproximating, we get the stated bound. \square

F.1 Proof of Corollary 5

The Lambert function $W(x) : [0, \infty) \rightarrow [0, \infty)$ is defined by the equality

$$x = W(x) \exp(W(x)) \quad \text{for } x \geq 0. \quad (21)$$

The following lemma provides bounds on $W(x)$.

Lemma 17. *The Lambert function satisfies $0.6321 \log(x+1) \leq W(x) \leq \log(x+1)$ for $x \geq 0$.*

Proof. The inequalities are satisfied for $x = 0$, hence we in the following we assume $x > 0$. We first prove the lower bound. From (21) we have

$$W(x) = \log \left(\frac{x}{W(x)} \right). \quad (22)$$

From the first equality, using the elementary inequality $\ln(x) \leq \frac{a}{e} x^{\frac{1}{a}}$ for any $a > 0$, we get

$$W(x) \leq \frac{1}{a e} \left(\frac{x}{W(x)} \right)^a \quad \forall a > 0,$$

that is

$$W(x) \leq \left(\frac{1}{a e} \right)^{\frac{1}{1+a}} x^{\frac{a}{1+a}} \quad \forall a > 0. \quad (23)$$

Using (23) in (22), we have

$$W(x) \geq \log \left(\frac{x}{\left(\frac{1}{a e} \right)^{\frac{1}{1+a}} x^{\frac{a}{1+a}}} \right) = \frac{1}{1+a} \log(a e x) \quad \forall a > 0.$$

Consider now the function $g(x) = \frac{x}{x+1} - \frac{b}{\log(1+b)(b+1)} \log(x+1)$, $x \geq b$. This function has a maximum in $x^* = (1 + \frac{1}{b}) \log(1+b) - 1$, the derivative is positive in $[0, x^*]$ and negative in $[x^*, b]$. Hence the minimum is in $x = 0$ and in $x = b$, where it is equal to 0. Using the property just proved on g , setting $a = \frac{1}{x}$, we have

$$W(x) \geq \frac{x}{x+1} \geq \frac{b}{\log(1+b)(b+1)} \log(x+1) \quad \forall x \leq b.$$

For $x > b$, setting $a = \frac{x+1}{e x}$, we have

$$W(x) \geq \frac{e x}{(e+1)x+1} \log(x+1) \geq \frac{e b}{(e+1)b+1} \log(x+1) \quad (24)$$

Hence, we set b such that

$$\frac{e b}{(e+1)b+1} = \frac{b}{\log(1+b)(b+1)}$$

Numerically, $b = 1.71825\dots$, so

$$W(x) \geq 0.6321 \log(x+1).$$

For the upper bound, we use Theorem 2.3 in [13], that says that

$$W(x) \leq \log \frac{x+C}{1+\log(C)}, \quad \forall x > -\frac{1}{e}, \quad C > \frac{1}{e}.$$

Setting $C = 1$, we obtain the stated bound. □

Lemma 18. *Define $f(x) = \beta \exp \frac{x^2}{2\alpha}$, for $\alpha, \beta > 0$, $x \geq 0$. Then*

$$f^*(y) = y \sqrt{\alpha W \left(\frac{\alpha y^2}{\beta^2} \right)} - \beta \exp \left(\frac{W \left(\frac{\alpha y^2}{\beta^2} \right)}{2} \right).$$

Moreover

$$f^*(y) \leq y \sqrt{\alpha \log \left(\frac{\alpha y^2}{\beta^2} + 1 \right)} - \beta.$$

Proof. From the definition of Fenchel dual, we have

$$f^*(y) = \max_x x y - f(x) = \max_x x y - \beta \exp \frac{x^2}{2\alpha} \leq x^* y - \beta$$

where $x^* = \arg \max_x x y - f(x)$. We now use the fact that x^* satisfies $y = f'(x^*)$, to have

$$x^* = \sqrt{\alpha W \left(\frac{\alpha y^2}{\beta^2} \right)},$$

where $W(\cdot)$ is the Lambert function. Using Lemma 17, we obtain the stated bound. \square

Proof of Corollary 5. Notice that the KT potential can be written as

$$F_t(x) = \epsilon \cdot \frac{2^t \cdot \Gamma(1) \Gamma \left(\frac{t+1}{2} + \frac{x}{2} \right) \cdot \Gamma \left(\frac{t+1}{2} - \frac{x}{2} \right)}{\Gamma(\frac{1}{2})^2 \Gamma(t+1)}.$$

Using Lemma 16 with $\delta = 0$ we can lower bound $F_t(x)$ with

$$H_t(x) = \epsilon \cdot \exp \left(\frac{x^2}{2t} + \frac{1}{2} \ln \left(\frac{1}{t} \right) - \ln(e\sqrt{\pi}) \right).$$

Since $H_t(x) \leq F_t(x)$, we have $F_t^*(x) \leq H_t^*(x)$. Using Lemma 18, we have

$$\forall u \in \mathcal{H} \quad F_T^*(\|u\|) \leq H_T^*(\|u\|) \leq \sqrt{T \log \left(\frac{24T^2 \|u\|^2}{\epsilon^2} + 1 \right)} + \epsilon \left(1 - \frac{1}{e\sqrt{\pi T}} \right).$$

An application of Theorem 3 completes the proof. \square

F.2 Proof of Corollary 6

Proof. Let

$$F_t(x) = \frac{2^t \cdot \Gamma(\delta+1) \Gamma \left(\frac{t+\delta+1}{2} + \frac{x}{2} \right) \Gamma \left(\frac{t+\delta+1}{2} - \frac{x}{2} \right)}{\Gamma(\frac{\delta+1}{2})^2 \Gamma(t+\delta+1)},$$

$$H_t(x) = \exp \left(\frac{x^2}{2(t+\delta)} + \frac{1}{2} \ln \left(\frac{1+\delta}{t+\delta} \right) - \ln(e\sqrt{\pi}) \right).$$

Let $f_t(x) = \ln(F_t(x))$ and $h_t(x) = \ln(H_t(x))$. By Lemma 16, $H_t(x) \leq F_t(x)$ and therefore $f_t^{-1}(x) \leq h_t^{-1}(x)$ for all $x \geq 0$. Theorem 4 implies that

$$\forall u \in \Delta_t \quad \text{Regret}_t(u) \leq f_t^{-1}(\text{D}(u\|\pi)) \leq h_t^{-1}(\text{D}(u\|\pi)).$$

Setting $t = T$ and $\delta = T/2$, and overapproximating $h_t^{-1}(\text{D}(u\|\pi))$ we get the stated bound. \square