

## Mathematical Arguments

### Proof of Lemma 1

We prove here the equivalence of propagation dynamics  $DTIC(\mathcal{P})$ ,  $CTIC(\mathcal{F}, \infty)$  and  $RN(\mathcal{P})$ , provided that for any  $(i, j) \in \mathcal{E}$ ,  $\int_0^\infty f_{ij}(t)dt = \mathcal{H}_{ij}$ . More specifically, we prove the following lemma, that will be useful in the subsequent proofs. In the following, we will denote by  $X_i$  the state of node  $i$  at the end of the infection process, i.e  $X_i = 1$  if the infection has reached node  $i$ , and  $X_i = 0$  otherwise.

**Lemma 5.** *Let  $\mathcal{G} = (\mathcal{V}, \mathcal{E})$  be a given directed network and  $A \subset \mathcal{V}$  a set of influencers. For any  $i \notin A$ , we denote by  $\mathcal{Q}_i$  the collection of directed paths (without loops) in  $\mathcal{G}$  from  $A$  to node  $i$ . Then, under the infection processes  $DTIC(\mathcal{P})$ ,  $CTIC(\mathcal{F}, \infty)$  and  $RN(\mathcal{P})$ , we have  $\forall i \notin A$ ,*

$$X_i = 1 - \prod_{q \in \mathcal{Q}_i} (1 - \prod_{(j,l) \in q} E_{jl}), \quad (1)$$

where the  $(E_{jl})_{jl}$  are independant Bernoulli random variables  $E_{jl} \sim \mathcal{B}(p_{jl})$  for infection processes  $DTIC(\mathcal{P})$  and  $RN(\mathcal{P})$ , and  $E_{jl} \sim \mathcal{B}(1 - \exp(-\int_0^\infty f_{jl}(t)dt))$  for infection process  $CTIC(\mathcal{F}, \infty)$ .

*Proof.* First, note that, for  $RN(\mathcal{P})$ , the random variables  $1_{\{(j,l) \in \mathcal{E}'\}}$  and, for  $DTIC(\mathcal{P})$ , the indicator function of the events that node  $j$  succeeds in infecting node  $l$  if  $j$  is infected during the process and  $l$  is still healthy at that time are independant Bernoulli variables  $E_{jl} \sim \mathcal{B}(p_{jl})$  and can all be drawn at  $t = 0$ . Moreover, by definition of the infection processes, a node  $i \in \mathcal{V}$  is reached by the contagion if and only if there exists a path from  $A$  to  $i$ , such that each of its edges transmitted the contagion. We thus have for  $DTIC(\mathcal{P})$  and  $RN(\mathcal{P})$ :

$$X_i = 1 - \prod_{q \in \mathcal{Q}_i} (1 - \prod_{(j,l) \in q} E_{jl}). \quad (2)$$

For  $CTIC(\mathcal{F}, \infty)$ , the variables drawn at the beginning of the infection process are the (possibly infinite) times  $\tau_{jl}$  such that node  $j$  will infect node  $l$  at time  $t_j + \tau_{jl}$  if node  $j$  has been infected at time  $t_j$ , and node  $l$  has not been infected by another node before time  $t_j + \tau_{jl}$ . By definition, these independent random variables have the following survival function:

$$P(\tau_{jl} < t) = 1 - \exp\left(-\int_0^t f_{jl}(s)ds\right) \quad (3)$$

Therefore, we have by the same arguments than previously,

$$X_i = 1 - \prod_{q \in \mathcal{Q}_i} (1 - \prod_{(j,l) \in q} 1_{\{\tau_{jl} < \infty\}}), \quad (4)$$

which proves the result for  $CTIC(\mathcal{F}, \infty)$ , defining  $E_{jl} = 1_{\{\tau_{jl} < \infty\}}$  □

Lemma 1 is then a direct corollary of Lemma 5 in the case where, for any  $(j, l) \in \mathcal{E}$ ,  $\int_0^\infty f_{jl}(t)dt = \mathcal{H}_{jl}$ .

### Proofs of Proposition 1 and Corollary 1

We develop here the full proofs for Proposition 1 and Corollary 1 that apply to any set of initially infected nodes. We will first need to prove two useful results: Lemma 6, that proves for  $j \in \mathcal{V}$  a positive correlation between the events 'node  $j$  did not infect node  $i$  during the epidemic' and Lemma 8, that bound the probability that a given node gets infected during the infection process.

**Lemma 6.**  $\forall i \notin A$ ,  $\{1 - X_j E_{ji}\}_{j \in \mathcal{V}}$  are positively correlated.

*Proof.* We will make use of the FKG inequality ([1]):

**Lemma 7.** (FKG inequality) Let  $L$  be a finite distributive lattice, and  $\mu$  a nonnegative function on  $L$ , such that, for any  $(x, y) \in L^2$ ,

$$\mu(x \vee y)\mu(x \wedge y) \leq \mu(x)\mu(y) \quad (5)$$

Then, for any non-decreasing function  $f$  and  $g$  on  $L$

$$\left( \sum_{x \in L} f(x)g(x) \right) \left( \sum_{x \in L} \mu(x) \right) \geq \left( \sum_{x \in L} f(x)\mu(x) \right) \left( \sum_{x \in L} g(x)\mu(x) \right) \quad (6)$$

For a given set of influencers  $A$ , the  $X_j$  are deterministic functions of the independent random variables  $(E_{ij})_{ij}$ . Thus, let  $f_{ij}(\{E_{i'j'}\}_{(i',j')}) = 1 - X_j E_{ji}$ . In order to apply the FKG inequality, we first need to show that each  $f_{ij} : \{0, 1\}^{n^2} \rightarrow \{0, 1\}$  is decreasing with respect to the natural partial order on  $\{0, 1\}^{n^2}$  (i.e.  $X \leq Y$  if  $X_i \leq Y_i$  for all  $i$ ). Let  $u \in \{0, 1\}^{n^2}$  be a given transmission state of the edges of the network. In order to prove the decreasing behavior of  $f_{ij}$ , it is sufficient to show that  $f_{ij}(u)$  is decreasing with respect to every  $u_{(i,j)}$ .

But from Lemma 5, it is obvious that  $X_i(u) = 1 - \prod_{q \in \mathcal{Q}_i} (1 - \prod_{(j,l) \in q} u_{(j,l)})$  is increasing with respect to every  $u_{(i,j)}$ . This implies that  $f_{ij}(u) = 1 - X_j(u)u_{(j,i)}$  is decreasing with respect to every  $u_{(i,j)}$  and that  $f_{ij} : \{0, 1\}^{n^2} \rightarrow \{0, 1\}$  is decreasing with respect to the natural partial order on  $\{0, 1\}^{n^2}$ .

Finally, since we consider a product measure (due to the independence of the  $E_{ij}$ ) on a product space, we can apply the FKG inequality to  $\{1 - X_j E_{ji}\}_{j \in \{1, \dots, N\}}$ , and these random variables are positively correlated.  $\square$

The next lemma ensures that the variables  $X_i$  satisfy an implicit inequation that will be the starting point of the proof of Proposition 1.

**Lemma 8.** For any  $A$  such that  $|A| = n_0 < n$  and for any  $i \notin A$ , the probability  $\mathbb{E}[X_i]$  that node  $i$  will be reached by the contagion originating from  $A$  verifies:

$$\mathbb{E}[X_i] \leq 1 - \exp\left(-\sum_j \mathcal{H}_{ji} \mathbb{E}[X_j]\right) \quad (7)$$

*Proof.* We first note that a node is infected if and only if one of its neighbors is infected, and the respective ingoing edge transmitted the contagion. Thus

$$X_i = 0 \Leftrightarrow \forall j \in \{1, \dots, n\}, X_j = 0 \text{ or } E_{ji} = 0, \quad (8)$$

which implies the following alternative expression for  $X_i$ :

$$1 - X_i = \prod_j (1 - X_j E_{ji}). \quad (9)$$

Moreover, the positive correlation of  $\{1 - X_j E_{ji}\}_{j \in \{1, \dots, N\}}$  implies that

$$\mathbb{E}\left[\prod_j (1 - X_j E_{ji})\right] \geq \prod_j \mathbb{E}[1 - X_j E_{ji}] \quad (10)$$

which leads to

$$\begin{aligned} \mathbb{E}[X_i] &\leq 1 - \prod_j \mathbb{E}[1 - X_j E_{ji}] \\ &= 1 - \prod_j (1 - \mathbb{E}[X_j] \mathbb{E}[E_{ji}]) \\ &= 1 - \exp\left(\sum_j \ln(1 - \mathbb{E}[X_j] \mathbb{E}[E_{ji}])\right) \\ &\leq 1 - \exp\left(\sum_j \ln(1 - \mathbb{E}[E_{ji}] \mathbb{E}[X_j])\right) \\ &= 1 - \exp\left(-\sum_j \mathcal{H}_{ji} \mathbb{E}[X_j]\right) \end{aligned} \quad (11)$$

since we have on the one hand, for any  $x \in [0, 1]$  and  $a < 1$ ,  $\ln(1 - ax) \geq \ln(1 - a)x$ , and on the other hand  $\mathbb{E}[E_{ji}] = 1 - \exp(-\mathcal{H}_{ji})$  by definition of  $\mathcal{H}$ .  $\square$

Using Lemma 8, we are now ready to start the proof of Proposition 1.

*Proof of Proposition 1.* In order to simplify notations, we define  $Z_i = (\mathbb{E}[X_i])_i$  that we collect in the vector  $Z = (Z_i)_{i \in [1..n]}$ . Using lemma 8 and convexity of exponential function, we have for any  $u \in R^n$  such that  $\forall i \in A, u_i = 0$  and  $\forall i \notin A, u_i \geq 0$ ,

$$u^\top Z \leq |u|_1 \left( 1 - \sum_{i=1}^{n-1} \frac{u_i}{|u|_1} \exp(-(\mathcal{H}^\top Z)_i) \right) \leq |u|_1 \left( 1 - \exp\left(-\frac{Z^\top \mathcal{H} u}{|u|_1}\right) \right) \quad (12)$$

where  $|u|_1 = \sum_i |u_i|$  is the  $L_1$ -norm of  $u$ .

Now taking  $u = (1_{i \notin A} Z_i)_i$  and noting that  $\forall i \in \{1, \dots, n\}, \forall j \in A, \mathcal{H}(A)_{ij} = 0$ , we have

$$\frac{Z^\top Z - n_0}{|Z|_1 - n_0} \leq 1 - \exp\left(-\frac{Z^\top \mathcal{H}(A) Z}{|Z|_1 - n_0}\right) \leq 1 - \exp\left(-\frac{\rho_c(A)(Z^\top Z - n_0)}{|Z|_1 - n_0} - \frac{\rho_c(A)n_0}{|Z|_1 - n_0}\right) \quad (13)$$

where  $\rho_c(A) = \rho\left(\frac{\mathcal{H}(A) + \mathcal{H}(A)^\top}{2}\right)$ . Defining  $y = \frac{Z^\top Z - n_0}{|Z|_1 - n_0}$  and  $z = |Z|_1 - n_0 = \sigma(A) - n_0$ , the aforementioned inequation rewrites

$$y \leq 1 - \exp\left(-\rho_c(A)y - \frac{\rho_c(A)n_0}{z}\right) \quad (14)$$

But by Cauchy-Schwarz inequality applied to  $u$ ,  $(n - n_0)(Z^\top Z - n_0) \geq (|Z|_1 - n_0)^2$ , which means that  $z \leq y(n - n_0)$ . We now consider the equation

$$x - 1 + \exp\left(-\rho_c(A)x - \frac{\rho_c(A)n_0}{x(n - n_0)}\right) = 0 \quad (15)$$

Because the function  $f : x \rightarrow x - 1 + \exp\left(-\rho_c(A)x + \frac{\rho_c(A)n_0}{x(n - n_0)}\right)$  is continuous, verifies  $f(1) > 0$  and  $\lim_{x \rightarrow 0^+} f(x) = -1$ , equation 15 admits a solution  $\gamma_1$  in  $]0, 1[$ .

We then prove by contradiction that  $z \leq \gamma_1(n - n_0)$ . Let us assume  $z > \gamma_1(n - n_0)$ . Then  $y \leq 1 - \exp\left(-\rho_c(A)y - \frac{\rho_c(A)n_0}{\gamma_1(n - n_0)}\right)$ . But the function  $h : x \rightarrow x - 1 + \exp\left(-\rho_c(A)x + \frac{\rho_c(A)n_0}{\gamma_1(n - n_0)}\right)$  is convex and verifies  $h(0) < 0$  and  $h(\gamma_1) = 0$ . Therefore, for any  $y > \gamma_1$ ,  $0 = f(\gamma_1) \leq \frac{\gamma_1}{y} f(y) + (1 - \frac{\gamma_1}{y}) f(0)$ , and therefore  $f(y) > 0$ . Thus,  $y \leq \gamma_1$ . But  $z \leq y(n - n_0) \leq \gamma_1(n - n_0)$  which yields the contradiction.  $\square$

*Proof of Corollary 1.* We distinguish between the cases  $\rho_c(A) > 1$  and  $\rho_c(A) \leq 1$ .

**Case  $\rho_c(A) < 1$ .** Using Eq. 15 and the fact that  $\exp(z) \geq 1 + z$ , we get  $\gamma_1 \leq \rho_c(A)\gamma_1 + \frac{\rho_c(A)n_0}{\gamma_1(n - n_0)}$  which rewrites  $\gamma_1 \leq \sqrt{\frac{\rho_c(A)n_0}{(1 - \rho_c(A))(n - n_0)}}$  in the case  $\rho_c < 1$ . Therefore,

$$\sigma(A) \leq n_0 + \sqrt{\frac{\rho_c(A)}{1 - \rho_c(A)}} \sqrt{n_0(n - n_0)} \quad (16)$$

**Case  $\rho_c(A) \geq 1$ .** Using Eq. 15, we get  $\gamma_1 - 1 + \exp\left(-\frac{\rho_c(A)n_0}{\gamma_1(n - n_0)}\right) \geq 0$ , which implies  $\gamma_1 \ln\left(\frac{1}{1 - \gamma_1}\right) \geq \frac{\rho_c(A)n_0}{n - n_0} \geq \frac{n_0}{n - n_0}$ . By concavity of the logarithm, we therefore have  $\gamma_1^2 \geq \frac{n_0(1 - \gamma_1)}{n - n_0}$  which means that  $\gamma_1(n - n_0) \geq \frac{n_0(\sqrt{4n/n_0 - 3} - 1)}{2}$ . By plugging this lower bound in Eq. 15, we obtain

$$\sigma(A) \leq n_0 + \left(1 - \exp\left(-\rho_c(A) - \frac{2\rho_c(A)}{\sqrt{4n/n_0 - 3} - 1}\right)\right)(n - n_0) \quad (17)$$

$\square$

## Proofs of Proposition 2 and Corollary 2

In this subsection, we develop the proofs for Proposition 2 and Corollary 2 in the case when the set of initially infected node is drawn from a uniform distribution over  $\mathcal{P}_{n_0}(\mathcal{V})$ .

We start with an important lemma that will play the same role in the proof of Proposition 2 than Lemma 8 in the proof of Proposition 1.

**Lemma 9.** Define  $\rho_c = \rho(\frac{\mathcal{H} + \mathcal{H}^\top}{2})$ . Assume  $A$  is drawn from an uniform distribution over  $\mathcal{P}_{n_0}(\mathcal{V})$ . Then, for any  $i \in \mathcal{V}$ , the probability  $\mathbb{E}[X_i]$  that node  $i$  will be reached by the contagion satisfies the following implicit inequation:

$$\mathbb{E}[X_i] \leq 1 - \frac{n - n_0}{n} \exp\left(-\frac{n}{n - n_0} \sum_j \mathcal{H}_{ji} \mathbb{E}[X_j]\right) \quad (18)$$

*Proof.*

$$\begin{aligned} \mathbb{E}[X_i] &= \mathbb{E}[1_{\{i \in A\}}] + \mathbb{E}[1_{\{i \notin A\}}] \mathbb{E}[\mathbb{E}[X_i | A] | i \notin A] \\ &\leq \frac{n_0}{n} + \frac{n - n_0}{n} \left(1 - \mathbb{E}[\exp\left(-\sum_j \mathcal{H}_{ji} \mathbb{E}[X_j | A]\right) | i \notin A]\right) \\ &\leq \frac{n_0}{n} + \frac{n - n_0}{n} \left(1 - \exp\left(-\mathbb{E}[\sum_j \mathcal{H}_{ji} \mathbb{E}[X_j | A] | i \notin A]\right)\right) \\ &= 1 - \frac{n - n_0}{n} \exp\left(-\sum_j \mathcal{H}_{ji} \mathbb{E}[X_j | i \notin A]\right) \\ &\leq 1 - \frac{n - n_0}{n} \exp\left(-\frac{n}{n - n_0} \sum_j \mathcal{H}_{ji} \mathbb{E}[X_j]\right) \end{aligned} \quad (19)$$

where the first inequality is Lemma 8 and the second one is Jensen inequality for conditional expectations.  $\square$

*Proof of Proposition 2.* We define  $Z_i = (\mathbb{E}[X_i])_i$  that we collect in the vector  $Z = (Z_i)_{i \in [1 \dots n]}$ . Then, using Lemma 9, and convexity of exponential function, we have:

$$\frac{Z^\top Z}{|Z|_1} \leq \left(1 - \frac{n - n_0}{n} \sum_{i=1}^n \frac{Z_i}{|Z|_1} \exp\left(-\frac{n}{n - n_0} (\mathcal{H}^\top Z)_i\right)\right) \leq \left(1 - \frac{n - n_0}{n} \exp\left(-\frac{n}{n - n_0} \frac{Z^\top \mathcal{H} Z}{|Z|_1}\right)\right) \quad (20)$$

Now, defining  $y = \frac{Z^\top Z}{|Z|_1}$ , we have by Cauchy-Schwarz inequality  $|Z|_1 \leq ny$  where  $y \leq 1 - \frac{n - n_0}{n} \exp\left(-\frac{n}{n - n_0} \rho_c y\right)$ . Because function  $f : x \rightarrow x - 1 + \frac{n - n_0}{n} \exp\left(-\frac{n}{n - n_0} \rho_c x\right)$  is continuous and convex over  $]0, 1[$ ,  $f(0) < 0$  and  $f(1) > 0$ , there exists a solution  $\gamma \in ]0, 1[$  of the equation  $f(x) = 0$ . By the same arguments than in proof of Proposition 1, we have that, for any  $z \in [0, 1]$ ,  $f(z) \leq 0 \Rightarrow z \leq \gamma$ . This proves the uniqueness of  $\gamma$  as well as the fact that  $y \leq \gamma$ . Now, defining  $\gamma_2 = \frac{n_0}{n} + \frac{n - n_0}{n} \gamma$ , we have on the one hand

$$\sigma_{\text{uniform}} \leq n_0 + \gamma_2(n - n_0) \quad (21)$$

and on the other hand

$$\gamma_2 - 1 + \exp\left(-\rho_c \gamma_2 - \frac{\rho_c n_0}{n - n_0}\right) = 0 \quad (22)$$

which proves the proposition.  $\square$

*Proof of Corollary 2.* In the case  $\rho_c < 1$ , using Proposition 2 and the fact that  $\exp(z) \geq 1 + z$ , we get  $\gamma_2 \leq \rho_c \gamma_2 + \frac{\rho_c n_0}{n - n_0}$  which rewrites  $\gamma_2 \leq \frac{\rho_c n_0}{(1 - \rho_c)(n - n_0)}$  in the case  $\rho_c < 1$ . Therefore,

$$\sigma_{\text{uniform}} \leq n_0 \left(1 + \frac{\rho_c}{1 - \rho_c}\right) = \frac{n_0}{1 - \rho_c} \quad (23)$$

The second claim is straightforward from Proposition 2, using the fact that  $\gamma_2 \leq 1$ .  $\square$

## Proofs of Lemma 2, Lemma 3, Proposition 3 and Corollary 4

*Proof of Lemma 2.* Because matrices  $\frac{\mathcal{H}(A)+\mathcal{H}(A)^\top}{2}$  and  $\frac{\beta}{\delta}\mathcal{A}$  are symmetric and verify  $0 \leq \frac{\mathcal{H}(A)+\mathcal{H}(A)^\top}{2} \leq \frac{\beta}{\delta}\mathcal{A} = \mathcal{H}$  where  $\leq$  stands for the coefficient-wise inequality, we have  $\rho(\frac{\mathcal{H}(A)+\mathcal{H}(A)^\top}{2}) \leq \frac{\beta}{\delta}\rho(\mathcal{A})$  as a direct consequence of the Perron-Frobenius theorem (see e.g [2]). We now introduce the function

$$f : \rho \rightarrow n_0 + \sqrt{\frac{\rho}{1-\rho}}\sqrt{n_0(n-n_0)} - \frac{\sqrt{nn_0}}{1-\rho}$$

We have  $f(0) < 0$  and  $f'(\rho) = \sqrt{n_0(n-n_0)}\frac{\rho}{(1-\rho)^{3/2}} - \sqrt{n_0n}\frac{1}{(1-\rho)^2} < 0$ . Therefore,  $f(\rho) < 0$  for any  $\rho \in [0, 1]$ , which proves the Lemma.  $\square$

*Proof of Lemma 3.* First, note that, for bond percolation, the random variables  $1_{\{\{j,l\} \in \mathcal{E}'\}}$  are independent Bernoulli variables  $F_{\{j,l\}} \sim \mathcal{B}(p_{jl})$ . We therefore have, similarly than in the proof of Lemma 5

$$X_i = 1 - \prod_{q \in \mathcal{Q}_i} (1 - \prod_{\{j,l\} \in q} F_{\{j,l\}}). \quad (24)$$

where  $X_i$  is 1 if node  $i$  belongs to the connected component containing the influencer node  $v$ , and is 0 otherwise. We then show that, because  $\mathcal{P}$  is symmetric, for any infection process  $DTIC(\mathcal{P})$  on the directed graph  $\mathcal{G}_d$ , we can also define independent variables  $F'_{\{j,l\}} \sim \mathcal{B}(p_{jl})$  such that the final infection state  $X'_i$  of node  $i$  is:

$$X'_i = 1 - \prod_{q \in \mathcal{Q}_i} (1 - \prod_{\{j,l\} \in q} F'_{\{j,l\}}), \quad (25)$$

which proves that  $X_i$  and  $X'_i$  have the same probability distribution.

Indeed, the event that node  $j$  makes an attempt to infect node  $l$  will never occur in the same epidemic than the event that node  $l$  makes an attempt to infect node  $j$ . Therefore, drawing two variables  $E_{jl}$  and  $E_{lj}$  at the beginning of each epidemic and letting the dynamic decide which of the two results will be used, or drawing only one variable  $F'_{\{j,l\}} \sim \mathcal{B}(p_{jl})$  and using it for each epidemic to decide whether the infection can spread along the edge  $\{j, l\}$  or not is strictly equivalent, given that  $E_{jl}$  and  $E_{lj}$  are independent and have the same distribution. From equations 24 and 25, we see that, for any  $i \in \mathcal{V}$ , the probability that a node  $i$  is infected is the same for the two processes.  $\square$

*Proof of Proposition 3.* By proposition 2 applied to the case  $n_0 = 1$  with the notation  $\gamma_3 = \frac{(n-1)\gamma_2+1}{n}$ , we get  $\sigma_{\text{uniform}} \leq n\gamma_3$ . We then use the fact that, when the influencer node is uniformly randomly drawn on  $\mathcal{V}$ , it belongs to the largest connected component and therefore creates an infection of  $C_1(\mathcal{G}')$  nodes with probability  $\frac{C_1(\mathcal{G}')}{n}$ . Therefore,  $\mathbb{E}[\frac{C_1(\mathcal{G}')}{n}C_1(\mathcal{G}')] \leq \sigma_{\text{uniform}} \leq n\gamma_3$ . But  $\mathbb{E}[C_1(\mathcal{G}')^2] \geq \mathbb{E}[C_1(\mathcal{G}')]^2$  which yields  $\mathbb{E}[C_1(\mathcal{G}')] \leq n\sqrt{\gamma_3}$ . Moreover, denoting as  $C_A(\mathcal{G}')$  the size of the connected component containing the influencer node, we have  $\sigma_{\text{uniform}} = \mathbb{E}[C_A(\mathcal{G}')] = \sum_i i\mathbb{P}(C_A(\mathcal{G}') = i) \geq n\mathbb{P}(C_A(\mathcal{G}') = n) = n\mathbb{P}(\mathcal{G}' \text{ is connected})$ , and therefore  $\mathbb{P}(\mathcal{G}' \text{ is connected}) \leq \gamma_3$ .  $\square$

*Proof of Corollary 4.* According to Eq. 15 of the article, there exists  $m \in \mathbb{N}$  and  $\eta < 1$  such that for any  $n \geq m$ ,  $\rho(\mathcal{H}^n) \leq \eta$ . Therefore, Corollary 3 implies  $\mathbb{E}[C_1(\mathcal{G}'_n)] \leq \sqrt{\frac{n}{1-\eta}}$ . But for any  $\delta > 0$ ,  $\mathbb{P}(C_1(\mathcal{G}'_n) > \delta n^{1/2+\epsilon}) \leq \frac{\mathbb{E}[C_1(\mathcal{G}'_n)]}{\delta n^{1/2+\epsilon}} = o(1)$  which proves the corollary.  $\square$

## Additional references

- [1] Cees M Fortuin, Pieter W Kasteleyn, and Jean Ginibre. Correlation inequalities on some partially ordered sets. *Communications in Mathematical Physics*, 22(2):89–103, 1971.
- [2] Carl D Meyer. *Matrix analysis and applied linear algebra*, volume 2. Siam, 2000.