The Kernel Beta Process

Lu Ren, Yingjian Wang, David Dunson, and Lawrence Carin

Presented by: David Carlson
Duke University

david.carlson@duke.edu
The kernel beta process

**Theorem:** Assume parameters \( \{x_i^*, \psi_i^*, \pi_i, \omega_i\} \) are drawn from measure \( \nu_X = H(dx^*)Q(d\psi^*)\nu(d\pi, d\omega) \), and that the following measure is constituted for any covariate \( x \in \mathcal{X} \):

\[
B_x = \sum_{i=1}^{\infty} \pi_i K(x, x_i^*; \psi_i^*) \delta_{\omega_i}
\]

For any finite set of covariates \( S = \{x_1, \ldots, x_{|S|}\} \), define the random vector \( K = (K(x_1, x^*; \psi^*), \ldots, K(x_{|S|}, x^*; \psi^*))^T \). For \( \forall A \subset \mathcal{F} \), the characteristic function for measures at covariates in \( S \) satisfies

\[
\mathbb{E}[e^{j<u,B(A)>}] = \exp\left\{ \int_{\mathcal{X} \times \Psi \times [0,1] \times A} (e^{j<u,K\pi>} - 1) \nu_X(dx^*, d\psi^*, d\pi, d\omega) \right\}
\]

with \( \nu_X \) the Lévy measure of the kernel beta process.
Properties of KBP

If $B$ is drawn from KBP, $x, x' \in \mathcal{X}$, for $\forall \mathcal{A} \in \mathcal{F}$:

- **Expectation:** $\mathbb{E}[B_x(\mathcal{A})] = B_0(\mathcal{A})\mathbb{E}(K_x)$
  with $\mathbb{E}(K_x) = \int_{\mathcal{X} \times \psi} K(x, x^*; \psi^*)H(dx^*)Q(d\psi^*)$.

- **Covariance:** $\text{Cov}(B_x(\mathcal{A}), B_{x'}(\mathcal{A})) = \mathbb{E}(K_x K_{x'}) \int_{\mathcal{A}} \frac{B_0(d\omega)(1-B_0(d\omega))}{c(\omega)+1} - \text{Cov}(K_x, K_{x'}) \int_{\mathcal{A}} B_0^2(d\omega)$
  (If $K(x, x^*; \psi^*) = 1$ for all $x \in \mathcal{X}$, $\mathbb{E}(K_x) = \mathbb{E}(K_x K_{x'}) = 1$, and $\text{Cov}(K_x, K_{x'}) = 0$, and the above results reduce to beta process.)

- **Conditional covariance:** With the kernel vectors $K_x, K_{x'}$ fixed, the conditional covariance is given as:
  $\text{Corr}(B_x(\mathcal{A}), B_{x'}(\mathcal{A})) = \frac{\langle K_x, K_{x'} \rangle}{\|K_x\|_2 \cdot \|K_{x'}\|_2}$
Experiment - music analysis and image denoising

Figure: Music temporal correlation: (a) KBP-FA, (b) BP-FA, (c) dHDP-HMM.

Figure: Image denoising result