A Appendix

A.1 Appendix A : Algorithm

The structure of the neural network (VNN) mentioned in Section 3.2 for simultaneously learning the neural Lyapunov function and the nonlinear controller is detailed in Fig. 4 [5].

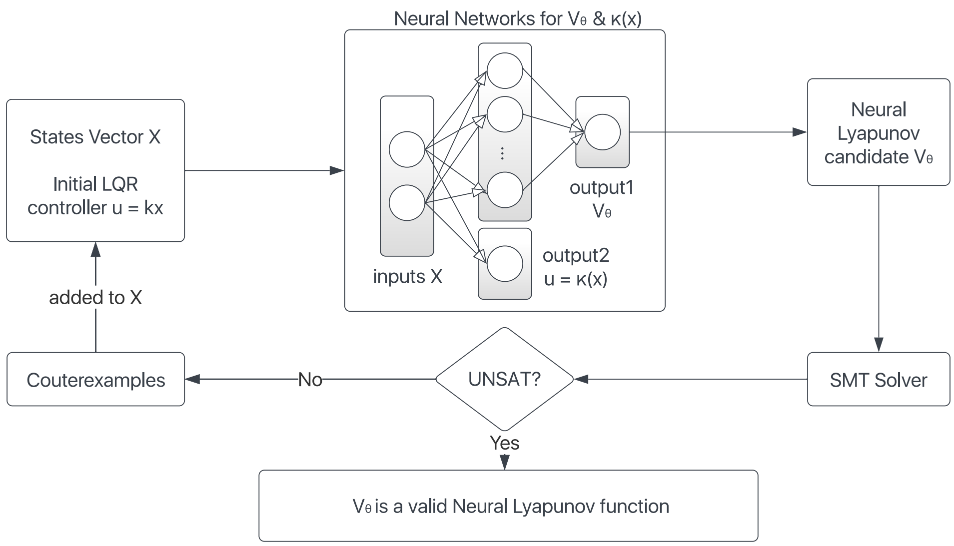

Figure 4: Algorithmic structure of learning a neural Lyapunov function and the corresponding nonlinear controller with a one-hidden layer neural network and an SMT solver.

A.2 Appendix B: Proofs

We first begin by defining notions of stability that are necessary to the proofs.

Definition 4 (Set stability). *A closed set $A \subseteq \mathbb{R}^n$ is said to be uniformly asymptotically stable (UAS) for the closed-loop system (2), if the following two conditions are met:*
(1) (Uniform stability) For every $\epsilon > 0$, there exists a $\delta_\epsilon > 0$ such that $\|x(0)\|_A < \delta_\epsilon$ implies that $x(t)$ is defined for $t \geq 0$ and $\|x\|_A < \epsilon$ for any solution x of (2) for all $t \geq 0$; and
(2) (Uniform attractivity) There exists some $\rho > 0$ such that, for every $\epsilon > 0$, there exists some $T > 0$ such that $x(t)$ is defined for $t \geq 0$ and $\|x(t)\|_A < \epsilon$ for any solution $x(t)$ of (2) whenever $\|x(0)\|_A < \rho$ and $t \geq T$.

Definition 5 (Reachable Set). *Let $R^t(x_0)$ denote the point $x(t)$ reached by the solution of (2) at time t starting at x_0. For $T \geq 0$ define the finite time horizon reachable set as*

$$R^{0 \leq t \leq T}(x_0) = \cup_{0 \leq t \leq T} R^t(x_0).$$

Similarly, for a set $W \subset \mathcal{D}$, define

$$R^{0 \leq t \leq T}(W) := \cup_{x_0 \in W} R^{0 \leq t \leq T}(x_0).$$

Similarly, if solutions are defined for all $t \geq 0$, then the reachable set is defined as

$$R(W) := \cup_{x_0 \in W} \cup_{t \geq 0} R^t(x_0).$$

Before we prove Theorem 3 we introduce some lemmas. First we state an extension of the universal approximation theorem that states it is possible to simultaneously pointwisely approximate a function and its partial derivatives by a neural network. The proof of this result can be found in [31].

Theorem 5. *Let $K \subset \mathbb{R}^m$ be a compact set and suppose $f : K \to \mathbb{R}^n \in C^1(\mathbb{R}^n)$. Then, for every $\epsilon > 0$ there exists a neural network $\phi : K \to \mathbb{R}$ of the form $\phi(x) = C(\sigma \circ (\omega x + b))$ for $\sigma \in C^1(\mathbb{R})$ and not a polynomial, $\omega \in \mathbb{R}^{k \times m}, b \in \mathbb{R}^k$ and $C \in \mathbb{R}^{k \times n}$ for some $k \in \mathbb{N}$ such that*

$$\|f - \phi\|_\infty := \sup_{x \in K} |f(x) - \phi(x)| < \epsilon \tag{16}$$

and for all $i = 1, \dots, n$, the following simultaneously holds

$$\left\| \frac{\partial f}{\partial x_i} - \frac{\partial \phi}{\partial x_i} \right\|_\infty < \epsilon. \tag{17}$$

To make the connection between the topology of the dynamical system and the compact set guaranteed by Lemma 2 we consider the reachable set. The following result states that the finite time horizon reachable set is compact and can be found in [14].

Lemma 3. *Suppose that $K \subset \mathbb{R}^n$ is a compact set, then the set $R^{0\le t\le T}(K)$ is compact for any $T \geq 0$.*

We recall theorem 3:

Theorem 3. *Suppose that the origin is UAS for system (2) and $\mathcal{I}$ is a forward invariant set contained in the ROA of the origin. Fix any $\gamma_1, \gamma_2 > 0$. There exists a forward invariant and compact set $K \subset \mathcal{I}$ satisfying the under approximation $\mu(\mathcal{I} \setminus K) < \gamma_1$. On K there exists a neural network V_ϕ that satisfies the Lyapunov conditions on $K \setminus \mathcal{A}$, where $\mathcal{A}$ is a closed neighborhood of the origin. The neural Lyapunov function V_ϕ can certify that a closed invariant set $\mathcal{B}$ containing $\mathcal{A}$ and satisfying $\mu(\mathcal{B} \setminus \mathcal{A}) < \gamma_2$ is UAS. Furthermore, the set K is contained in the ROA of $\mathcal{B}$.*

Proof. By the converse Lyapunov theorem [22], there exists a function V satisfying the Lyapunov conditions on $\mathcal{I}$. Lemma 2 states that there exists a compact set W such that $\mu(\mathcal{I} \setminus W) < \gamma/2$. Since unions preserve compactness we can suppose without loss of generality that W contains the closed ball of radius r for $r > 0$ sufficiently small, denoted as B_r, which lies in the interior of $\mathcal{I}$. By virtue of $\mathcal{I}$ being a forward invariant set contained in the region of attraction, for autonomous systems, asymptotic stability is equivalent to uniform attractive stability, so in particular there exists a time $T > 0$ such that all solutions starting in W will enter $\mathcal{A}_\rho := \{x \in B_r : V(x) \leq \rho\}$ for any $\rho \geq 0$ without leaving $\mathcal{I}$. The continuity of measure and the Lyapunov condition $V(0) = 0$ implies there exists a constant $\rho_0 > 0$ such that $\mu(\mathcal{A}_{\rho_0}) < \gamma/2$ since $\cap_{\rho>0}\mathcal{A}_\rho = \{0\}$ and this is a set of measure zero. For ease of notation, we simply refer to $\mathcal{A}_{\rho_0}$ as $\mathcal{A}$. Let $T \geq 0$ be the time such that all solutions starting in W enter $\mathcal{A}$. By Lemma 3, the reachable set $R^{0\le t\le T}(W)$ is compact and satisfies $\mu(\mathcal{I} \setminus R^{0\le t\le T}(W)) < \gamma/2$. Denote

$$K := R^{0\le t\le T}(W) \cup \mathcal{A}.$$

We see that K which contains the origin is compact and forward invariant. Similarly, by the continuity of of V and the continuity of measure, it follows that $\cap_{\rho>\rho_0}\mathcal{A}_\rho = \mathcal{A}$ and this implies there exists a level set $\mathcal{A}_{\rho_1}$ such that $\mu(\mathcal{A}_{\rho_1} \setminus \mathcal{A}) < \gamma_2$.

By the continuity of V there exists a constant $\delta > 0$ such that $V > \delta$ and $\nabla_f V(x) < -\delta$ on $K \setminus \mathcal{A}$. We can suppose that $\delta < \rho_1 - \rho_0$ in the inequality above. By Theorem 5, there exists a neural network approximation of V denoted V_ϕ satisfying the pointwise bounds $\|V_\phi - V\|_\infty < \delta/2$ and $\|\nabla_f V - \nabla_f V_\phi\|_\infty < \delta/2$ on $K \setminus \mathcal{A}$. This proves that V_ϕ satisfies the Lyapunov conditions on $K \setminus \mathcal{A}$. To summarize we have established that

$$V_\phi(x) > \delta/2 \;\forall x \in K \setminus \mathcal{A},$$

and

$$\nabla_f V_\phi(x) < -\delta/2; \;\forall x \in K \setminus \mathcal{A}.$$

By this pointwise bound $\|V_\phi - V\|_\infty < \delta/2$ it follows that

$$\mathcal{A} \subset \mathcal{B} := \{x \in B_r : V_\phi \leq \rho_0 + \delta_2\} \subset \mathcal{A}_{\rho_1}.$$

This proves that $\mu(\mathcal{B} \setminus \mathcal{A}) < \gamma_2$. Now we show that V_ϕ verifies that the set $\mathcal{B}$ is uniformly asymptotically stable.

(**Uniform Stability**) Given $\epsilon > 0$ as per the definition of uniform stability. Denote

$$B_\epsilon(\mathcal{B}) := \cup_{x\in\mathcal{B}} B_\epsilon(x).$$

Without loss of generality by taking $\epsilon < r$ we can assume that $B_\epsilon(\mathcal{B}) \subset K$. Choose $c > 0$ such that

$$0 < c < \min_{\|x\|_{\mathcal{B}}=\epsilon} V_\phi(x)$$

holds. Then by a contradiction argument the set

$$\Omega^c := \{x \in B_\epsilon(\mathcal{B}) : V_\phi(x) \le c\}$$

is contained in the interior of $B_\epsilon(\mathcal{B})$. By the continuity of V_ϕ and compactness of $\mathcal{B}$, V_ϕ is uniformly continuous on $\mathcal{B}$. Thus, there exists $0 < \delta_\epsilon < \epsilon$ such that

$$\|x_0 - x\| < \delta_\epsilon \implies |V(x) - V(x_0)| < c - \rho_0/2 \text{ for all } x_0 \in \mathcal{B}.$$

In particular, this prove that $B_{\delta_\epsilon}(\mathcal{B}) \subset \Omega^c \subset B_\epsilon(\mathcal{B})$. A standard argument shows that the set Ω^c is forward invariant and hence for all $x_0 \in B_{\delta_\epsilon}(\mathcal{B})$ this implies that $x(t) \in \Omega^c$ for all $t \ge 0$ and proves uniform stability.

(**Uniform Attractivity**). Given that K is a compact positive invariant set that contains $\mathcal{A}$ we claim that there exists some time $T > 0$ for which the solution enters $\mathcal{A}$. Indeed, suppose otherwise this means that $x(t) \in K \setminus \mathcal{A}$ for all $t \ge 0$. By Lemma 5 we get that $\nabla_f V_\phi(x(t)) < -\delta/2$ for all $t \ge 0$ on $K \setminus \mathcal{A}$. It follows that

$$V(x(t)) = V(x(0)) + \int_0^t \nabla_f V_\phi(x(\tau))d\tau \le V(x(0)) - \delta t/2.$$

As the right hand side will eventually become negative, this is a contradiction to the $V_\phi > 0$ on $K \setminus \mathcal{A}$. To conclude, note that the set $\mathcal{A}$ is forward invariant which implies that $\|x(t)\|_{\mathcal{A}} = 0$ for all $t \ge T$. In particular, as $\mathcal{A}$ is contained in $\mathcal{B}$ this proves the uniform attractivity of $\mathcal{B}$. □

Suppose further that $\mathcal{I}$ is the ROA of the system (2). It was mentioned as a closing remark that if the Lyapunov function V is *radially unbounded*, this means that $V(x) \to \infty$ when $x \to \delta\mathcal{I}$ (the boundary of $\mathcal{I}$), then the level sets of V approach the ROA. If the origin is UAS for system (2), then by the converse Lyapunov theorem, it follows that $V(x)$ is radially unbounded. We show that the neural Lyapunov function inherits a similar property where the level sets approach K.

Theorem 6. *In addition to the assumptions of Theorem 3, suppose that $\mathcal{I}$ is the region of attraction which is bounded. Set $W^c := \{x \in K : V_\phi(x) \le c\}$. Then, for any sequence $k_n \to \infty$, $\cup_{n\in\mathbb{N}} W^{k_n} = K$.*

Proof. Without loss of generality suppose that k_n is an increasing sequence and $\epsilon < k_1$. Again, define $V^c = \{x \in \mathcal{D} : V(x) \le c\}$. Note that if $x \in K$ satisfies $V_\phi(x) \le c$, then $V(x) \le c + \epsilon$. Therefore, $W^{k_n} \subset V^{k_n+\epsilon} \cap K$. A similar argument shows that $V^{k_n-\epsilon} \cap K \subset W^{k_n} \subset V^{k_n+\epsilon} \cap K$. Since $K \subset \mathcal{D}$ this implies that $\cup_{n\in\mathbb{N}} V^{k_n-\epsilon} \cap K = K \cap \cup_{n\in\mathbb{N}} V^{k_n-\epsilon} = K$. Therefore, $\cup_{n\in\mathbb{N}} W^{k_n} = K$. □

Now that we have established guarantees for the existence of neural Lyapunov functions. The proof of Theorem 4 is analogous, so we defer the proof of Theorem 2 to the end of this section.

Theorem 4. *(Stability Guarantees for the Unknown System) Let ϕ be the approximated dynamics of right-hand side of the closed-loop system (2) trained by the first neural network. There exists a neural Lyapunov function V which is learned using ϕ and verified by an SMT solver that satisfies the Lyapunov conditions with respect to the actual dynamics f. Furthermore, if the system satisfies Assumption 3 and V satisfies Assumption 4, then the origin is UAS for the closed-loop system* (2).

Proof. Fix $\beta > 0$ and let $M > 0$ be chosen such that $\|\frac{\partial V}{\partial x}\| < M$. As V is learned using the learned dynamics ϕ, V satisfies $V \ge 0$ and $-\nabla_\phi V(x) < -\beta$ on $\mathcal{D} \setminus B_\varepsilon$. To certify that V satisfies the Lyapunov conditions on $\mathcal{D} \setminus B_\varepsilon$, it suffices to verify that $\nabla_f V < 0$. By the universal approximation theorem, there exists a neural network ϕ approximating f such that $\|f(x, \kappa(x) - \phi(x, \kappa(x)\|_\infty < \frac{\beta}{M}$ on the $\mathcal{D} \setminus \{\|x\| < \varepsilon\}$. As in (10), the following holds

$$\nabla_f V(x) < \nabla_\phi V(x) + \beta < -\beta + \beta = 0, \quad \forall x \in \mathcal{D}\backslash\{0\}. \tag{18}$$

Therefore, V satisfies the neural Lyapunov conditions on $\mathcal{D} \setminus B_\varepsilon$.

(**Uniform Stability**). The uniform stability property follows from the Assumption 3 as the quadratic Lyapunov function guarantees uniform stability at the origin.

(**Uniform Attractivity**). We show that under these assumptions, the neural Lyapunov function is able to verify uniform attractivity. As uniform attractivity is equivalent to attractivity for autonomous systems, it suffices to verify that any level set of V, denoted V^c, which contains B_ε is a ROA for this dynamical system. By a similar argument to the Uniform Stability part of Theorem 3, any trajectory starting in V^c must eventually enter B_ε. Since B_ε is contained in the ROA of the closed loop system provided by the quadratic Lyapunov function, it follows that $\|x(t)\| \to 0$ as $t \to \infty$. $\square$

As a remark, it can be seen from this proof that if Assumption 3 is satisfied then any level set containing the ball must be a ROA of the system. This is because trajectories flow from one level set to a sublevel set. Eventually trajectories must enter the level set contained in B_ε.

To prove Theorem 2, we introduce concepts that will allow us to approximate Lipschitz functions smoothly. Let us first restate this theorem.

Theorem 2. *(Approximation of Lipschitz constants). Suppose that $K \subset \mathbb{R}^n$ is a compact set.*
(a) If $f : K \to \mathbb{R}^m$ is L-Lipschitz in the uniform norm, i.e.

$$\|f(x) - f(y)\|_\infty \leq L\|x - y\|_\infty, \tag{13}$$

then for every $\epsilon > 0$ there exists a neural network of the form $\phi(x) = C(\sigma \circ (\omega x + b))$ for $\sigma \in C^1(\mathbb{R})$ and not a polynomial, $\omega \in \mathbb{R}^{k\times m}, b \in \mathbb{R}^k$ and $C \in \mathbb{R}^{k\times n}$ for some $k \in \mathbb{N}$ such that $\sup_{x\in K} |f(x) - \phi(x)| < \epsilon$ and ϕ has a Lipschitz constant of $L + \epsilon$ in the same norms as (13).
(b) If $f : K \to \mathbb{R}^m$ is L-Lipschitz in the two norm, i.e.

$$\|f(x) - f(y)\|_\infty \leq L\|x - y\|_2, \tag{14}$$

then for every $\epsilon > 0$ there exists a neural network ϕ of the same form such that $\sup_{x\in K} |f(x) - \phi(x)| < \epsilon$ and ϕ has a Lipschitz constant of $L + \epsilon\left(\frac{\sqrt{n+n/\epsilon}}{2} + L\right)$ in the same norms as (14).

The idea will be to first approximate f by a smooth function and then approximate this smooth function by a neural network. In this end, define $\eta \in C^\infty(\mathbb{R}^n)$ by

$$\eta(x) := \begin{cases} C\exp\left(\frac{1}{|x|^2-1}\right) & \text{if } |x| < 1 \\ 0 & \text{if } |x| \geq 1 \end{cases}$$

where the constant C is some normalizing constant, that is $C > 0$ is selected so that $\int_{\mathbb{R}^n} \eta dx = 1$. Some standard properties of $\eta(x)$ is that $\eta \geq 0$, $\eta \in C^\infty(\mathbb{R}^n)$ and spt$(\eta) \subset B_1(0)$ which is the unit ball in $\mathbb{R}^n$. For each $\epsilon > 0$, set

$$\eta_\epsilon(x) := \frac{1}{\epsilon^n}\eta\left(\frac{x}{\epsilon}\right).$$

We call η the standard mollifier. The functions η_ϵ are C^∞ and satisfy

$$\int_{\mathbb{R}^n} \eta_\epsilon dx = 1, \operatorname{spt}(\eta_\epsilon) \subset B(0, \epsilon).$$

Then, by taking the convolution of f with the mollifier η_ϵ, denote $f_\epsilon = f \star \eta_\epsilon$ which can be further simplified to

$$\begin{aligned} f_\epsilon(x) &= \int_{\mathbb{R}^n} f(y)\eta_\epsilon(x-y)dy \\ &= \int_{\mathbb{R}^n} f(x-y)\eta_\epsilon(y)dy \\ &= \frac{1}{\epsilon^n}\int_{B(0,\epsilon)} f(x-y)\eta(\frac{y}{\epsilon}) \\ &= \int_{B(0,1)} f(x-\epsilon y)\eta(y)dy. \end{aligned}$$

It is well known that $f_\epsilon \in C^\infty$. Additionally, by the Lipschitz continuity of f, uniform convergence holds on $\mathbb{R}^n$:

$$\begin{aligned} |f(x) - f_\epsilon(x)| &\leq \int_{B(0,1)} |(f(x-\epsilon y) - f(x))\eta_\epsilon(y)|dy \leq L\int_{B(0,1)} \|\epsilon y\|\eta_\epsilon(y)dy \\ &\leq L\epsilon\int_{B(0,1)} \eta_\epsilon(y)dy \leq L\epsilon. \end{aligned}$$

However, to define f_ϵ, we need this function to be defined on $\mathbb{R}^n$. Therefore, we need a specific case of the following lemma called the Kirszbraun theorem. The proof of this result can be found in [34].

Lemma 4. *Suppose that $U \subset \mathbb{R}^n$. For any Lipschitz map $f : U \to \mathbb{R}^m$ there exists a Lipschitz-continuous map*

$$F : \mathbb{R}^n \to \mathbb{R}^m$$

that extends f and has the same Lipschitz constant as f.

We are now ready to prove Theorem 2.

Proof. As the proof of (a) is analogous to the proof of (b) and simpler, we elect to prove (b) only. Since f is L-Lipschitz in the uniform norm, we have that the component functions f_i are L-Lipschitz. Since neural networks can be stacked in parallel, it suffices to prove this result for the case that $m = 1$. Moreover, since K is compact, by taking the approximation on some hypercube containing K and then restricting onto K we can also assume without loss of generality that K is convex. The uniform norm is still well-defined as the restriction of a continuous function is continuous. By the Kirszbraun theorem there exists an extension F to $\mathbb{R}^n$ with the same Lipschitz constant L so we can suppose without loss of generality that f is defined on $\mathbb{R}^n$ and has Lipschitz constant L. Denote $f_\epsilon := f \star \eta_\epsilon$. Since we have that $f_\epsilon(x) = \int_{\mathbb{R}^n} f(x-y)\eta_\epsilon(y)dy$ from the above calculation, we see that

$$\begin{aligned}
|f_\epsilon(x_1) - f_\epsilon(x_2)| &\leq \left| \int_{\mathbb{R}^n} |\left(f(x_1 - y) - f(x_2 - y)\right)\eta_\epsilon(y)dy \right| \\
&\leq \int_{\mathbb{R}^n} |\left(f(x_1 - y) - f(x_2 - y)\right)\eta_\epsilon(y)|\, dy \\
&\leq L\|x_1 - x_2\| \int_{\mathbb{R}^n} \eta_\epsilon(y)dy \\
&= L\|x_1 - x_2\|.
\end{aligned}$$

This shows that f_ϵ is L-Lipschitz. Since $f_\epsilon \to f$ uniformly we can choose $\epsilon > 0$ sufficiently small so that $\|f - f_\epsilon\| < \epsilon/2$. Therefore, by Theorem 5, there exists a neural network ϕ such that $\sup_{x\in K} |f(x) - \phi(x)| < \epsilon/2$ and $\sup_{x\in K} |\frac{\partial f}{\partial x_i}(x) - \frac{\partial \phi}{\partial x_i}(x)| < \epsilon/2$ for all $i = 1, \ldots, n$. Since f is L-Lipschitz it follows that $\|\nabla f\|_2 \leq L$. By the uniform bound on the partial derivatives and the following inequality, $(a+b)^2 \leq (1+\epsilon)a^2 + (1+1/\epsilon)b^2$, this gives

$$\begin{aligned}
\|\nabla \phi\|_2 &= \sqrt{\left(\frac{\partial \phi}{\partial x_1}\right)^2 + \ldots + \left(\frac{\partial \phi}{\partial x_n}\right)^2} \\
&\leq \sqrt{\left(\frac{\partial f}{\partial x_1} \pm \frac{\epsilon}{2}\right)^2 + \ldots + \left(\frac{\partial f}{\partial x_n} \pm \frac{\epsilon}{2}\right)^2} \\
&\leq \sqrt{(1+\epsilon)\left(\left(\frac{\partial f}{\partial x_1}\right)^2 + \ldots + \left(\frac{\partial f}{\partial x_n}\right)^2\right)} + \frac{\sqrt{n}\epsilon}{2}\sqrt{1+1/\epsilon} \\
&\leq L + \epsilon\left(\frac{\sqrt{n+n/\epsilon}}{2} + L\right)
\end{aligned}$$

Therefore, by the mean value theorem and the convexity of K, it follows that ϕ is $L + \epsilon\left(\frac{\sqrt{n+n/\epsilon}}{2} + L\right)$-Lipschitz. □

A.3 Appendix C: Experiments

As stated in Section 5, the learned dynamics is of the format $\phi = W_2 \tanh(W_1 X + B_1) + B_2$ where $X = [x, u]$. In VNN, the valid neural Lyapunov function is of the following form $V_\theta = \tanh(W_2 \tanh(W_1 x + B_1) + B_2)$. It is worth mentioning that we have access to the nonlinear dynamics for the cases in Section 5 and consequently, we are able to test the neural Lyapunov functions on the actual dynamics. In all three experiments, we observe that the neural Lyapunov functions are indeed valid Lyapunov functions for the actual dynamics, which shows the effectiveness of the proposed algorithm. The code is open sourced at `https://github.com/RuikunZhou/Unknown_Neural_Lyapunov`.

A.3.1 Van der Pol oscillator

The dynamics of the Van der Pol oscillator are:

$$\begin{aligned} \dot{x}_1 &= -x_2 \\ \dot{x}_2 &= x_1 + \left(x_1^2 - 1\right) x_2. \end{aligned} \tag{19}$$

Correspondingly, the phase plot and the limit cycle of this nonlinear system are shown in Figure 5 [22].

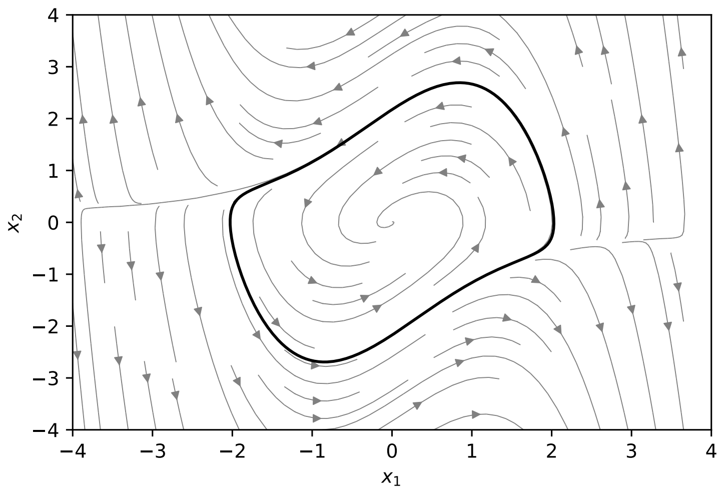

Figure 5: Phase space plot and the limit cycle (bold black line) of Val Der Pol oscillator without controller, where the area within the bold black curve forms the actual ROA.

Clearly, we can write $x = [x_1 \quad x_2]^T$, and we use 100 hidden neurons to learn the dynamics in FNN. With the learned dynamics, the weights and biases matrices of obtained neural Lyapunov function in VNN are:

$$W_1 = \begin{bmatrix} -1.82994 & -0.70762 & 3.35979 & -6.42827 & -1.14237 & 0.39034 \\ 1.30866 & 0.57501 & 0.27398 & 0.32546 & -1.16843 & -0.03503 \end{bmatrix}^T,$$

$$W_2 = \begin{bmatrix} -1.32270 & -0.73489 & 1.87897 & 0.89612 & 1.65451 & 1.17499 \end{bmatrix},$$

$B_1 = \begin{bmatrix} -2.30191 & 0.38658 & 0.47604 & 0.83902 & 0.87791 & 1.18262 \end{bmatrix}$ and $B_2 = [0.62172]$.

A.3.2 Unicycle path following

In this case, we have two state variables, the angle error θ_e and the distance error d_e, and the dynamics of this system can be written as:

$$\begin{aligned} \dot{s} &= \frac{v\cos(\theta_e)}{1 - d_e\kappa(s)}, \\ \dot{d}_e &= v\sin(\theta_e), \\ \dot{\theta}_e &= \omega - \frac{v\kappa(s)\cos(\theta_e)}{1 - d_e\kappa(s)}. \end{aligned} \tag{20}$$

Here we assume the target path is a unit circle $\kappa(s) = 1$ and take ω as the input u with $x = [d_e \quad \theta_e]^T$, consequently the dynamical system is of the format $\dot{x} = f(x, u)$. Similarly, after obtaining the learned dynamics $\phi(x, u)$ with 200 hidden neurons, the weights and biases matrices of V_θ for this experiment are recorded below.

$$W_1 = \begin{bmatrix} -2.13787 & -0.02771 & 2.83659 & -3.33855 & 0.61321 & 4.98050 \\ 1.07949 & -0.25036 & 0.69794 & -2.23639 & -1.62861 & 0.11680 \end{bmatrix}^T,$$

$$W_2 = \begin{bmatrix} -1.23695 & 1.08396 & -2.13833 & -0.76877 & -0.84737 & 1.47562 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} -1.90726 & 0.87544 & 0.18892 & 0.73855 & 1.09844 & -0.79774 \end{bmatrix} \text{ and } B_2 = [0.59095],$$

and the nonlinear controller function is $u = 5\tanh(-5.95539d_e - 4.034264\theta_e + 0.19740)$

A.3.3 Inverted pendulum

The system dynamics of inverted pendulum can be described as

$$\ddot{\theta} = \frac{mg\ell\sin(\theta) + u - 0.1\dot{\theta}}{m\ell^2}. \tag{21}$$

In this example, the only nonlinear function we need to learn for FNN is (21). Therefore, the input $[x \quad u]^T$ is 3-dimensional and the output is 1-dimensional. Using constants $g = 9.81, m = 0.15$ and $\ell = 0.5$, by the same process as the previous experiment, the weights and bias matrices of the neural Lyapunov function for this experiment are listed below, and the corresponding parameters can be found in Table 3.

$$W_1 = \begin{bmatrix} 0.03331 & 0.03467 & 2.12564 & -0.39925 & 0.12885 & 0.95375 \\ -0.03113 & -0.01892 & 0.02354 & -0.10678 & -0.32245 & 0.01298 \end{bmatrix}^T,$$

$$W_2 = \begin{bmatrix} -0.33862 & 0.65177 & -0.52607 & 0.23062 & -0.04802 & 0.66825 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} -0.48061 & 0.88048 & 0.86448 & -0.87253 & 0.81866 & -0.26619 \end{bmatrix} \text{ and } B_2 = [0.22032],$$

and the nonlinear controller function is $u = 20\tanh\left(-23.28632\theta - 5.27055\dot{\theta}\right)$

Table 3: Parameters in inverted pendulum case

K_f	K_ϕ	δ	α	$\|\frac{\partial V}{\partial x}\|$	β	ε
<33.214	633.806	5e-5	5e-3	0.51	0.02	0.4

A.4 Appendix D: Limitations and future work

This work is concerned with formulating an algorithm to learn the unknown dynamics with stability guarantees using Lyapunov functions with nonlinear controllers. Since the experiments are run in an ideal setting, we acknowledge the following limitations, which will be further addressed or taken into consideration in future work.

1. The data set used to train the unknown dynamics and learn the Lyapunov function is generated from the trajectories of solutions to ordinary differential equations, but in practice noise typically pollutes state measurements, and sometimes it is difficult to have direct access to the states measurements and obtain a significant number of data points. In this paper, we first try to prove the proposed approach works well with ideal measurements both practically and theoretically. In the future, we will study the question of how to learn the unknown dynamics and a robust Lyapunov function with different values of β in (11) to guarantee stability with noisy measurements. Further, the implementation of this algorithm on real dynamical systems will be investigated as well afterwards.
2. Although all the nonlinear systems in Section 5 are widely studied and standard nonlinear problems, the systems considered here are relatively low-dimensional. This decision is made in view of the lack of expressibility of shallow neural networks with limited width and the scalability of the SMT solvers. As in all tasks, we have to consider the trade off between computational time and performance so the proposed number of neurons in Section 5 achieves this balance. For now, the main bottleneck we experienced is approximating

high-dimensional dynamics with the one-hidden-layer shallow neural networks. Regarding the scalability of the SMT solver, we have seen dReal works well with learned complex high-dimensional system in [5] and [36]. In our future work, we will try to learn high-dimensional unknown dynamics, for instance quadrature dynamics in six dimensions with deeper neural networks. We believe that dReal should be able to handle such a case.

3. Computing a valid Lyapunov function is challenging and has been a well-studied topic for dynamical systems. We admit that our method is not complete, that is, it does not guarantee that we can obtain a valid Lyapunov function after running the algorithm. The original paper of this framework [5] similarly suffers from this same issue and to the best of our knowledge, we are unaware of any papers in the literature that have addressed this issue. Finding a complete algorithm for computing Lyapunov functions is an interesting topic for future research.