A   Notation and Auxiliary Results

This section gives an overview of the notations used in the paper and summarizes some basic results in partial differential equation and functional analysis.

A.1   Basic notations

For $n \in \mathbb{N}$, we denote $\{1, 2, ...n\}$ by $[n]$ for simplicity in the paper.

For two Banach spaces X, Y, $\mathscr{L}(X, Y)$ refers to the set of continuous linear operator mapping from X to Y.

For a mapping $f : X \to Y$, and $u, v \in X$, $df(u, v)$ denotes Gateaux differential of f at u in the direction of v, and $f'(u)$ denotes Fréchet derivative of f at u.

For $r > 0$, and $x_0 \in X$, where X is a Banach space equipped with norm $\| \cdot \|_X$, $B_r(x_0)$ refers to $\{x \in X : \|x - x_0\|_X < r\}$.

For a function $f : X \to \mathbb{R}$, where X is a measurable space. We denote by $\mathrm{supp} f$ the support set of f, i.e. the closure of $\{x \in X : f(x) \neq 0\}$.

For a measurable set $\Omega \subset \mathbb{R}^n$, define the parabolic region $Q_t = Q_t(\Omega)$ as $\Omega \times [0, t]$.

The parabolic boundary $\partial_p Q_t$ is then defined as $\Omega \times \{0\} \cup \partial\Omega \times [0, t]$.

For $R > 0$, the parabolic neighborhood of 0 (denoted by $Q(R)$) is defined as $\{(x, t) \in \mathbb{R}^n \times \mathbb{R}_{\geq 0} : |x| < R, \ t < R^2\}$.

Its parabolic boundary $\partial_p(Q(R))$ is defined as $\{(x, t) : |x| < R, \ t = 0 \text{ or } |x| = R, \ t \in [0, R^2]\}$.

A.2   Multi-index notations

For $n \in \mathbb{N}$, we call an $n-$tuple of non-negative integers $\alpha \in \mathbb{N}^n$ a multi-index. We use the notation $|\alpha| = \Sigma_{i=1}^n \alpha_i$, $\alpha! = \Pi_{i=1}^n \alpha_i!$. For $x = (x_1, x_2, ...x_n) \in \mathbb{R}^n$, we denote by $x^\alpha = \Pi_{i=1}^n x_i^{\alpha_i}$ the corresponding multinomial. Given two multi-indices $\alpha, \beta \in \mathbb{N}^d$, we say $\alpha \leq \beta$ if and only if $\alpha_i \leq \beta_i, \ \forall i \in [n]$.

For an open set $\Omega \subset \mathbb{R}^n$, $T \in \mathbb{R}^+$ and a function $f(x) : \Omega \to \mathbb{R}$ or $f(x, t) : \Omega \times [0, T] \to \mathbb{R}$, we denote by

$$D^\alpha f = \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} ... \partial x_n^{\alpha_n}} \tag{14}$$

the classical or weak derivative of f.

For $k \in \mathbb{R}^n$, we denote by $D^k f$ (or $\nabla^k f$) the vector whose components are $D^\alpha f$ for all $|\alpha| = k$, and we abbreviate $D^1 f$ as Df.

A.3   Norm notations

Let $n \in \mathbb{N}^*$, $m \in \mathbb{N}$, $T \in \mathbb{R}^+$, and $\Omega \subset \mathbb{R}^n$, $Q \subset \mathbb{R}^n \times [0, T]$ be open sets. We denote by $L^p(\Omega)$ and $L^p(Q)$ the usual Lebesgue space.

The Sobolev space $W^{m,p}(\Omega)$ is defined as

$$\{f(x) \in L^p(\Omega) : D^\alpha f \in L^p(\Omega), \ \forall \alpha \in \mathbb{N}^n \ with \ |\alpha| \leq m\}. \tag{15}$$

And we define $W^{m,p}(Q)$ as

$$\{f(x) \in L^p(Q) : D^\alpha f \in L^p(Q), \ \forall \alpha \in \mathbb{N}^n \ with \ |\alpha| \leq m\}. \tag{16}$$

We define

$$\|f\|_{W^{m,p}(\Omega)} := \left(\sum_{|\alpha| \leq m} \|D^\alpha f\|_{L^p(\Omega)}^p \right)^{\frac{1}{p}} \tag{17}$$

for $1 \leq p < \infty$, and

$$\|f\|_{W^{m,\infty}(\Omega)} := \max_{|\alpha| \leq m} \|D^\alpha f\|_{L^\infty(\Omega)} \tag{18}$$

for $p = \infty$.

We define $\|f\|_{W^{m,p}(Q)}$ for $p \in [1, \infty]$ similarly.

We will use simplified notations $\|f\|_p$ and $\|f\|_{m,p}$ for L^p-norm and $W^{m,p}-$norm when the domain is whole space ($\mathbb{R}^n$, $\mathbb{R}^n \times [0,T]$ or $\mathbb{R}^n \times \mathbb{R}_{\geq 0}$) or when it is clear to the reader.

$W_0^{m,p}(\Omega)$ is defined as the completion of $C_0^\infty(\Omega)$ under $\|\cdot\|_{m,p}$ norm. In the similar way we define $W_0^{m,p}(Q_t)$.

A.4 Auxiliary results

In this section, we list out several fundamental yet important results in the field of PDE and functional analysis.

To begin with, we would like to recall three useful inequalities in real analysis.

Lemma A.1 (Young's convolution inequality). *In $\mathbb{R}^n$, we define the convolution of two functions f and g as $(f * g)(x) := \int_{\mathbb{R}^n} f(y)g(x-y)dy$. Suppose $f \in L^p(\mathbb{R}^n)$, $g \in L^q(\mathbb{R}^n)$, and $\frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1$ with $p,q,r \in [1,\infty]$, then $\|f * g\|_r \leq \|f\|_p \|g\|_q$.*

Lemma A.2 (Sobolev embedding theorem). *Let Ω be an open set in $\mathbb{R}^n$, $p \in [1,\infty]$, and $m \leq k$ be a non-negative integer.*

(i) *If $\frac{1}{p} - \frac{k}{n} > 0$, and set $q = \frac{np}{n-pk}$, then $W^{m,p} \subset W^{m-k,q}$ and the embedding is continuous, i.e. there exists a constant $c > 0$ such that $\|u\|_{m-k,q} \leq c\|u\|_{m,p}$, $\forall u \in W^{m,p}$.*

(ii) *If $\frac{1}{p} - \frac{k}{n} \leq 0$, then for any $q \in [1,\infty)$, $W^{m,p} \subset W^{m-k,q}$ and the embedding is continuous.*

Lemma A.3 (A special case of Gagliardo-Nirenberg inequality). *Ω is an open set in $\mathbb{R}^n$, Let $q \in [1,\infty]$ and $j,k \in \mathbb{N}$, and suppose $j \neq 0$ and*

$$\begin{cases} 1 < r < \infty \\ k - j - \frac{n}{r} \notin \mathbb{N} \\ \frac{j}{k} \leq \theta < 1. \end{cases}$$

If we set

$$\frac{1}{p} = \frac{j}{n} + \theta\left(\frac{1}{r} - \frac{k}{n}\right) + \frac{1-\theta}{q}, \tag{19}$$

then there exists a constant C independent of u such that

$$\|\nabla^j u\|_p \leq C\|\nabla^k u\|_r^\theta \|u\|_q^{1-\theta}, \quad \forall u \in L^q(\Omega) \cap W^{k,r}(\Omega). \tag{20}$$

Next, we list out several basic results in second order parabolic equations, which will be of great use in Appendix C.

In the following lemmas, we denote by $\mathcal{L}_0$ the operator $\frac{\partial}{\partial t} - \Delta$.

Lemma A.4. *Suppose Ω is bounded, and $Q_T = \Omega \times [0,T]$.*

Let $u \in W^{2,2}(Q_T)$ be the solution to

$$\begin{cases} \mathcal{L}_0 u = f(x,t), & (x,t) \in Q_T \\ u = 0, & (x,t) \in \partial_p Q_T, \end{cases} \tag{21}$$

then for $2 \leq p < \infty$, if $f \in L^p(Q_T)$, we have $u \in W^{2,p}(Q_T)$ and there exists a constant C such that $\|u\|_{2,p} \leq C\|f\|_p$.

Proof. Since Ω is bounded, we can choose an $R_0 > 1$ such that $Q_T \subset Q(R_0)$. Set $\hat{u}(x,t) = u(x,t)\mathbf{1}_{Q_T}$ and $\hat{f}(x,t) = f(x,t)\mathbf{1}_{Q_T}$, which are extensions of u and f in $\mathbb{R}^n \times \mathbb{R}_{\geq 0}$, respectively. The boundary condition in Eq. (21) implies that $\hat{u} \in W^{2,p}(\mathbb{R}^n \times \mathbb{R}_{\geq 0})$.

Furthermore, for any $R > R_0$, it holds that,

$$\begin{cases} \mathcal{L}_0 \hat{u} = \hat{f}(x,t), & (x,t) \in Q(R) \\ \hat{u} = 0, & (x,t) \in \partial_p Q(R). \end{cases}$$

From proposition 7.18 in [19], and in light of the fact that both $\hat{u}$ and $\hat{f}$ are supported on Q_T, we obtain that $\|D^2\hat{u}\|_p \leq C(\|\hat{f}\|_p + \frac{1}{R}\|D\hat{u}\|_p + \frac{1}{R^2}\|\hat{u}\|_p)$ holds for $\forall R > R_0$. Additionally, Poincaré inequality guarantees that there exists a constant $C' > 0$ depending only on Ω and n, such that $\|\hat{u}\|_{2,p} \leq C'\|D^2\hat{u}\|_p$.

Therefore we have

$$\frac{1}{C'}\|\hat{u}\|_{2,p} \leq C\left(\|\hat{f}\|_p + \frac{1}{R}\|\hat{u}\|_{2,p} + \frac{1}{R^2}\|\hat{u}\|_{2,p}\right) \leq C\left(\|\hat{f}\|_p + \frac{2}{R}\|\hat{u}\|_{2,p}\right) \tag{22}$$

$$\left(\frac{1}{C'} - \frac{2C}{R}\right)\|\hat{u}\|_{2,p} \leq C\|\hat{f}\|_p. \tag{23}$$

Let $R \to \infty$ and we derive $\|\hat{u}\|_{2,p} \leq CC'\|\hat{f}\|_p$. Since $\|\hat{u}\|_{W^{2,p}(\mathbb{R}^n \times \mathbb{R}_{\geq 0})} = \|u\|_{W^{2,p}(Q_T)}$ and $\|\hat{f}\|_{L^p(\mathbb{R}^n \times \mathbb{R}_{\geq 0})} = \|f\|_{L^p(Q_T)}$, we completes the proof. □

Lemma A.5. *Let u be the solution to*

$$\begin{cases} \mathcal{L}_0 u(x,t) = 0, & (x,t) \in \mathbb{R}^n \times [0,T] \\ u(x,0) = g(x), & x \in \mathbb{R}^n. \end{cases}$$

For any compact set $Q \subset \mathbb{R}^n \times [0,T]$ and $p \geq 1$, let $r < \frac{(n+2)p}{n+p}$, then

(i) *there exists a constant C such that $\|u\|_{W^{1,r}(Q)} \leq C\|g\|_{L^p(\mathbb{R}^n)}$,*

(ii) *there exists a constant C' such that $\|u\|_{W^{2,r}(Q)} \leq C'\|g\|_{W^{1,p}(\mathbb{R}^n)}$.*

Proof. u have the explicit form

$$u(x,t) = \int_{\mathbb{R}^n} \frac{a}{t^{\frac{n}{2}}} e^{-b\frac{|x-y|^2}{t}} g(y)dy, \tag{24}$$

where $a = (4\pi)^{-\frac{n}{2}}$ and $b = \frac{1}{4}$.

Note that u is the convolution between heat kernel $K(x,t) := \frac{a}{t^{\frac{n}{2}}} e^{-b\frac{|x|^2}{t}}$ and $g(x)$, with Lemma A.1, we derive

$$\|u(\cdot,t)\|_{r_0} \leq \|g\|_{p_0}\|K(\cdot,t)\|_{q_0} \tag{25}$$

for $p_0, q_0, r_0 \in [1,\infty]$ satisfying $\frac{1}{p_0} + \frac{1}{q_0} = \frac{1}{r_0} + 1$.

Due to the uniform convergence of (24), we have $\frac{\partial u}{\partial x_i} = (\frac{\partial}{\partial x_i}K(x,t)) * g(x)$ for all $i \in [n]$. Thus we have

$$\|\frac{\partial u(\cdot,t)}{\partial x_i}\|_{r'} \leq \|g\|_{p'}\|\frac{\partial}{\partial x_i}K(x,t)\|_{q'}. \tag{26}$$

for $p', q', r' \in [1,\infty]$ satisfying $\frac{1}{p'} + \frac{1}{q'} = \frac{1}{r'} + 1$.

It is enough to decide appropriate tuples of $(p_0, q_0, r_0), (p', q', r')$.

For $q \geq 1$,

$$\|K(\cdot,t)\|_q^q = \frac{a^q}{t^{\frac{nq}{2}}} \int_{\mathbb{R}^n} e^{-bq\frac{\|x\|^2}{t}} dx = \frac{a^q}{t^{\frac{nq-n}{2}}} \int_{\mathbb{R}^n} e^{-bq\|y\|^2} dy \quad (y = \frac{1}{\sqrt{t}}x). \tag{27}$$

Thus $\|K(\cdot, t)\|_q = \frac{C}{t^{\frac{nq-n}{2q}}}$, where C is a constant.

As a result, for any $p, q, r \in [1, \infty]$ satisfying $\frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1$,

$$\|u\|_{L^r(\mathbb{R}^n \times [0,T])}^r = \int_0^T \|u(\cdot, t)\|_r^r dt \leq \|g\|_p^r \int_0^T \|K(\cdot, t)\|_q^r dt = C^r \|g\|_p^r \int_0^T \frac{dt}{t^{\frac{nq-n}{2q}r}}. \qquad (28)$$

Here we have $\|u\|_r \leq C_1 \|g\|_p$ for a constant $C_1 \iff$ the integral in the R.H.S. of Eq. (28) converges $\iff \frac{nq-n}{2q}r < 1$. Then we could decide appropriate tuple (p_0, q_0, r_0) for (25): tuples that satisfy $\frac{n}{n+2} \frac{1}{p_0} < \frac{1}{r_0} \leq \frac{1}{p_0}$. (The second inequality comes from the constraint $q_0 \in [1, \infty]$. It could be removed when these L^p−norms are calculated in a bounded domain).

We handle (p', q', r') in (26) with exactly the same method and find that tuples which satisfy $\frac{n}{n+2} \frac{1}{p'} + \frac{1}{n+2} < \frac{1}{r'} \leq \frac{1}{p'}$ gives

$$\left\| \frac{\partial u}{\partial x_i} \right\|_{r'} \leq C_2 \|g\|_{p'}, \qquad (29)$$

where C_2 is a constant.

Finally, note that for any bounded set Ω, and $1 \leq q < p$, there is a constant C such that $\|v\|_{L^q(\Omega)} \leq C\|v\|_{L^p(\Omega)}$ for all $v \in L^p(\Omega)$. Together with the inequalities (28) and (29), this means that for any compact set $Q \subset \mathbb{R}^n \times [0, T]$, $p \geq 1$, and $r < \frac{(n+2)p}{n+p}$,

$$\|u\|_{W^{1,r}(Q)} = (\|u\|_{L^r(Q)}^r + \sum_{i=1}^n \|\frac{\partial u}{\partial x_i}\|_{L^r(Q)}^r)^{\frac{1}{r}} \qquad (30)$$

$$\leq (C_3 \|u\|_{L^{r_0}(Q)}^r + C_4 \sum_{i=1}^n \|\frac{\partial u}{\partial x_i}\|_{L^{r'}(Q)}^r)^{\frac{1}{r}} \qquad (31)$$

$$\leq (C_3 \|u\|_{L^{r_0}(\mathbb{R}^n \times [0,T])}^r + C_4 \sum_{i=1}^n \|\frac{\partial u}{\partial x_i}\|_{L^{r'}(\mathbb{R}^n \times [0,T])}^r)^{\frac{1}{r}} \qquad (32)$$

$$\leq (C_5 \|g\|_{L^p(\mathbb{R}^n)}^r + C_6 \sum_{i=1}^n \|g\|_{L^p(\mathbb{R}^n)}^r)^{\frac{1}{r}} \qquad (33)$$

$$= C_7 \|g\|_{L^p(\mathbb{R}^n)}, \qquad (34)$$

where $r_0, r' \in [p, \frac{(n+2)p}{n+p}) \cap [r, +\infty)$ and all C_i are constants.

This gives the first statement.

Next, we prove the second statement.

Note that $\frac{\partial^2 u}{\partial x_i x_j} = (\frac{\partial}{\partial x_i} K(x, t)) * \frac{\partial g(x)}{\partial x_j}$, $\forall i, j \in [n]$.

With the same argument for (29), we obtain that for any r'', p'' satisfying $\frac{n}{n+2} \frac{1}{p''} + \frac{1}{n+2} < \frac{1}{r''} \leq \frac{1}{p''}$,

$$\left\| \frac{\partial^2 u}{\partial x_i x_j} \right\|_{L^{r''}(\mathbb{R}^n \times [0,T])} \leq C \left\| \frac{\partial g(x)}{\partial x_j} \right\|_{L^{p''}(\mathbb{R}^n)}, \quad \forall i, j \in [n], \qquad (35)$$

where C is a constant.

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Therefore, for any compact set $Q \subset \mathbb{R}^n \times [0, T]$, $p \geq 1$, and $r < \frac{(n+2)p}{n+p}$,

$$\|u\|_{W^{2,r}(Q)} \tag{36}$$

$$= \left(\|u\|_{L^r(Q)}^r + \sum_{i=1}^{n} \left\| \frac{\partial u}{\partial x_i} \right\|_{L^r(Q)}^r + \sum_{i,j=1}^{n} \left\| \frac{\partial^2 u}{\partial x_i x_j} \right\|_{L^r(Q)}^r \right)^{\frac{1}{r}} \tag{37}$$

$$\leq \left(C_1 \|u\|_{L^{r_0}(Q)}^r + C_2 \sum_{i=1}^{n} \left\| \frac{\partial u}{\partial x_i} \right\|_{L^{r'}(Q)}^r + C_3 \sum_{i,j=1}^{n} \left\| \frac{\partial^2 u}{\partial x_i x_j} \right\|_{L^{r''}(Q)}^r \right)^{\frac{1}{r}} \tag{38}$$

$$\leq \left(C_1 \|u\|_{L^{r_0}(\mathbb{R}^n \times [0,T])}^r + C_2 \sum_{i=1}^{n} \left\| \frac{\partial u}{\partial x_i} \right\|_{L^{r'}(\mathbb{R}^n \times [0,T])}^r + C_3 \sum_{i,j=1}^{n} \left\| \frac{\partial^2 u}{\partial x_i x_j} \right\|_{L^{r''}(\mathbb{R}^n \times [0,T])}^r \right)^{\frac{1}{r}} \tag{39}$$

$$\leq \left(C_4 \|g\|_{L^p(\mathbb{R}^n)}^r + C_5 \sum_{i=1}^{n} \|g\|_{L^p(\mathbb{R}^n)}^r + C_6 \sum_{i,j=1}^{n} \left\| \frac{\partial g}{\partial x_j} \right\|_{L^p(\mathbb{R}^n)}^r \right)^{\frac{1}{r}} \tag{40}$$

$$\leq C_7 \|g\|_{W^{1,p}(\mathbb{R}^n)}, \tag{41}$$

where $r_0, r', r'' \in [p, \frac{(n+2)p}{n+p}) \cap [r, +\infty)$ and all C_i are constants.

This completes the proof. □

At last, we present it here a well-known result in functional analysis.

Lemma A.6 (Inverse function theorem in Banach space). *Let X, Y be two Banach spaces, $V \subset X$ be an open set, and $g \in C^1(V, Y)$ be a mapping. Assume $x_0 \in V$, $y_0 = g(x_0)$ and the inverse of the Fréchet derivative $(g'(x_0))^{-1} \in \mathscr{L}(Y, X)$. Then there exists $r > 0$ and $s > 0$ such that $B_r(y_0) \subset g(V)$, $B_s(x_0) \subset V$ and $g : B_s(x_0) \to g(B_s(x_0))$ is a differmorphism.*

B Derivation of a Class of Hamilton-Jacobi-Bellman (HJB) Equations

For the sake of completeness of the paper, we give the derivation of a class of Hamilton-Jacobi-Bellman (HJB) Equations as below.

To start with, we derive the general form of HJB Equation in stochastic control problem.

In stochastic control, the state function $\{X_t\}_{0 \leq t \leq T}$ is a stochastic process, where T is the time horizon of the control problem. The evolution of the state function is governed by the following stochastic differential equation:

$$\begin{cases} \mathrm{d}X_s = m(s, X_s)\mathrm{d}s + \sigma \mathrm{d}W_s & s \in [t, T] \\ X_t = x \end{cases}, \tag{42}$$

where $m : [t, T] \times \mathbb{R}^n \to \mathbb{R}^n$ is the control function and $\{W_s\}$ is a standard n-dimensional Brownian motion.

Given a control function $m = (m_1(s, y), m_2(s, y), \dots m_n(s, y))$, $s \in [t, T]$, $y \in \mathbb{R}^n$, its total cost is defined as $J_{x,t}(m) = \mathbb{E} \int_t^T r(X_s, m(s, X_s), s)\mathrm{d}s + g(X_T)$, where $r : \mathbb{R}^n \times \mathbb{R}^n \times [0, T] \to \mathbb{R}$ measures the cost rate during the process and $g : \mathbb{R}^n \to \mathbb{R}$ measures the final cost at the terminal state. The expectation is taken over the randomness of the trajectories.

We are interested in finding a control function that minimizes the total cost for a given initial state. Formally speaking, we define the *value function* of the control problem (42) as $u(x, t) = \min_{m \in \mathcal{M}} J_{x,t}(m)$, where $\mathcal{M}$ denotes the set of possible control functions that we take into consideration.

It is obvious that u satisfies $u(x, T) = g(x)$. In addition, according to dynamical programming principle, we have

$$u(x, t) = \min_{m \in \mathcal{M}} \mathbb{E}\left(\int_t^{t+h} r(X_s, m(s, X_s), s)ds + u(X_{t+h}, t + h) \right), \tag{43}$$

With Ito's formula, we derive

$$u(X_{t+h}, t+h) = u(x,t) + (\partial_t u + \frac{1}{2}\sigma^2 \Delta u)h + \nabla u \cdot (m(t,x)h + \sigma(W_{t+h} - W_t)) + o(h) \quad (44)$$

After taking expectation and some calculation, we derive from (44),

$$0 = (\partial_t u + \frac{1}{2}\sigma^2 \Delta u)h + \min_{m \in \mathcal{M}} \mathbb{E}(\int_t^{t+h} r(X_s, m(s, X_s), s)ds + \nabla u \cdot m(t,x)h) + o(h) \quad (45)$$

$$0 = \partial_t u(x,t) + \frac{1}{2}\sigma^2 \Delta u(x,t) + \min_{m \in \mathcal{M}}(r(x, m(t,x), t) + \nabla u \cdot m(t,x)) \quad (46)$$

Then we get HJB equation

$$\begin{cases} \partial_t u(x,t) + \frac{1}{2}\sigma^2 \Delta u(x,t) + \min_{m \in \mathcal{M}}(r(x, m(t,x), t) + \nabla u \cdot m(t,x)) = 0 \\ u(x,T) = g(x). \end{cases} \quad (47)$$

Next, we further simplify this equation in some special cases.

In practice, different components of the state have different meanings, and thus the effects of controlling corresponding components have different significance. Therefore, the cost function's dependence on each component of m_t takes a very different form.

Based on this argument, we consider the case when $r(x, y)$ takes the form

$$r(x, y, t) = \sum_{i=1}^n a_i |y_i|^{\alpha_i} - \varphi(x, t) \quad (48)$$

for some appropriate function φ and $a_i \geq 0, \alpha_i > 1$ (if $\alpha_i \leq 1$, the minimizing term might be $-\infty$), $\forall i \in [n]$.

Denote $m(t,x) = (m_1(t,x), m_2(t,x), ...m_n(t,x))$ as $y \in \mathbb{R}^n$, and $\frac{\partial u(x,t)}{\partial x_i}$ as $\partial_i u$, and suppose that $\mathcal{M}$ is so large that it includes the global minimizor of (43), then the third term in HJB equation (47) could be written as

$$\min_{y \in \mathbb{R}^n}(-\varphi(x,t) + \sum_{i=1}^n (a_i|y_i|^{\alpha_i} + y_i \partial_i u)) = \varphi(x,t) + \sum_{i=1}^n \min_{y_i \in \mathbb{R}}(a_i|y_i|^{\alpha_i} + y_i \partial_i u). \quad (49)$$

With some simple computation, we get

$$\min_{y_i \in \mathbb{R}}(a_i|y_i|^{\alpha_i} + y_i \partial_i u) = \left(\frac{a_i}{(a_i\alpha_i)^{\frac{\alpha_i}{\alpha_i-1}}} - \frac{1}{(a_i\alpha_i)^{\frac{1}{\alpha_i-1}}}\right)|\partial_i u|^{\frac{\alpha_i}{\alpha_i-1}}. \quad (50)$$

As a result, HJB equation in this case is

$$\begin{cases} \partial_t u(x,t) + \frac{1}{2}\sigma^2 \Delta u(x,t) - \varphi(x,t) - \sum_{i=1}^n A_i|\partial_i u|^{c_i} = 0 \\ u(x,T) = g(x), \end{cases} \quad (51)$$

where $A_i = (a_i\alpha_i)^{-\frac{1}{\alpha_i-1}} - a_i(a_i\alpha_i)^{-\frac{\alpha_i}{\alpha_i-1}} \in (0, +\infty)$ and $c_i = \frac{\alpha_i}{\alpha_i-1} \in (1, +\infty)$.

Remark B.1. *After taking the transform $v(x,t) := u(x, T-t)$, the equation above becomes*

$$\begin{cases} \partial_t v(x,t) - \frac{1}{2}\sigma^2 \Delta v(x,t) + \sum_{i=1}^n A_i|\partial_i v|^{c_i} = -\varphi(x, T-t) \\ v(x,0) = g(x). \end{cases} \quad (52)$$

We will study this equation in the rest of the paper instead.

Remark B.2. *The minimizer of (49) is $y_i^* = (\frac{|\partial_i u|}{a_i\alpha_i})^{\frac{1}{\alpha_i-1}}$ and this gives the i-th component of the optimal control m^*. Based on the fact that both the value function u and the optimal control m^* are of interest in applications, it is necessary to study this equation in $W^{1,p}$ space.*

Moreover, in most cases, only a bounded domain $\Omega \subset \mathbb{R}^n$ is taken into consideration in both real applications and numerical experiments. Therefore, we study this equation in the space of $W^{1,p}(\Omega \times [0,T])$ for a bounded domain Ω, instead of $W^{1,p}(\mathbb{R}^n \times [0,T])$.

Remark B.3. *The form of cost function (48) we investigate in the paper is a generalization of the widely-used power-law cost (or utility) function, which is representative in optimal control. For example, in financial markets, we often face power-law trading cost in optimal execution problems [10, 30]. The cost function in Linear–Quadratic–Gaussian control and Merton's portfolio model (constant relative risk aversion utility function in [22]) is also of this form. Therefore, we believe our theoretical analysis for this class of HJB equation is relevant for practical applications.*

C Proof of Theorem 4.3

In this section, we give the proof of an equivalent statement of Theorem 4.3.

In light of remark B.1, it is equivalent to consider the stability property (as is defined in Definition 4.1) for the following equation:

$$\begin{cases} \partial_t u(x,t) - \frac{1}{2}\sigma^2 \Delta u(x,t) + \sum_{i=1}^n A_i |\partial_i u|^{c_i} = h(x,t) & (x,t) \in \mathbb{R}^n \times [0,T] \\ u(x,0) = g(x), \end{cases} \tag{53}$$

where $A_i > 0$, $c_i \in (1,\infty)$, and $h(x,t)$ corresponds to $-\varphi(x, T-t)$ in Eq. (52). Without loss of generality, we assume $\sigma = \sqrt{2}$ for simplicity in the discussion below.

We define operators $\mathcal{L}_0 := \frac{\partial}{\partial t} - \Delta$, $\tilde{\mathcal{L}}_{\text{HJB}} u := \mathcal{L}_0 u + \sum_{i=1}^n A_i |\partial_i u|^{c_i}$ and $\tilde{\mathcal{B}}_{\text{HJB}} u(x,t) := u(x,0)$ for clarity. We define $\bar{c}$ as $\max_{i \in [n]} c_i$.

We start with the proof of some auxiliary results.

Lemma C.1. *For every $c > 1$, there exist $k \in \mathbb{N}$, $\{t_i\}_{i=1}^k$ satisfying $1 \leq t_1 < t_2 < ... < t_k < c$, and k power functions $f_1, ...f_k$ whose orders are strictly less than c and no smaller than 0, such that*

$$(b+w)^c - b^c - w^c \leq \sum_{i=1}^k f_i(b) w^{t_i}, \ \forall b, w \geq 0. \tag{54}$$

Proof. Obviously, the inequality holds when $c \in \mathbb{N}$ because of the binomial expansion. We will only consider the case when $c \notin \mathbb{N}$.

*Step 1.*We prove that for any $b, w \geq 0$ such that $\max\{b, x\} \geq 1$, $F_{b,w}(c) := (b+w)^c - b^c - w^c$ is monotone increasing.

Without loss of generality, suppose $b \geq w$. Then

$$F'(c) = (b+w)^c \ln(b+w) - b^c \ln b - w^c \ln w \tag{55}$$

$$= (b+\eta)^{c-1}(1 + c \ln(b+\eta))w - w^c \ln w \tag{56}$$

holds for an $\eta \in (0, w)$ (mean value theorem).

Thus

$$F'(c) \geq w^c(1 + c \ln(b+\eta)) - w^c \ln w > w^c(1 + \ln(b+\eta) - \ln w) > w^c > 0. \tag{57}$$

The inequalities rely on the assumption $b \geq 1$, which means $\ln(b+\eta) > 0$, and the first inequality comes from $b \geq w$.

This completes the proof in this step.

Step 2. We then construct k, $\{t_i\}_{i=1}^k$, $\{f_i\}_{i=1}^k$ stated in the lemma.

Set $n = \lceil c \rceil$. By virtue of the increasing property of $F(c)$ when $\max\{b, w\} \geq 1$, we get

$$F_{b,w}(c) \leq F_{b,w}(n) = \sum_{i=1}^{n-1} \binom{n}{i} b^{n-i} \cdot w^i. \tag{58}$$

When b, w satisfies $\max\{b, w\} < 1$,

$$F_{b,w}(c) \leq (b+w)^c - b^c = c(b+\eta)^{c-1} w < c2^{c-1} w \tag{59}$$

holds for an $\eta \in (0, w)$. The equality is an application of mean value theorem, and the second inequality comes from the fact that $\eta \in (0, w)$.

To conclude, $(b + w)^c - b^c - w^c \le c2^{c-1}w + \sum_{i=1}^{n-1} \binom{n}{i} b^{n-i} \cdot w^i, \forall b, w \ge 0$. Since $c \notin \mathbb{N}$, which means $n - 1 < c$, this completes the proof. $\qquad \square$

Lemma C.2. *Suppose u^* is the exact solution to*

$$\begin{cases} \tilde{\mathcal{L}}_{\text{HJB}} u = h & (x, t) \in \mathbb{R}^n \times [0, T] \\ \tilde{\mathcal{B}}_{\text{HJB}} u = g \end{cases} .$$

Fix a bounded open set $\Omega \subset \mathbb{R}^n$. Suppose u_1 satisfies $\tilde{\mathcal{B}}_{\text{HJB}} u^* = \tilde{\mathcal{B}}_{\text{HJB}} u_1$ and that $\text{supp}(u_1 - u^*) \subset Q_T(\Omega)$. Recall c_i are parameters in the operator $\tilde{\mathcal{L}}_{\text{HJB}}$. Let $p \in [2, \infty)$.

If $p \ge n \cdot \max_{i \in [n]} \frac{c_i - 1}{c_i} = (1 - \bar{c}^{-1})n$, then there exists $\delta_0 > 0$ such that, when $\|\tilde{\mathcal{L}}_{\text{HJB}} u_1 - h\|_p < \delta_0$,

we have $\|u^* - u_1\|_{2,p} \le C\|\tilde{\mathcal{L}}_{\text{HJB}} u_1 - h\|_p$ for a constant C independent of u_1.

Proof. Define $w = w_{u_1} := u_1 - u^*$, then $\text{supp}(w)$ is compact and $w(x, 0) = 0$. We further define $f = f_{u_1} := \tilde{\mathcal{L}}_{\text{HJB}} u_1 - h$. Since $u_1 = u^*$ in $\mathbb{R}^n \times [0, T] \backslash Q_T(\Omega)$ and $f = \tilde{\mathcal{L}}_{\text{HJB}} u_1 - \tilde{\mathcal{L}}_{\text{HJB}} u^*$, we have $\text{supp}(f) \subset Q_T(\Omega)$. The $W^{m,p}$ and L^p norm in the rest of the proof is defined on the domain $Q_T(\Omega)$.

Compute

$$f = \tilde{\mathcal{L}}_{\text{HJB}} u_1 - \tilde{\mathcal{L}}_{\text{HJB}} u^* = \mathcal{L}_0 w + \sum_{i=1}^{n} A_i |\partial_i(u^* + w)|^{c_i} - A_i |\partial_i u^*|^{c_i} \tag{60}$$

Thus, for any (x, t),

$$|\mathcal{L}_0 w(x, t)| = \left| f - \sum_{i=1}^{n} (A_i |\partial_i(u^* + w)|^{c_i} - A_i |\partial_i u^*|^{c_i}) \right|\Big|_{(x,t)} \tag{61}$$

$$\le |f(x, t)| + \sum_{i=1}^{n} A_i \left| |\partial_i(u^* + w)|^{c_i} - |\partial_i u^*|^{c_i} \right|\Big|_{(x,t)} \tag{62}$$

$$\le |f(x, t)| + \sum_{i=1}^{n} A_i \left((|\partial_i u^*| + |\partial_i w|)^{c_i} - |\partial_i u^*|^{c_i} \right)\Big|_{(x,t)} \tag{63}$$

where the second inequality could be derived from the fact that $(a + b)^c - a^c \ge a^c - (a - b)^c$ for $a \ge b \ge 0$ and $c \ge 1$.

For $i \in [n]$, apply Lemma C.1 for $c = c_i$ and we obtain k_i and $\{t_{ij}\}_{j=1}^{k_i}$, $\{f_{ij}\}_{j=1}^{k_i}$ satisfying corresponding properties.

We have

$$|\mathcal{L}_0 w(x, t)| \le |f(x, t)| + \sum_{i=1}^{n} A_i \left(|\partial_i w|^{c_i} + \sum_{j=1}^{k_i} f_{ij}(|\partial_i u^*|)|\partial_i w|^{t_{ij}} \right)\Big|_{(x,t)}. \tag{64}$$

With Lemma A.4 and triangle inequality, we obtain

$$\|w\|_{2,p} \le C\|\mathcal{L}_0 w\|_p \le C(\|f\|_p + \sum_{i=1}^{n} A_i(\||\partial_i w|^{c_i}\|_p + \sum_{j=1}^{k_i} \|f_{ij}(|\partial_i u^*|)|\partial_i w|^{t_{ij}}\|_p)). \tag{65}$$

We will handle each term respectively.

Using Lemma A.2, we have

$$\||\partial_i w|^{c_i}\|_p = \|\partial_i w\|_{c_i p}^{c_i} \le \|w\|_{1, c_i p}^{c_i} \le \hat{C}_i \|w\|_{2, \frac{nc_i p}{n + c_i p}}^{c_i}. \tag{66}$$

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for constants $\hat{C}_i$.

Using Lemma A.2 and Hölder inequality, we have

$$\||f_{ij}(|\partial_i u^*|)|\partial_i w|^{t_{ij}}\|_p \leq \|f_{ij}(|\partial_i u^*|)\|_\infty \||\partial_i w|^{t_{ij}}\|_p \tag{67}$$

$$= \|f_{ij}(|\partial_i u^*|)\|_\infty \|\partial_i w\|_{t_{ij}p}^{t_{ij}} \leq \tilde{C}_{ij} \|w\|_{2,\frac{nt_{ij}p}{n+t_{ij}p}}^{t_{ij}}. \tag{68}$$

for constants $\tilde{C}_{ij}$ (Since $\overline{Q_T(\Omega)}$ is compact, we can tell that $\|f_{ij}(|\partial_i u^*|)\|_\infty < \infty$ and thus $\tilde{C}_{ij}$ are well-defined).

When $p \geq n \cdot \max_{i \in [n]} \frac{c_i - 1}{c_i}$, because of $t_{ij} < c_i$, we have $\frac{nc_i p}{n + c_i p} \leq p$, $\frac{nt_{ij}p}{n + t_{ij}p} \leq p$ for all i, j.

Note that Ω is bounded, so for $1 \leq q < p$, there is a constant C such that $\|v\|_{L^q(\Omega)} \leq \|v\|_{L^p(\Omega)}$ for all $v \in L^p(\Omega)$.

As a consequence, we can derive from Eq. (65,66,68) that $M := \|w\|_{2,p}$ satisfies the inequality

$$K_0 M - \sum_{i=1}^n \left(K_i M^{c_i} + \sum_{1 \leq j \leq k_i, t_{ij} > 1} K_{ij} M^{t_{ij}} \right) \leq \|f\|_p, \tag{69}$$

where all K_i and K_{ij} are positive constants depending only on p, n, u^* and Ω.

For clarity, We define the L.H.S. of (69) as a function F with variable M.

With the observations (i)$F(0) = 0$, (ii) $F'(0) > 0$, (iii)$F''(M) < 0$, (iv)$F(+\infty) = -\infty$, we could tell that $F(M)$ has a unique zero m_0 in $\mathbb{R}^+$. We could further tell that for any non-negative number $C \leq \max_{M \in [0, m_0]} F(M)$, solving $F(M) \leq C$ ($M \geq 0$) derives $M \in [0, a] \cap [b, \infty]$ for some $0 < a < b$ depending on C and that $a \to 0, b \to m_0$ monotonously as C decreases to 0. Note that there exists $\delta > 0$ such that $a \leq \frac{2}{K_0} C$ for $\forall C \in [0, \delta]$. In order to prove $\|w\|_{2,p} = O(\|f\|_p)$, it suffices to show that $\|w\|_{2,p}$ (i.e. M in the discussion above) would not fall in the second interval providing C (or $\|f\|_p$, correspondingly) is sufficiently small. We will prove by contradiction.

Note that $\tilde{\mathcal{L}}_{\text{HJB}}$ is a continuous injection from $W_0^{2,p}(Q_T(\Omega))$ to $L^p(Q_T(\Omega))$, and that there exists $r_0^* > 0$ such that $\tilde{\mathcal{L}}_{\text{HJB}}$ is a differmorphism from $B_{r_0}(u^*)$ (in $W_0^{2,p}(Q_T(\Omega))$) to $\tilde{\mathcal{L}}_{\text{HJB}}(B_{r_0}(u^*)) \supset B_{r_1}(h)$ (in $L^p(Q_T(\Omega))$) for any $r_0 \in (0, r_0^*)$ and any $r_1 \in (0, r_1^*)$ with r_1^* depending on r_0 (this comes from an application of Lemma A.6).

Select $r_0 < \frac{m_0}{2}$ and determine the corresponding r_1^*. By the property of b, there exists $\delta_0 \in (0, \min\{\frac{r_1^*}{2}, \delta\})$ such that for any $C < \delta_0$, the corresponding b is larger than $\frac{3}{4}m_0$. If there exists $w \in W_0^{m,p}(Q_T(\Omega))$ satisfying $\|f\|_p = \|\tilde{\mathcal{L}}_{\text{HJB}}(u^* + w) - \tilde{\mathcal{L}}_{\text{HJB}}u^*\|_p := C < \delta_0$ while $\|w\|_{2,p} > b$ (b depends on C), then there will also be a $w' \in B_{r_0}(0)$ such that $\tilde{\mathcal{L}}_{\text{HJB}}(u^* + w') - \tilde{\mathcal{L}}_{\text{HJB}}u^* = f$. We could tell from the difference between their norm that $w \neq w'$. This contradicts the property of injection.

The proof is completed. $\qquad\square$

Lemma C.3. *Suppose u^* follows Lemma C.2, and Ω is a fixed bounded open set in $\mathbb{R}^n$. Suppose u_1 satisfies $\tilde{\mathcal{L}}_{\text{HJB}}u^* = \tilde{\mathcal{L}}_{\text{HJB}}u_1$ and that $\text{supp}(u_1 - u^*) \subset Q_T(\Omega)$. Let $q \in [1, \infty)$.*

If $\bar{c} \leq 2$ and $q > \frac{(\bar{c}-1)n^2}{(2-\bar{c})n+2}$, there exists $\delta_0 > 0$ such that when $\|\tilde{\mathcal{B}}_{\text{HJB}}u_1 - \tilde{\mathcal{B}}_{\text{HJB}}u^*\|_q < \delta_0$, we have $\|u^* - u_1\|_{1,r} \leq C\|\tilde{\mathcal{B}}_{\text{HJB}}u_1 - \tilde{\mathcal{B}}_{\text{HJB}}u^*\|_q$ for a constant C independent of u_1, where $r < \frac{(n+2)q}{n+q}$.

Proof. Define $w = w_{u_1} := u_1 - u^*$ and $f = f_{u_1} := \tilde{\mathcal{B}}_{\text{HJB}}u_1 - \tilde{\mathcal{B}}_{\text{HJB}}u^*$. Let w_1 be the solution to

$$\begin{cases} \mathcal{L}_0 u = 0, & (x, t) \in \mathbb{R}^n \times [0, T] \\ \tilde{\mathcal{B}}_{\text{HJB}}u = f. \end{cases}$$

The $W^{m,p}$ and L^p norm in the rest of the proof is defined on the domain $Q_T(\Omega)$.

Since the conditions $\bar{c} \leq 2$ and $q > \frac{(\bar{c}-1)n^2}{(2-\bar{c})n+2}$ hold, we have $[(\bar{c}-1)n, \frac{(n+2)q}{n+q}) \neq \varnothing$. Thus we could choose $r' \in [(\bar{c}-1)n, \frac{(n+2)q}{n+q}) \cap [r, \infty)$. Since $Q_T(\Omega)$ is bounded, it suffices to bound $\|u_1 - u^*\|_{r'}$.

To start with, from Lemma A.5, we have $\|w_1\|_{1,r'} \leq C\|f\|_q$.

Then we bound the difference between w_1 and w. Define $v = w_1 - w$, then v satisfies

$$\begin{cases} \mathcal{L}_0 v = \sum_{i=1}^n A_i |\partial_i(u^* + w)|^{c_i} - A_i |\partial_i u^*|^{c_i} \\ \tilde{\mathcal{B}}_{\text{HJB}} v = 0. \end{cases}$$

By Lemma A.2 and Lemma A.4 , we get $\|v\|_{1,r'} \leq C\|v\|_{2,\frac{nr'}{n+r'}} \leq C'\|\mathcal{L}_0 v\|_{\frac{nr'}{n+r'}}$.

Therefore we have $\|w\|_{1,r'} \leq \|w_1\|_{1,r'} + \|v\|_{1,r'} \leq C\|f\|_q + C'\|\mathcal{L}_0 v\|_{\frac{nr'}{n+r'}}$.

Next, we give an estimation for $\|\mathcal{L}_0 v\|_{\frac{nr'}{n+r'}}$.

Following from the proof in Lemma C.2, we obtain $\{k_i\}_{i=1}^n \subset \mathbb{N}$, $\{t_{ij}\}_{1 \leq i \leq n, 1 \leq j \leq k_i} \subset \mathbb{R}$, and power functions $\{f_{ij}\}_{1 \leq i \leq n, 1 \leq j \leq k_i}$.

With similar computations, we have

$$\|\mathcal{L}_0 v\|_{\frac{nr'}{n+r'}} \leq \sum_{i=1}^n A_i (\||\partial_i w|^{c_i}\|_{\frac{nr'}{n+r'}} + \sum_{j=1}^{k_i} \|f_{ij}(|\partial_i u^*|)|\partial_i w|^{t_{ij}}\|_{\frac{nr'}{n+r'}}) \tag{70}$$

$$\leq \sum_{i=1}^n A_i (\|\partial_i w\|_{\frac{c_i nr'}{n+r'}}^{c_i} + \sum_{j=1}^{k_i} \|f_{ij}(|\partial_i u^*|)\|_\infty \|\partial_i w\|_{\frac{n t_{ij} r'}{n+r'}}^{t_{ij}}). \tag{71}$$

Because of $r' \geq (\bar{c} - 1)n$ and $t_{ij} < c_i$, we have $\frac{nc_i r'}{n+r'} \leq r'$, $\frac{n t_{ij} r'}{n+r'} \leq r'$ for all i, j.

Thus, due to the fact that $Q_T(\Omega)$ is bounded, all $\|\partial_i w\|_{\frac{c_i nr'}{n+r'}}$ and $\|\partial_i w\|_{\frac{n t_{ij} r'}{n+r'}}$ could be bounded by $C_{i,j}\|w\|_{1,r'}$, where $C_{i,j}$ are constants.

With similar methods applied in Lemma C.2, we could prove this lemma. $\qquad\square$

Lemma C.4. *We denote by u^* the exact solution to*

$$\begin{cases} \tilde{\mathcal{L}}_{\text{HJB}} u(x,t) = h(x,t) & (x,t) \in \mathbb{R}^n \times [0,T], \\ \tilde{\mathcal{B}}_{\text{HJB}} u(x,t) = g(x) & x \in \mathbb{R}^n. \end{cases}$$

Fix Ω, which is an arbitrary bounded open set in $\mathbb{R}^n$. For two functions $\hat{f}_1(x,t)$, $\hat{f}_2(x)$, denote by u_1 the solution to

$$\begin{cases} \tilde{\mathcal{L}}_{\text{HJB}} u(x,t) = h(x,t) + \hat{f}_1(x,t), \ \text{in } \mathbb{R}^n \times [0,T] \\ \tilde{\mathcal{B}}_{\text{HJB}} u(x,t) = g(x) + \hat{f}_2(x), \ \text{in } \mathbb{R}^n. \end{cases}$$

For $p, q \geq 1$, let $r_0 = \frac{(n+2)q}{n+q}$. Assume the following inequalities hold for p, q and r_0:

$$p \geq \max\left\{2, \left(1 - \frac{1}{\bar{c}}\right) n\right\}; \ q > \frac{(\bar{c} - 1)n^2}{(2 - \bar{c})n + 2}; \ \frac{1}{r_0} \geq \frac{1}{p} - \frac{1}{n}, \tag{72}$$

Further assume that $\bar{c} \leq 2$ and $\operatorname{supp}(u_1 - u^*) \subset Q_T(\Omega)$.

Then for $\forall r \in [1, r_0)$, there exists $\delta_0 > 0$ such that, when $\|\hat{f}_1\|_p < \delta_0$ and $\|\hat{f}_2\|_q < \delta_0$, $\|u_1 - u^*\|_{1,r} \leq C(\|\hat{f}_1\|_p + \|\hat{f}_2\|_q)$ for a constant C independent of u_1.

Proof. It is straight-forward to define u_2 as the solution to

$$\begin{cases} \tilde{\mathcal{L}}_{\text{HJB}} u(x,t) = h(x,t) + \hat{f}_1(x,t) & (x,t) \in \mathbb{R}^n \times [0,T] \\ \tilde{\mathcal{B}}_{\text{HJB}} u(x,t) = g(x) & x \in \mathbb{R}^n \end{cases}$$

and bound $\|u^* - u_2\|_{1,r}$, $\|u_2 - u_1\|_{1,r}$ respectively.

From Lemma C.2, there exists $\delta_1 > 0, C_1 > 0$ such that $\|\hat{f}_1\|_p < \delta_1$ implies $\|u^* - u_2\|_{2,p} \leq C_1 \|\hat{f}_1\|_p$. And from Lemma C.3, there exists $\delta_2 > 0, C_2 > 0$ such that $\|\hat{f}_2\|_q < \delta_2$ implies $\|u_2 - u_1\|_{1,r} \leq C_2 \|\hat{f}_2\|_q$. By virtue of the condition $\frac{1}{r} > \frac{1}{r_0} \geq \frac{1}{p} - \frac{1}{n}$, with Lemma A.2 and the fact that we are considering $\|u^* - u_2\|_{1,r}$, $\|u_2 - u_1\|_{1,r}$ on a compact domain, providing $\|\hat{f}_1\| < \delta_1$ and $\|\hat{f}_2\| < \delta_2$, we derive

$$\|u^* - u_1\|_{1,r} \leq \|u^* - u_2\|_{1,r} + \|u_2 - u_1\|_{1,r} \tag{73}$$

$$\leq C\|u^* - u_2\|_{2,p} + \|u_2 - u_1\|_{1,r} \tag{74}$$

$$\leq CC_1\|\hat{f}_1\|_p + C_2\|\hat{f}_2\|_q, \tag{75}$$

where C is a constant.

This concludes the proof. $\qquad\square$

Finally, we give the proof of an equivalent statement of Theorem 4.3.

Theorem C.5. *Let $\hat{f}_1, \hat{f}_2, u^*$ and u_1 follow from Lemma C.4. Let p, q, r_0 satisfy the conditions in Lemma C.4. Assume $\bar{c} \leq 2$. For any bounded open set $Q \subset \mathbb{R}^n \times [0, T]$, it holds that for any $r \in [1, r_0)$, there exists $\delta > 0$ and a constant C independent of u_1, $\hat{f}_1$ and $\hat{f}_2$, such that $\max\{\|\hat{f}_1\|_{L^p(\mathbb{R}^n \times [0,T])}, \|\hat{f}_2\|_{L^q(\mathbb{R}^n)}\} < \delta$ implies $\|u_1 - u^*\|_{W^{1,r}(Q)} \leq C(\|\hat{f}_1\|_{L^p(\mathbb{R}^n \times [0,T])} + \|\hat{f}_2\|_{L^q(\mathbb{R}^n)})$.*

Proof. Since Q is bounded, there exists $R > 0$ such that $Q \subset Q(R)$.

Let $\hat{u}_1$ be the constraint of u_1 in $Q(R)$. Construct an extension v of $\hat{u}_1$ to $\mathbb{R}^n \times \mathbb{R}_{\geq 0}$ such that

(i) $v = u^*$ in $(\mathbb{R}^n \times \mathbb{R}_{\geq 0})\backslash Q(2R)$,

(ii) $\|\tilde{f}_1\|_p \leq C'\|\hat{f}_1\|_{L^p(Q)}$ and $\|\tilde{f}_2\| \leq C'\|\hat{f}_2\|_{L^q(B_R(0))}$ for a constant C' depending only on n, R, p, q, Q, where $\tilde{f}_1 := \tilde{\mathcal{L}}_{\mathrm{HJB}}v - \hat{f}_1$, $\tilde{f}_2 := \tilde{\mathcal{B}}_{\mathrm{HJB}}v - \hat{f}_2$.

Note that $\mathrm{supp}(\tilde{f}_1) \subset \overline{Q(2R)}$ and $\mathrm{supp}(\tilde{f}_2) \subset B_{2R}(0)$, the existence of v is obvious.

From Lemma C.4, there exists $C > 0$ and $\delta > 0$ such that $\|\tilde{f}_1\|_p < \delta$ and $\|\tilde{f}_2\|_q < \delta$ imply $\|v - u^*\|_{1,r} \leq C(\|\tilde{f}_1\|_p + \|\tilde{f}_2\|_q)$. Thus

$$\|u_1 - u^*\|_{W^{1,r}(Q)} = \|v - u^*\|_{W^{1,r}(Q)} \leq \|v - u^*\|_{1,r} \leq C(\|\tilde{f}_1\|_p + \|\tilde{f}_2\|_q) \tag{76}$$

$$\leq CC'(\|\hat{f}_1\|_{L^p(Q)} + \|\hat{f}_2\|_{L^q(B_R(0))}) \leq CC'(\|\hat{f}_1\|_p + \|\hat{f}_2\|_q). \tag{77}$$

The proof is completed. $\qquad\square$

D Proof of Theorem 4.4

In this section, we give the proof of Theorem 4.4.

Based on remark B.1, it is equivalent to consider Eq. (53). We will show that the following equation satisfies the properties stated in Theorem 4.4,

$$\begin{cases} \partial_t u - \Delta u + |Du|^2 = 0 \;\; in \; \mathbb{R}^n \times [0, T] \\ u(x, 0) = g(x). \end{cases} \tag{78}$$

This equation is a special case for Eq. (53) with $A_i = 1$, $c_i = 2$, $\forall i \in [n]$ and $h(x, t) \equiv 0$.

Denote by u^* the exact solution to the equation above. The notations $\mathcal{L}_0, \tilde{\mathcal{L}}_{\mathrm{HJB}}, \tilde{\mathcal{B}}_{\mathrm{HJB}}$ in the following discussion have the same meaning as in section C.

We prove some auxiliary results first.

Lemma D.1. *For $p \in [2, 2n)$, and an open set or a parabolic region $\mathfrak{A}$, we denote the function space $W^{2,p}(\mathfrak{A})$ as X and $L^{\frac{np}{2n-p}}(\mathfrak{A})$ as Y. For any $u \in X$, we have $\|u - u'\|_X \geq A\sqrt{\|\tilde{\mathcal{L}}_{\text{HJB}}u - \tilde{\mathcal{L}}_{\text{HJB}}u'\|_Y + B} - C$ holds for $\forall u' \in X$, where A, B, C are positive constants depending on u.*

Proof. We divide the proof into two steps.

Step 1. We check that $\tilde{\mathcal{L}}_{\text{HJB}}$ as an operator mapping from X to Y is Fréchet-differentiable.

For any $u, v \in X$, $t \in \mathbb{R}$, since $v \in X$, which means $|Dv|^2 \in W^{1,\frac{p}{2}} \subset Y$, we have

$$\|\tilde{\mathcal{L}}_{\text{HJB}}(u + tv) - \tilde{\mathcal{L}}_{\text{HJB}}u - t \cdot (\mathcal{L}_0 v + 2Du \cdot Dv)\|_Y = \|t^2|Dv|^2\|_Y = o(t) \ (t \to 0). \tag{79}$$

Therefore, $d\tilde{\mathcal{L}}_{\text{HJB}}(u, v) = \mathcal{L}_0 v + 2Du \cdot Dv$ by definition.

Define operator $A(u) \in \mathscr{L}(X, Y)$ as $A(u)v = dL(u, v)$. For $u, u' \in X$ and any $v \in X$, note that

$$\|A(u')v - A(u)v\|_Y = 2\|D(u' - u) \cdot Dv\|_Y \leq C_0\|D(u' - u) \cdot Dv\|_{1, \frac{p}{2}} \tag{80}$$

$$\leq C_0\|D(u' - u)\|_{1,p}\|Dv\|_{1,p} \leq C_0\|u' - u\|_X\|v\|_X \tag{81}$$

for a constant C_0, where the second inequality comes from Cauchy-Schwarz inequality. Thus we have $\|A(u) - A(u')\|_{\mathscr{L}(X, Y)} \leq C_0\|u - u'\|_X$, which means A is continuous with regard to u. As a result, $\tilde{\mathcal{L}}_{\text{HJB}}$ is Fréchet-differentiable and $\tilde{\mathcal{L}}'_{\text{HJB}}(u) = A(u)$. Moreover, we derive $\|L'(u)\| \leq \|L'(0)\| + \|L'(0) - L'(u)\| \leq C_0\|u\|_X + C_1$.

Step 2. For any $u, u' \in X$, let $y = \tilde{\mathcal{L}}_{\text{HJB}}u$, $y' = \tilde{\mathcal{L}}_{\text{HJB}}u'$.

Define $u_\eta = (1 - \eta)u + \eta u'$ and $y_\eta = \tilde{\mathcal{L}}_{\text{HJB}}u_\eta$ for $\eta \in [0, 1]$. Fix a number $m \in \mathbb{N}$.

From the property proved in step 1, for any $\eta \in [0, 1]$, there exists $r_\eta \in (0, \frac{1}{m})$ such that

$$\|\tilde{\mathcal{L}}_{\text{HJB}}v - \tilde{\mathcal{L}}_{\text{HJB}}u_\eta\| \leq 2\|L'(u_\eta)\| \cdot \|u_\eta - v\| \leq 2(C_0\|u_\eta\| + C_1)\|v - u_\eta\|, \ \ \forall v \in B_{r_\eta}(u_\eta). \tag{82}$$

Note that $\{B_{\frac{r_\eta}{2}}(u_\eta) : \eta \in [0, 1]\}$ is an open cover of $\{u_\eta : \eta \in [0, 1]\}$. Because of the compactness of $\{u_\eta : \eta \in [0, 1]\}$, we obtain an increasing finite sequence $\{\eta_i\}_{i=0}^N$ with $\eta_0 = 0, \eta_N = 1$ such that either $u_{\eta_i} \in B_{r_{\eta_{i-1}}}(u_{\eta_{i-1}})$ or $u_{\eta_{i-1}} \in B_{r_{\eta_i}}(u_{\eta_i})$ for $\forall i \in [N]$. This means that

$$\|y_{\eta_i} - y_{\eta_{i-1}}\| \leq 2(C_0\|u_{\eta_j}\| + C_1)\|u_{\eta_i} - u_{\eta_{i-1}}\|, \ j \in \{i - 1, i\} \tag{83}$$

holds for $\forall i \in [N]$.

Therefore

$$\|y' - y\| \leq \sum_{i=1}^N \|y_{\eta_i} - y_{\eta_{i-1}}\| \leq \sum_{i=1}^N 2(C_0\|u_{\eta_j}\| + C_1)\|u_{\eta_i} - u_{\eta_{i-1}}\| \tag{84}$$

Note that this inequality holds for every m. As $m \to \infty$, R.H.S. of (84) converges to

$$\|u - u'\| \int_0^1 2(C_0\|u + s(u' - u)\| + C_1)ds. \tag{85}$$

$$= \|u - u'\|(C_0\|u + \theta(u' - u)\| + C_1), \ (\theta \in (0, 1)) \tag{86}$$

$$\leq \|u - u'\|(C_0(\|u\| + \|u' - u\|) + C_1). \tag{87}$$

The equality comes from mean value theorem for integral, and the inequality comes from triangular inequality.

Combining Eq. (84) and (87) together completes the proof. $\square$

Lemma D.2. *Let u_1 satisfies $\tilde{\mathcal{B}}_{\text{HJB}}u^* = \tilde{\mathcal{B}}_{\text{HJB}}u_1$ and that $\text{supp}(u_1 - u^*)$ is compact in $\mathbb{R}^n \times \mathbb{R}^+$. Define $f(x, t) := \tilde{\mathcal{L}}_{\text{HJB}}u_1$. Let $p \in [2, \infty), m \in \mathbb{N}$. If $p \geq \frac{n}{2}$ then there exists $\delta_0 > 0$ such that $\|f\|_{m,p} < \delta_0$ implies $\|u^* - u_1\|_{m+2,p} = O(\|f\|_{m,p})$.*

Proof. When $m = 0$, this statement is a direct consequence of Lemma C.2.

When $m > 0$, for every multi-index α with $|\alpha| \leq m$, operate D^α on both sides of $\tilde{\mathcal{L}}_{\text{HJB}} u_1 = f$ and $\tilde{\mathcal{L}}_{\text{HJB}} u^* = 0$. We then obtain

$$\mathcal{L}_0 D^\alpha u_1 + D^\alpha |Du_1|^2 = D^\alpha f \tag{88}$$

$$\mathcal{L}_0 D^\alpha u^* + D^\alpha |Du^*|^2 = 0. \tag{89}$$

Define $w := u_1 - u^*$, and compute the difference between (88) and (89), we get

$$\mathcal{L}_0 D^\alpha w = D^\alpha f - \sum_{i=1}^n (2 D^\alpha (\partial_i u^* \partial_i w) + D^\alpha (\partial_i w)^2). \tag{90}$$

With similar methods used in Lemma C.2, we could bound $\|D^\alpha w\|_{2,p}$ with $\|D^\alpha f\|_p$, based on which we complete the proof. $\qquad\square$

Finally, we show that Eq. (78) satisfies the properties stated in Theorem 4.4, which will conclude the proof for Theorem 4.4.

Theorem D.3. *For any $\varepsilon > 0, A > 0, r \geq 1, m \in \mathbb{N}$ and $p \in \left[1, \frac{n}{4}\right]$, there exists a function $u \in C^\infty (\mathbb{R}^n \times (0, T])$ which satisfies the following conditions:*

- $\|\tilde{\mathcal{L}}_{\text{HJB}} u\|_{L^p(\mathbb{R}^n \times [0,T])} < \varepsilon$, $\tilde{\mathcal{B}}_{\text{HJB}} u = \tilde{\mathcal{B}}_{\text{HJB}} u^*$, *and* $\text{supp}(u - u^*)$ *is compact.*

- $\|u - u^*\|_{W^{m,r}(\mathbb{R}^n \times [0,T])} > A$.

Proof. Since $L^1(\Omega)$ has the weakest topology in function spaces $W^{m,r}(\Omega)$ when Ω is bounded, it is enough to consider the case for $r = 1$, $m = 0$.

Set $p_0 = \frac{5}{9}n$, $p_1 = \frac{11}{9}n$ and $p_2 = \frac{11}{7}n$.

Step 1. We construct two families of functions $\{v_{a,c}\}, \{F_{a,c}\}$ as the basis of proof.

For any $a, c > 0$, define $f_{a,c}(x) = c|x|^{-0.7}, x \in \overline{B_1(0)} \backslash B_a(0)$ in $\mathbb{R}^n$. We could extend it to a C^∞ function $\tilde{f}_{a,c}(x)$ defined on $\mathbb{R}^n$ such that $(\text{i})\|\tilde{f}_{a,c}\|_\infty < \|f_{a,c}\|_\infty + \min\{1, c\}$ and $(\text{ii})\text{supp}(\tilde{f}_{a,c}) \subset B_{1.1}(0)$.

We could further construct a C^∞ function $\hat{f}_{a,c}(x, t)$ such that

(i) $\text{supp}(\hat{f}_{a,c}) \subset B_{1.1}(0) \times (0, T]$,

(ii) $\hat{f}_{a,c}(x, t) = \tilde{f}_{a,c}(x), \forall t \in [\frac{T}{2}, T]$,

(iii) $\|\hat{f}_{a,c}(x, t)\|_{L^\infty(\mathbb{R}^n) \times [0,T]} \leq \|\tilde{f}_{a,c}\|_{L^\infty(\mathbb{R}^n)}$.

Define $u_{a,c}$ as the solution to

$$\begin{cases} \tilde{\mathcal{L}}_{\text{HJB}} u = \hat{f}_{a,c} & in \ \mathbb{R}^n \times [0, T] \\ \tilde{\mathcal{B}}_{\text{HJB}} u = g. \end{cases}$$

Select a function $w_{a,c} \in C^\infty(\mathbb{R}^n \times \mathbb{R})$ with compact support $\text{supp}(w_{a,c}) \subset \mathbb{R}^n \times (0, T]$, such that $\|u_{a,c} - u^* - w_{a,c}\| < \epsilon_c$ in $W^{3,4p_0}(\mathbb{R}^n \times [0, T]), W^{2,4p_1}(\mathbb{R}^n \times [0, T])$ and $W^{2,4p}(\mathbb{R}^n \times [0, T])$, where ϵ_c is an small value depending on c and is to be decided later.

We define $v_{a,c} = u^* + w_{a,c}$ and $F_{a,c} = \tilde{\mathcal{L}}_{\text{HJB}} v_{a,c}$.

Step 2. We show that $\{v_{a,c}\}$ and $\{F_{a,c}\}$ have following properties:

(i) $\text{supp}(v_{a,c} - u^*)$ is compact in $\mathbb{R}^n \times (0, T]$.

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(ii) $\tilde{\mathcal{B}}_{\text{HJB}} v_{a,c} = \tilde{\mathcal{B}}_{\text{HJB}} u^*$.

(iii) $\text{supp}(F_{a,c})$ is compact in $\mathbb{R}^n \times (0, T]$.

(iv) There exists a constant $M < \infty$ such that $\|F_{a,c}\|_q < cM$ and $\|F_{a,c}\|_{1,p_0} < cM$.

(v) For any $c > 0$, $\|F_{a,c}\|_{p_2} \to \infty$ as $a \to 0$.

(i) and (ii) comes directly from the construction of $v_{a,c}$.

Because $\text{supp}(v_{a,c} - u^*)$ is close, for any $(x, t) \in (\mathbb{R}^n \times [0, T]) \backslash \text{supp}(v_{a,c} - u^*)$, there exists $r > 0$ such that $(B_r(x, t) \cap (\mathbb{R}^n \times [0, T])) \subset (\mathbb{R}^n \times [0, T]) \backslash \text{supp}(v_{a,c} - u^*)$, which means $v_{a,c} = u^*$ in $B_r(x, t) \cap (\mathbb{R}^n \times [0, T])$ and thus $\tilde{\mathcal{L}}_{\text{HJB}} v_{a,c} = \tilde{\mathcal{L}}_{\text{HJB}} u^* = 0$. This gives (iii).

Due to the fact that the function $|x|^{-0.7} \in L^p(B_2(0)) \cap W^{1,p_0}(B_2(0))$, there exists a constant $M < \infty$ such that $\|\hat{f}_{a,1}\|_p < M - 1$ and $\|\hat{f}_{a,1}\|_{1,p_0} < M - 1$ holds for any a and any construction of $\hat{f}_{a,1}$ based on $f_{a,1}$. Due to the linearity of norms, we derive $\|\hat{f}_{a,c}\|_p < c(M - 1)$ and $\|\hat{f}_{a,c}\|_{1,p_0} < c(M - 1)$.

It is easy to check that $\tilde{\mathcal{L}}_{\text{HJB}}$ is a continuous mapping from $W^{3,4p_0}(\Omega)$ to $W^{1,p_0}(\Omega)$, from $W^{2,4p_1}(\Omega)$ to $L^{p_2}(\Omega)$ and from $W^{2,4p}(\Omega)$ to $L^p(\Omega)$ for any compact set $\Omega \subset \mathbb{R}^n \times [0, T]$. Therefore, $\|F_{a,c} - \hat{f}_{a,c}\|$ is small in $W^{1,p_0}(\mathbb{R}^n \times [0, T])$, $L^{p_2}(\mathbb{R}^n \times [0, T])$, and $L^p(\mathbb{R}^n \times [0, T])$.

Since $\||x|^{-0.7}\|_{L^{p_2}(B_1(0))} = +\infty$, by the construction of $\hat{f}_{a,c}$ we have $\|\hat{f}_{a,c}\|_{p_2} \to +\infty$ as $a \to 0$.

As a result of the continuity of $\tilde{\mathcal{L}}_{\text{HJB}}$, we could guarantee $\|F_{a,c}\|_p < cM$, $\|F_{a,c}\|_{1,p_0} < cM$ and $\|F_{a,c}\|_{p_2} > \frac{1}{2}\|\hat{f}_{a,c}\|_{p_2}$ by choosing ϵ_c sufficiently small previously. This gives (iv) and (v).

Step 3. We give an estimation for $\|v_{a,c} - u^*\|_1$, i.e., $\|w_{a,c}\|_1$).

We mention at the beginning of this part that all C_i appeared below are positive constants.

For any $\epsilon > 0$, set c to $\frac{1}{2M} \min\{\epsilon, \delta_0\}$, where δ_0 follows from an application of Lemma D.2 for the case $p = p_0$. Then for any $a > 0$, $\|F_{a,c}\|_p < \epsilon$ and we obtain $\|w_{a,c}\|_{3,p_0} \leq C_0 \|F_{a,c}\|_{1,p_0}$.

In Lemma A.3, we choose $j = 2$, $k = 3$, $\theta = \frac{n + \frac{13}{11}}{n + \frac{6}{5}}$, $r = p_0$, $q = 1$, $p = \frac{11}{9}n$ and derive $\|\nabla^2 w_{a,c}\|_{p_1} \leq C_1 \|\nabla^3 w_{a,c}\|_{p_0}^\theta \|w_{a,c}\|_1^{1-\theta}$. Since $\|\nabla^3 w_{a,c}\|_{p_0} \leq \|w_{a,c}\|_{3,p_0}$, we get

$$\|w_{a,c}\|_1 \geq \left(\frac{\|\nabla^2 w_{a,c}\|_{p_1}}{C_1 \|w_{a,c}\|_{3,p_0}^\theta} \right)^{\frac{1}{1-\theta}} \geq C_2 \left(\frac{\|\nabla^2 w_{a,c}\|_{p_1}}{\|F_{a,c}\|_{1,p_0}^\theta} \right)^{\frac{1}{1-\theta}} \geq \frac{C_2}{(C_3 M)^{\frac{1-\theta}{\theta}}} \|\nabla^2 w_{a,c}\|_{p_1}^{\frac{1}{1-\theta}}, \quad (91)$$

where the last inequality comes from property (iv) in Step 2.

By virtue of property (i) in Step 2, we have $\|\nabla^2 w_{a,c}\|_{p_1} \geq C_4 \|w_{a,c}\|_{2,p_1}$, which is an application of Poincaré inequality. Together with Lemma D.1,

$$\|w_{a,c}\|_1 \geq \frac{C_2}{(C_3 M)^{\frac{1-\theta}{\theta}}} \|\nabla^2 w_{a,c}\|_{p_1}^{\frac{1}{1-\theta}} \geq C_5 \|w_{a,c}\|_{2,p_1}^{\frac{1}{1-\theta}} \geq C_5 \left(C_6 \sqrt{\|F_{a,c}\|_{p_2}} - C_7 \right)^{\frac{1}{1-\theta}}. \quad (92)$$

Since R.H.S. above goes to $+\infty$ as $a \to 0$ due to property (v), for any $A > 0$, there exists $a_0 > 0$ such that $\|w_{a_0,c}\|_1 > A$.

Setting $u = v_{a_0,c} = u^* + w_{a_0,c}$ completes the proof. $\qquad \square$

E Improved Theorem 4.3

In this section, we give the stability result for Eq. (53) (Theorem E.3). Different from Theorem C.5, the constraint $\bar{c} \leq 2$ is released here.

The notations u^*, $\bar{c}$, $\mathcal{L}_0$, $\tilde{\mathcal{L}}_{\text{HJB}}$, $\tilde{\mathcal{B}}_{\text{HJB}}$ in the following discussion come from C.

The proof of Theorem E.3 is quite similar to that of Theorem C.5.

We begin with some auxiliary results.

Lemma E.1. *Suppose Ω is a fixed bounded open set in $\mathbb{R}^n$. Suppose u_1 satisfies $\tilde{\mathcal{L}}_{\mathrm{HJB}}u^* = \tilde{\mathcal{L}}_{\mathrm{HJB}}u_1$ and that $\mathrm{supp}(u_1 - u^*) \subset Q_T(\Omega)$. Let $q \in [1, \infty)$.*

If $q > \frac{(\bar{c}-1)n^2}{n+2\bar{c}}$, then there exists $\delta_0 > 0$ such that when $\|\tilde{\mathcal{B}}_{\mathrm{HJB}}u_1 - \tilde{\mathcal{B}}_{\mathrm{HJB}}u^*\|_{1,q} < \delta_0$, we have $\|u^* - u_1\|_{2,r} \leq C\|\tilde{\mathcal{B}}_{\mathrm{HJB}}u_1 - \tilde{\mathcal{B}}_{\mathrm{HJB}}u^*\|_{1,q}$ for a constant C independent of u_1, where $r < \frac{(n+2)q}{n+q}$.

Proof. The proof is almost the same as that for Lemma C.3.

Following its proof, we define w, f, w_1 and v similarly. And the $W^{m,p}$ and L^p norm in the rest of the proof will also be defined on the domain $Q_T(\Omega)$.

Since $q > \frac{(\bar{c}-1)n^2}{n+2\bar{c}}$, we have $[(1 - \bar{c}^{-1})n, \frac{(n+2)q}{n+q}) \neq \varnothing$.

Thus we could choose $r' \in [(1 - \bar{c}^{-1})n, \frac{(n+2)q}{n+q}) \cap [r, \infty)$.

Since $Q_T(\Omega)$ is bounded, it suffices to bound $\|u_1 - u^*\|_{r'}$.

From Lemma A.5, we have $\|w_1\|_{2,r'} \leq C\|f\|_{1,q}$. And from Lemma A.4 , we get $\|v\|_{2,r'} \leq C'\|\mathcal{L}_0 v\|_{r'}$.

Therefore we have $\|w\|_{2,r'} \leq \|w_1\|_{2,r'} + \|v\|_{2,r'} \leq C\|f\|_{1,q} + C'\|\mathcal{L}_0 v\|_{r'}$.

Next, we give an estimation for $\|\mathcal{L}_0 v\|_{r'}$.

Following from the proof in Lemma C.2, we obtain $\{k_i\}_{i=1}^n \subset \mathbb{N}$, $\{t_{ij}\}_{1 \leq i \leq n, 1 \leq j \leq k_i} \subset \mathbb{R}$, and power functions $\{f_{ij}\}_{1 \leq i \leq n, 1 \leq j \leq k_i}$.

With similar computations, we have

$$\|\mathcal{L}_0 v\|_{r'} \leq \sum_{i=1}^n A_i(\||\partial_i w|^{c_i}\|_{r'} + \sum_{j=1}^{k_i} \|f_{ij}(|\partial_i u^*|)|\partial_i w|^{t_{ij}}\|_{r'}) \tag{93}$$

$$\leq \sum_{i=1}^n A_i(\|\partial_i w\|_{c_i r'}^{c_i} + \sum_{j=1}^{k_i} \|f_{ij}(|\partial_i u^*|)\|_\infty \|\partial_i w\|_{t_{ij} r'}^{t_{ij}}). \tag{94}$$

Due to the fact that $Q_T(\Omega)$ is bounded, all $\|\partial_i w\|_{c_i r'}$ and $\|\partial_i w\|_{t_{ij} r'}$ could be bounded by $C_{i,j}\|w\|_{1,\bar{c}r'}$, where $C_{i,j}$ are constants.

Moreover, since $r' \geq (1 - \bar{c}^{-1})n$, we could tell from Lemma A.2 that

$$\|w\|_{1,\bar{c}r'} \leq \hat{C}\|w\|_{2,\frac{n\bar{c}r'}{n+\bar{c}r'}} \leq \hat{C}'\|w\|_{2,r'}, \tag{95}$$

where $\hat{C}$ and $\hat{C}'$ are constants.

With similar methods applied in Lemma C.2, we could prove this lemma. $\square$

Lemma E.2. *Fix Ω, which is an arbitrary bounded open set in $\mathbb{R}^n$. For two functions $\hat{f}_1(x,t)$, $\hat{f}_2(x)$, denote by u_1 the solution to*

$$\begin{cases} \tilde{\mathcal{L}}_{\mathrm{HJB}}u(x,t) = h(x,t) + \hat{f}_1(x,t), \text{ in } \mathbb{R}^n \times [0,T] \\ \tilde{\mathcal{B}}_{\mathrm{HJB}}u(x,t) = g(x) + \hat{f}_2(x), \text{ in } \mathbb{R}^n. \end{cases}$$

For $p, q \geq 1$, let $r_0 = \frac{(n+2)q}{n+q}$. Assume the following inequalities hold for p, q and r_0:

$$p \geq \max\left\{2, \left(1 - \frac{1}{\bar{c}}\right)n\right\}; \quad q > \frac{(\bar{c}-1)n^2}{n+2\bar{c}}. \tag{96}$$

Further assume $\mathrm{supp}(u_1 - u^*) \subset Q_T(\Omega)$.

Then for $\forall r \in [1, \min\{r_0, p\})$, there exists $\delta_0 > 0$ such that, when $\|\hat{f}_1\|_p < \delta_0$ and $\|\hat{f}_2\|_q < \delta_0$, $\|u_1 - u^*\|_{2,r} \leq C(\|\hat{f}_1\|_p + \|\hat{f}_2\|_{1,q})$ for a constant C independent of u_1.

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Proof. The proof follows as in Lemma C.4 by replacing the use of Lemma C.3 with Lemma E.1. $\quad\square$

Theorem E.3. *Let $\hat{f}_1$, $\hat{f}_2$ and u_1 follow from Lemma E.2. For $p, q, r \geq 1$, let $r_0 = \frac{(n+2)q}{n+q}$. Assume the following inequalities hold for p, q, r and r_0:*

$$p \geq \max\left\{2, \left(1 - \frac{1}{\bar{c}}\right)n\right\}\ ;\ q > \frac{(\bar{c}-1)n^2}{n+2\bar{c}}\ ;\ \frac{1}{r} > \frac{1}{\min\{r_0, p\}} - \frac{1}{n}. \tag{97}$$

Then for any bounded open set $Q \subset \mathbb{R}^n \times [0, T]$, there exists $\delta > 0$ and a constant C independent of u_1, $\hat{f}_1$ and $\hat{f}_2$, such that $\max\{\|\hat{f}_1\|_{L^p(\mathbb{R}^n \times [0,T])}, \|\hat{f}_2\|_{W^{1,q}(\mathbb{R}^n)}\} < \delta$ implies $\|u_1 - u^*\|_{W^{1,r}(Q)} \leq C(\|\hat{f}_1\|_{L^p(\mathbb{R}^n \times [0,T])} + \|\hat{f}_2\|_{W^{1,q}(\mathbb{R}^n)})$.

Proof. By replacing the use of Lemma C.4 in the proof for Theorem C.5 with Lemma E.2, we can bound $\|u_1 - u^*\|_{W^{2,r'}(Q)}$ with $\|\hat{f}_1\|_{L^p(\mathbb{R}^n \times [0,T])}$ and $\|\hat{f}_2\|_{W^{1,q}(\mathbb{R}^n)}$ for any $r' \in [1, \min\{r_0, p\})$.

We could further bound $\|u_1 - u^*\|_{W^{1,r}(Q)}$ with the help of Lemma A.2.

This concludes the proof. $\quad\square$

F   Experimental Settings

Hyperparameters.   The hyperparameters used in our experiment is described in Table 3.

Table 3: **Derailed experimental settings** of Section 6.

	$n = 100$	$n = 250$
Model Configuration		
Layers	4	
Hidden dimension	4096	
Activation	tanh	
Hyperparameters		
Toal iterations	5000	10000
Domain Batch Size N_1	100	50
Boundary Batch Size N_2	100	50
Inner Loop Iterations K	20	
Inner Loop Step Size η	0.05	
Learning Rate	$7e - 4$	
Learning Rate Decay	Linear	
Adam ε	$1e - 8$	
Adam(β_1, β_2)	$(0.9, 0.999)$	

Training data.   In all the experiments, the training data is sampled *online*. Specifically, in each iteration, we sample N_1 i.i.d. data points, $(x^{(1)}, t^{(1)}), \cdots, (x^{(N_1)}, t^{(N_1)})$, from the domain $\mathbb{R}^n \times [0, T]$, and N_2 i.i.d. data points, $(\tilde{x}^{(1)}, T), \cdots, (\tilde{x}^{(N_2)}, T)$, from the boundary $\mathbb{R}^n \times \{T\}$, where $(x^{(i)}, t^{(i)}) \sim \mathcal{N}(\mathbf{0}, \mathbf{I}_n) \times \mathcal{U}(0, 1)$ and $\tilde{x}^{(j)} \sim \mathcal{N}(\mathbf{0}, \mathbf{I}_n)$.

Evaluation metrics.   We use L^1, L^2, and $W^{1,1}$ relative errors to evaluate the quality of the learned solution.

L^1 and L^2 relative errors are two popular evaluation metrics, which are defined as

$$\frac{\sum_{j=1}^{S} |u^*(x_j) - u_\theta(x_j)|^p}{\sum_{j=1}^{S} |u^*(x_j)|^p},\ p = 1, 2, \tag{98}$$

where u_θ is the learned approximate solution, u^* is the exact solution and $\{x_j\}_{j=1}^{S}$ are S i.i.d. uniform samples from the domain $[0,1]^n \times [0,T]$.

Since the gradient of the solution to HJB equations plays an important role in applications, we also evaluate the solution using $W^{1,1}$ relative error, which is defined as

$$\frac{\sum_{j=1}^{S}(|u^*(x_j) - u_\theta(x_j)| + \sum_{i=1}^{n}|\partial_{x_i}u^*(x_j) - \partial_{x_i}u_\theta(x_j)|)}{\sum_{j=1}^{S}(|u^*(x_j)| + \sum_{i=1}^{n}|\partial_{x_i}u^*(x_j)|)}. \tag{99}$$

G   More experiments and visualizations
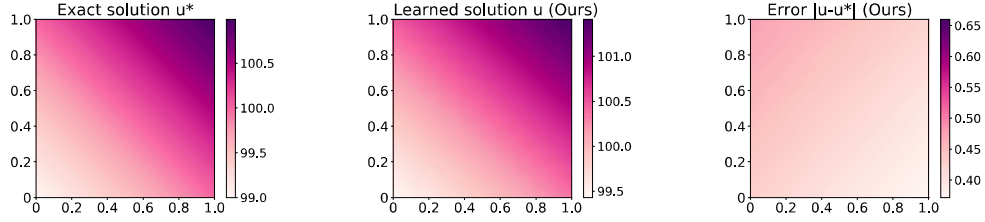
G.1   More instance of HJB Equations

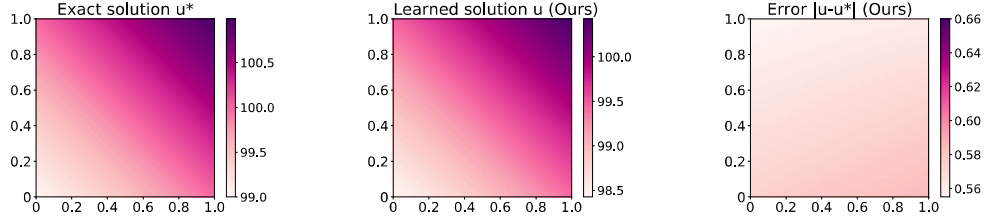Figure 3: **Visualization for the solution snapshot of Eq. (100).** c is set to 1.25.

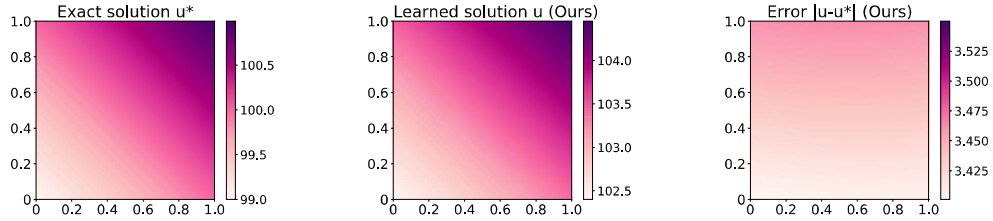Figure 4: **Visualization for the solution snapshot of Eq. (100).** c is set to 1.5.

Figure 5: **Visualization for the solution snapshot of Eq. (100).** c is set to 1.75.

To demonstrate the power of our method in solving general HJB Equations beyond classical LQG problems, we consider a more complicated HJB Equation as below:

$$\begin{cases} \partial_t u(x,t) + \Delta u(x,t) - \dfrac{1}{n}\sum_{i=1}^{n}|\partial_{x_i}u|^c = -2 & (x,t) \in \mathbb{R}^n \times [0,T] \\ u(x,T) = \sum_{i=1}^{n} x_i & x \in \mathbb{R}^n \end{cases}, \tag{100}$$

Eq. (100) has a unique solution $u(x,t) = x_1 + \cdots + x_n + T - t$. We consider to solve Eq. (100) for different valued of c using our method. We choose $c = 1.25, 1.5$ and 1.75 in the experiment. The neural network used for training is a 5-layer MLP with 4096 neurons and ReLU activation in each

Table 4: **Experimental results of solving the high dimensional HJB equations.** c is the parameter in Eq. (100). The dimensionality n is 100. Performances are measured by the L^1 relative error in the domain $[0,1]^n \times [0,T]$. Best performances are indicated in **bold**.

Method	$c = 1.25$	$c = 1.5$	$c = 1.75$
Original PINN [28]	1.11%	3.82%	2.73%
Adaptive time sampling [35]	1.18%	2.34%	7.94%
Learning rate annealing [34]	0.98%	1.13%	1.06%
Curriculum regularization [17]	6.27%	0.37%	3.51%
Adversarial training (ours)	**0.61%**	**0.15%**	**0.29%**

hidden layer. The training recipe, including the optimizer, learning rate, batch size, and the total iterations are the same as those in Appendix F. The number of inner-loop iterations K is set to 5, and the inner-loop step size η is searched from $\{2\mathrm{e}-1, 2\mathrm{e}-2, 2\mathrm{e}-3\}$.

Again, we examine the quality of the learned solution $u(x, t)$ by visualizing its snapshot on a two-dimensional space. Specifically, we consider the bivariate function $u(x_1, x_2, 1, 1, \cdots, 1; 0)$ and use a heatmap to show its function value given different x_1 and x_2. Figure 3-5 shows the ground truth u^*, the learned solutions u of our method, and the point-wise absolute error $|u - u^*|$ given different values of c.

From the above visualization, we can see that our method can solve Eq. (100) for different values of c effectively. Specifically, when $c = 1.25$ or 1.5, the point-wise absolute error is less than 0.5 for most of the area shown in the figures. When $c = 1.75$, the point-wise absolute error seems slightly larger, but it's still negligible compared with the scale of the learned solution. Thus, PINNs trained with our method fit the solution of Eq. (100) well, given different values of c.

We also compare our models with other baselines on these equations. The evaluation metric is L^1 relative error in the domain $[0,1]^n \times [0,1]$. The results are shown in Table 4. It's clear that our models outperform all the baselines on all these equations, showing the efficacy of our approach.

G.2 Tracing loss and error during the training

To give a more comprehensive comparison between original PINN and our method, we trace the loss and error during the training.

Table 5: **Error/loss-vs-time result of original PINN for Eq. (12).**

Iteration	1000	2000	3000	4000	5000
L^2 Loss	0.098	0.088	0.070	0.584	0.041
L^1 Relative Error	6.18%	5.36%	3.86%	3.94%	3.47%
$W^{1,1}$ Relative Error	17.53%	17.67%	14.83%	14.40%	11.31%

Table 6: **Error/loss-vs-time result of our method for Eq. (12).**

Iteration	1000	2000	3000	4000	5000
L^∞ Loss	11.841	9.352	2.404	1.605	0.711
L^1 Relative Error	15.22%	4.26%	0.97%	1.10%	0.27%
$W^{1,1}$ Relative Error	21.91%	18.62%	5.14%	4.96%	2.22%

It is clear that for the original PINN approach, the L^2 loss drops very quickly during training, while its $W^{1,1}$ relative error remains high. This result indicates the optimization is successful in this experiment, and that the stability property of the PDE leads to the high test error. By contrast, our proposed training approach enables the test error goes down steadily during training, which aligns with the theoretical claims.

H   Discussions on training with L^p loss

As is shown in the left panel of Table 2, directly optimizing L^p loss with large p fails to achieve a good approximator. This might seem to contradict our theoretical analysis in section 4. However, there is actually **no contradiction between our theorems and empirical results**. Theorem 4.3 focuses on the approximation ability, which indicates that if we have a model whose L^p loss is small, it will approximate the true solution well. The empirical results in Table 2 demonstrate the optimization difficulty of learning such a model.

Intuitively, we randomly sample points in each training iteration in the domain/boundary to calculate the loss. When p is large, most sampled points will hardly contribute to the loss, which leads to inefficiency and makes the training hard to converge. In Algorithm 1, we adversarially learn the points with large loss values, making all of them contribute to the model update (Step 8), significantly improving the model training.

Technically, directly applying Monte Carlo to compute L^p loss in experiments will lead to large variance estimations. For a function f,

$$\int |f|^p \mathrm{d}x = \frac{1}{N} \sum_{i=1}^{N} |f(X_i)|^p + \mathcal{O}\left(\sqrt{\frac{\mathrm{Var}|f(X)|^p}{N}}\right),$$

where $\{X_i\}_{i=1}^{N}$ are i.i.d. samplings in the domain.

Thus, $||f||_p$ suffers from an $\mathcal{O}((\mathrm{Var}|f(X)|^p/N)^{1/2p})$ error.

As $p \to \infty, \mathrm{Var}|f(X)|^p \sim ||f||_\infty^{2p}$. Therefore, the errors for estimating both Eq.(2,3) and the L^p norm of the residual are very large when p is large.