Supplementary Material

A Proof of Theorem 1

In order to prove Theorem 1 we first describe a technical hypothesis on the potential U. In detail, for all δ positive there exists two positive integrable functions $c_\delta(t)$ and $\kappa_\delta(t)$ such that for every $z \in \mathbb{R}^n$ and for all $t \in [0,T]$ we have

$$|\nabla U(z,t)| \leq \delta(U(z,t) + |z|^2) + c_\delta(t), \quad |\partial_t U(z,t)| \leq \delta(U(z,t) + |z|^2) + \kappa_\delta(t)\ . \tag{1}$$

Notice that here, in order to simplify the notation, we use the same symbol for U and for $\hat{U}(z,t) := U(z,u(t))$. We will also denote with w_ε the solution of problem (5).

As it is also remarked below the proof articulates as follow: first of all we asses the convergence of $w_\varepsilon \to w$ by compactness arguments, basically by performing an estimate on the solution w_ε; then the uniform estimate on the L^2 norm of $\dot{w}_\varepsilon$ is used to check that the limit w actually solves the problem (6).

Proof. The proof of this theorem follows the spirit of Theorem 4.2 of [26]. We will start with an uniform (in ε) estimate of $\|\dot{w}_\varepsilon\|^2_{L^2}$ and then we will use this estimate in weak form of the Euler equation to show the convergence of w_ε to the solution of (6). We will prove the theorem in the case $\alpha > 0$ and $\beta = 0$.

Uniform Estimate. Start form the differential equation in (5) and scalar multiply it by $(w'_\varepsilon - w^1)$:

$$\varepsilon^2 \alpha w_\varepsilon^{(4)} \cdot (w'_\varepsilon - w^1) - 2\varepsilon\alpha w^{(3)} \cdot (w'_\varepsilon - w^1) + \alpha \ddot{w} \cdot (w'_\varepsilon - w^1) + \nabla U \cdot (w'_\varepsilon - w^1) = 0,$$

then integrate this equation on the interval $(0,t)$, and using the boundary conditions (5) integrate by parts to obtain

$$\begin{aligned}
&\varepsilon^2 \alpha w_\varepsilon^{(3)}(t) \cdot (w'_\varepsilon - w^1) - \frac{\varepsilon^2\alpha}{2}|\ddot{w}_\varepsilon(t)|^2 + \frac{\varepsilon^2\alpha}{2}|\ddot{w}_\varepsilon(0)|^2 \\
&-2\varepsilon\alpha w_\varepsilon^{(3)}(t) \cdot (\dot{w}_\varepsilon(t) - w^1) + 2\varepsilon\alpha \int_0^t |\ddot{w}_\varepsilon(s)|^2\, ds + \frac{\alpha}{2}|\dot{w}_\varepsilon(t) - w^1|^2 \\
&+U(w_\varepsilon(t),t) - U(w^0,0) - \int_0^t \nabla U(w_\varepsilon(s),s) \cdot w^1\, ds - \int_0^t \partial_t U(w_\varepsilon(s),s)\, ds.
\end{aligned}$$

Now let us integrate this equality again in the interval $(0,T)$, therefore obtaining

$$\begin{aligned}
&\left(2\varepsilon - \frac{3}{2}\varepsilon^2\right) \int_0^T \alpha|\ddot{w}_\varepsilon(s)|\, ds + \frac{\varepsilon^2(1+T)}{2}\alpha|\ddot{w}_\varepsilon(0)| + \left(\frac{1}{2} - \varepsilon\right)\alpha|\dot{w}_\varepsilon(T) - w^1|^2 \\
&+2\varepsilon\alpha \int_0^T \int_0^\tau \ddot{w}_\varepsilon(s)\, ds d\tau + \frac{\alpha}{2}\int_0^T |\dot{w}_\varepsilon(s) - w^1|^2\, ds + U(w_\varepsilon(T),T) \\
&+\int_0^T U(w_\varepsilon(s),s)\, ds = \int_0^T \nabla U(w_\varepsilon(s),s) \cdot w^1 + \int_0^T \int_0^\tau \nabla U(w_\varepsilon(s),s) \cdot w^1\, ds d\tau \\
&+(1+T)U(w^0,0) + \int_0^T \int_0^\tau \partial_t U(w_\varepsilon(s),s)\, ds d\tau.
\end{aligned}$$

Now we can take all the positive (for ε small enough) terms to the right hand side to obtain

$$\begin{aligned}
\frac{\alpha}{2}\int_0^T |\dot{w}_\varepsilon - w^1|^2\, dt + \int_0^T U(w_\varepsilon(t),t)\, dt \leq &(1+T)U(w^0,0) \\
&+ (1+T)|w^1| \int_0^T |\nabla U(w_\varepsilon(t),t)|\, dt \\
&+ T\int_0^T |\partial_t U(w_\varepsilon(t),t)|\, dt.
\end{aligned}$$

Now using Eq. (1) we can choose δ to further reduce this inequality down to

$$\frac{\alpha}{2}\int_0^T |\dot{w}_\varepsilon - w^1|^2\,dt + \int_0^T U(w_\varepsilon(t),t)\,dt \leq c(T) + C(T)\int_0^T |w_\varepsilon(t)|^2\,dt, \tag{2}$$

where $c(T)$ and $C(T)$ are constant with respect to the parameter ε. Using Peter-Paul inequality we have that $|\dot{w}_\varepsilon - w^1|^2 \geq (1-\eta')|\dot{w}_\varepsilon|^2 + (1-1/\eta')|w^1|^2$ for all $\eta' > 0$. Similarly since $w_\varepsilon \in H^2$, we can write $w_\varepsilon(t) = w^0 + \int_0^t \dot{w}_\varepsilon$ and using Peter-Paul and Cauchy-Schwartz we also end up with the estimate $|w_\varepsilon - w^0| \geq (1-\eta)|w_\varepsilon| + (1-1/\eta)|w^0|$ for all $\eta > 0$, which implies

$$\int_0^T |w_\varepsilon(t)|^2\,dt \leq T\frac{1/\eta - 1}{1-\eta}|w^0|^2 + \frac{T^2}{1-\eta}\int_0^T |\dot{w}_\varepsilon(t)|^2\,dt. \tag{3}$$

Putting together Eq. (2) and (3) we finally obtain the wanted uniform bound $\alpha\|\dot{w}_\varepsilon\|_{L^2} \leq k(T)$, where $k(T)$ is a constant with respect to the parameter ε.

Convergence. Once we have this uniform bound we can complete the proof by arguing along the very same lines of the proof of Section 3.2 of [26] to obtain the thesis. $\square$