7 Supplementary Material

7.1 Proof of Proposition 2

Proof. The proof is a simpler, two-step variation of that of [5], which we refer to for additional details. For all $\varepsilon \geq 0$, let π_ε be the optimal plan for $d^2_{\mathbf{P}_\varepsilon}$, and suppose there exists π such that $\pi_\varepsilon \rightharpoonup \pi$ (which is possible up to subsequences). By definition of π_ε, we have that

$$\forall \varepsilon \geq 0, \int d^2_{\mathbf{P}_\varepsilon} \mathrm{d}\pi_\varepsilon \leq \int d^2_{\mathbf{P}_\varepsilon} \mathrm{d}\pi_{\mathrm{MK}}.$$

Since $d^2_{\mathbf{P}_\varepsilon}$ converges locally uniformly to $d^2_{\mathbf{V}_E} \stackrel{\text{def}}{=} (x, y) \rightarrow (x-y)^\top \mathbf{V}_E \mathbf{V}_E^\top (x-y)$, we get $\int d^2_{\mathbf{V}_E} \mathrm{d}\pi \leq \int d^2_{\mathbf{V}_E} \mathrm{d}\pi_{\mathrm{MK}}$. But by definition of π_{MK}, $(\pi_{\mathrm{MK}})_E \stackrel{\text{def}}{=} (p_E, p_E)_\sharp \pi_{\mathrm{MK}}$ is the optimal transport plan on E, therefore the last inequality implies $\pi_E = (\pi_{\mathrm{MK}})_E$.

Next, notice that the π_ε's all have the same marginals μ_E, ν_E on E and hence cannot perform better on E than π_{MK}. Therefore,

$$\begin{aligned}
\int_{E \times E} d^2_{\mathbf{V}_E} \mathrm{d}(\pi_{\mathrm{MK}}) + \varepsilon \int d^2_{\mathbf{V}_{E^\perp}} \mathrm{d}\pi_\varepsilon &\leq \int d^2_{\mathbf{P}_\varepsilon} \mathrm{d}\pi_\varepsilon \\
&\leq \int d^2_{\mathbf{P}_\varepsilon} \mathrm{d}\pi_{\mathrm{MK}} \\
&= \int_{E \times E} d^2_{\mathbf{V}_E} \mathrm{d}(\pi_{\mathrm{MK}})_E + \varepsilon \int d^2_{\mathbf{V}_{E^\perp}} \mathrm{d}\pi_{\mathrm{MK}}.
\end{aligned}$$

Hence, passing to the limit, $\int d^2_{\mathbf{V}_{E^\perp}} \mathrm{d}\pi \leq \int d^2_{\mathbf{V}_{E^\perp}} \mathrm{d}\pi_{\mathrm{MK}}$. Let us now disintegrate this inequality on $E \times E$ (using the equality $\pi_E = (\pi_{\mathrm{MK}})_E$):

$$\int \int_{E^\perp \times E^\perp} d^2_{\mathbf{V}_{E^\perp}} \mathrm{d}\pi_{(x_E, y_E)} \mathrm{d}(\pi_{\mathrm{MK}})_E \leq \int \int_{E^\perp \times E^\perp} d^2_{\mathbf{V}_{E^\perp}} \mathrm{d}(\pi_{\mathrm{MK}})_{(x_E, y_E)} \mathrm{d}(\pi_{\mathrm{MK}})_E.$$

Again, by definition, for (x_E, y_E) in the support of $(\pi_{\mathrm{MK}})_E$, $(\pi_{\mathrm{MK}})_{(x_E, y_E)}$ is the optimal transportation plan between μ_{x_E} and ν_{y_E}, and the previous inequality implies $\pi_{(x_E, y_E)} = (\pi_{\mathrm{MK}})_{(x_E, y_E)}$ for $(\pi_{\mathrm{MK}})_E$-a.e.(x_E, y_E), and finally $\pi = \pi_{\mathrm{MK}}$. Finally, by the a.c. hypothesis, all transport plans π_ε come from transport maps T_ε, which implies $T_\varepsilon \rightarrow T_{\mathrm{MK}}$ in $\mathrm{L}_2(\mu)$. ■

7.2 Proof of Proposition 3

Proof. Let $\mathbf{X} \subset \mathbb{R}^d$ be a compact, $\mu, \nu \in \mathcal{P}(\mathbf{X})$ be two a.c. measures, E a k-dimensional subspace which we identify w.l.o.g. with $\mathbb{R}^k$ and $\pi_{\mathrm{MI}} \in \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d)$ as in Definition 2. For $n \in \mathbb{N}$, let $\mu_n = \frac{1}{n}\sum_{i=1}^n \delta_{x_i}$, $\nu_n = \frac{1}{n}\sum_{i=1}^n \delta_{y_i}$ where the x_i (resp. y_i) are i.i.d. samples from μ (resp. ν). Let $t_n : \mathbb{R}^k \rightarrow \mathbb{R}^k$ be the Monge map from the projection on E $(p_E)_\sharp \mu_n$ of μ_n to that of ν_n, and $\pi_n \stackrel{\text{def}}{=} (\mathrm{Id}, t_n)_\sharp[(p_E)_\sharp \mu_n]$.

Up to points having the same projections on E (which under the a.c. assumption is a 0 probability event), t_n can be extended to a transport between μ_n and ν_n , whose transport plan we will denote γ_n.

Let $f \in C_b(\mathbf{X} \times \mathbf{X})$. Since we are on a compact, by density (given by the Stone-Weierstrass theorem) it is sufficient to consider functions of the form

$$f(x_1, ..., x_d; y_1, ..., y_d) = g(x_1, ..., x_k; y_1, ..., y_k) h(x_{k+1}, ..., x_d; y_{k+1}, ..., y_d).$$

We will use this along with the disintegrations of γ_n on $E \times E$ (denoted $(\gamma_n)_{x_{1:k}, y_{1:k}}$, $(x_{1:k}, y_{1:k}) \in E \times E$) to prove convergence:

$$\begin{aligned}
\int_{\mathbf{X} \times \mathbf{X}} f d\gamma_n &= \int_{\mathbf{X} \times \mathbf{X}} g(x_{1:k}, y_{1:k}) h(x_{k+1:d}, y_{k+1:d}) d\gamma_n \\
&= \int_{E \times E} g(x_{1:k}, y_{1:k}) d\pi_n \int h(x_{k+1:d}, y_{k+1:d}) d(\gamma_n)_{x_{1:k}, y_{1:k}} \\
&= \int_{E \times E} g(x_{1:k}, y_{1:k}) d\pi_n \int h(x_{k+1:d}, y_{k+1:d}) d(\mu_n)_{x_{1:k}} d(\nu_n)_{t_n(x_{1:k})}.
\end{aligned}$$

Then, we use (i) the Arzela-Ascoli theorem to get uniform convergence of t_n to T_E to get $d(\nu_n)_{t_n(x_{1:k})} \rightharpoonup d(\nu)_{T_E(x_{1:k})}$ and (ii) the convergence $\pi_n \rightharpoonup (p_E, p_E)_\sharp(\pi_{\text{MI}})$ to get

$$\begin{aligned}
&\int_{E\times E} g(x_{1:k}, y_{1:k}) d\pi_n \int h(x_{k+1:d}, y_{k+1:d}) d(\mu_n)_{x_{1:k}} d(\nu_n)_{t_n(x_{1:k})} \\
&\quad \to \int_{E\times E} g(x_{1:k}, y_{1:k}) d(p_E, p_E)_\sharp(\pi_{\text{MI}}) \int h(x_{k+1:d}, y_{k+1:d}) d(\mu)_{x_{1:k}} d(\nu)_{T_E(x_{1:k})} \\
&\quad = \int_{\mathbf{X}\times\mathbf{X}} f d\pi_{\text{MI}},
\end{aligned}$$

which concludes the proof in the compact case. ■

7.3 Proof of Proposition 4

Proof. Let $\mathbf{T}_E : \mathbf{A}_E^{-\frac{1}{2}}(\mathbf{A}_E^{\frac{1}{2}}\mathbf{B}_E\mathbf{A}_E^{\frac{1}{2}})^{\frac{1}{2}}\mathbf{A}_E^{-\frac{1}{2}}$ be the Monge map from $\mu_E \overset{\text{def}}{=} (p_E)_\sharp\mu$ and $\nu_E \overset{\text{def}}{=} (p_E)_\sharp\nu$. Let

$$V = \begin{pmatrix} | & & | & | & & | \\ v_1 & \dots & v_k & v_{k+1} & \dots & v_d \\ | & & | & | & & | \end{pmatrix} = (\mathbf{V}_E \quad \mathbf{V}_{E^\perp}) \in \mathbb{R}^{d\times d},$$

where $(v_1 \dots v_k)$ is an orthonormal basis of E and $(v_{k+1} \dots v_d)$ an orthonormal basis of $E^\perp$. Let us denote $X_E \overset{\text{def}}{=} p_E(X) \in \mathbb{R}^k$ and *mutatis mutandis* for $Y, E^\perp$. Denote $\mathbf{A}_E = p_E\mathbf{A}p_E^\top, \mathbf{A}_{E^\perp} = p_{E^\perp}\mathbf{A}p_{E^\perp}^\top, \mathbf{A}_{EE^\perp} = p_E\mathbf{A}p_{E^\perp}^\top$. With these notations, we decompose the derivation of $\mathbb{E}[XY^\top]$ along E and $E^\perp$:

$$\begin{aligned}
\mathbb{E}[XY^\top] = \mathbb{E}[\mathbf{V}_E X_E(\mathbf{V}_E Y_E)^\top] &+ \mathbb{E}[\mathbf{V}_{E^\perp} X_{E^\perp}(\mathbf{V}_{E^\perp} Y_{E^\perp})^\top] \\
&+ \mathbb{E}[\mathbf{V}_{E^\perp} X_{E^\perp}(\mathbf{V}_E Y_E)^\top] \\
&+ \mathbb{E}[\mathbf{V}_E X_E(\mathbf{V}_{E^\perp} Y_{E^\perp})^\top].
\end{aligned}$$

We can condition all four terms on X_E, and use point independence given coordinates on E which implies $(Y_E|X_E) = X_E$. The constraint $Y_E = \mathbf{T}_E X_E$ allows us to derive $\mathbb{E}\left[Y_{E^\perp}|X_E\right]$: indeed, it holds that

$$\begin{pmatrix} Y_E \\ Y_{E^\perp} \end{pmatrix} \sim \mathcal{N}\left(0_d, \begin{pmatrix} \mathbf{B}_E & \mathbf{B}_{EE^\perp} \\ \mathbf{B}_{EE^\perp}^\top & \mathbf{B}_{E^\perp} \end{pmatrix}\right),$$

which, using standard Gaussian conditioning properties, implies that

$$\mathbb{E}\left[Y_{E^\perp}|Y_E = \mathbf{T}_E X_E\right] = \mathbf{B}_{EE^\perp}^\top \mathbf{B}_E^{-1}\mathbf{T}_E X_E,$$

and therefore

$$\mathbb{E}\left[Y_{E^\perp}|\mathbf{P}_E(Y) = \mathbf{T}_E X_E\right] = V_{E^\perp}\mathbf{B}_{EE^\perp}^\top \mathbf{B}_E^{-1}\mathbf{V}_E^\top \mathbf{T}_E X_E.$$

Likewise,

$$\mathbb{E}\left[X_{E^\perp}|\mathbf{P}_E(X)\right] = \mathbf{V}_{E^\perp}\mathbf{A}_{EE^\perp}^\top \mathbf{A}_E^{-1}\mathbf{V}_E^\top X_E.$$

We now have all the ingredients necessary to the derivation of the four terms of $\mathbb{E}[XY^\top]$:

$$
\begin{aligned}
\mathbb{E}[\mathbf{V}_E X_E Y_E^\top \mathbf{V}_E^\top] &= \mathbf{V}_E \mathbb{E}_{X_E}\left[\mathbb{E}\left[X_E Y_E^\top | X_E\right]\right]\mathbf{V}_E^\top \\
&= \mathbf{V}_E \mathbb{E}_{X_E}\left[X_E \mathbb{E}\left[Y_E^\top | X_E\right]\right]\mathbf{V}_E^\top \\
&= \mathbf{V}_E \mathbb{E}_{X_E}\left[X_E X_E^\top \mathbf{T}_E^\top\right]\mathbf{V}_E^\top \\
&= \mathbf{V}_E \mathbb{E}_{X_E}\left[X_E X_E^\top\right]\mathbf{T}_E^\top \mathbf{V}_E^\top \\
&= \mathbf{V}_E \mathbf{A}_E \mathbf{T}_E \mathbf{V}_E^\top
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{E}[\mathbf{V}_E X_E Y_{E^\perp}^\top \mathbf{V}_{E^\perp}^\top] &= \mathbf{V}_E \mathbb{E}_{X_E}\left[\mathbb{E}[X_E Y_{E^\perp}^\top | X_E\right]\mathbf{V}_{E^\perp}^\top \\
&= \mathbf{V}_E \mathbb{E}_{X_E}\left[X_E \mathbb{E}\left[Y_{E^\perp}^\top | X_E = \mathbf{T}_E X_E\right]\right]\mathbf{V}_{E^\perp}^\top \\
&= \mathbf{V}_E \mathbb{E}_{X_E}\left[X_E \left(V_{E^\perp} \mathbf{B}_{EE^\perp}^\top \mathbf{B}_E^{-1} \mathbf{V}_E^\top \mathbf{T}_E X_E\right)\right)^\top\right]\mathbf{V}_{E^\perp}^\top \\
&= \mathbf{V}_E \mathbb{E}_{X_E}\left[X_E X_E^\top\right]\mathbf{T}_E^\top \mathbf{V}_E \mathbf{B}_{V_E}^{-\top} \mathbf{B}_{V_{EE^\perp}} \mathbf{V}_{E^\perp}^\top \\
&= \mathbf{V}_E \mathbf{A}_E \mathbf{T}_E \mathbf{V}_E \mathbf{B}_E^{-1} \mathbf{B}_{V_{EE^\perp}} \mathbf{V}_{E^\perp}^\top \\
&= \mathbf{V}_E \mathbf{A}_E \mathbf{T}_E \mathbf{V}_E \mathbf{B}_E^{-1} \mathbf{V}_E^\top \mathbf{B}_{EE^\perp} \mathbf{V}_{E^\perp}^\top
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{E}[\mathbf{V}_{E^\perp} X_{E^\perp} Y_E^\top \mathbf{V}_E^\top] &= \mathbf{V}_{E^\perp} \mathbb{E}_{X_E}\left[\mathbb{E}[X_{E^\perp} Y_E^\top | X_E\right]\mathbf{V}_E^\top \\
&= \mathbf{V}_{E^\perp} \mathbb{E}_{X_E}\left[\mathbb{E}\left[X_{E^\perp} | X_E\right] X_E^\top \mathbf{T}_E^\top\right]\mathbf{V}_E^\top \\
&= \mathbf{V}_{E^\perp} \mathbb{E}_{X_E}\left[\mathbf{A}_{EE^\perp}^\top \mathbf{A}_E^{-1} X_E X_E^\top \mathbf{T}_E^\top\right]\mathbf{V}_E^\top \\
&= \mathbf{V}_{E^\perp} \mathbf{V}_{E^\perp} \mathbf{A}_{EE^\perp}^\top \mathbf{A}_E^{-1} \mathbf{V}_E^\top \mathbf{A} \mathbf{T}_E \mathbf{V}_E^\top \\
&= \mathbf{V}_{E^\perp} \mathbf{V}_{E^\perp} \mathbf{A}_{EE^\perp}^\top \mathbf{T}_E \mathbf{V}_E^\top \\
&= \mathbf{V}_{E^\perp} \mathbf{A}_{EE^\perp}^\top \mathbf{T}_E \mathbf{V}_E^\top
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{E}[\mathbf{V}_{E^\perp} X_{E^\perp} Y_{E^\perp}^\top \mathbf{V}_{E^\perp}^\top] &= V_{E^\perp E} \mathbb{E}_{X_E}\left[\mathbb{E}[X_{E^\perp} | X_E\right]\mathbb{E}\left[Y_{E^\perp}^\top | X_E\right]\mathbf{V}_{E^\perp}^\top \\
&= \mathbf{V}_{E^\perp} \mathbb{E}_{X_E}\left[\mathbf{V}_{E^\perp} \mathbf{A}_{EE^\perp}^\top \mathbf{A}_E^{-1} \mathbf{V}_E^\top X_E X_E^\top \mathbf{T}_E^\top \mathbf{V}_E \mathbf{B}_{V_E}^{-\top} \mathbf{B}_{EE^\perp}\right]\mathbf{V}_{E^\perp}^\top \\
&= \mathbf{V}_{E^\perp} \mathbf{A}_{EE^\perp}^\top \mathbf{A}_E^{-1} \mathbf{V}_E^\top \mathbf{A}_E \mathbf{T}_E \mathbf{V}_E \mathbf{B}_E^{-1} \mathbf{B}_{EE^\perp} \mathbf{V}_{E^\perp}^\top \\
&= \mathbf{V}_{E^\perp} \mathbf{A}_{EE^\perp}^\top \mathbf{T}_E \mathbf{B}_E^{-1} \mathbf{B}_{EE^\perp} \mathbf{V}_{E^\perp}^\top \\
&= V_{E^\perp} \mathbf{A}_{EE^\perp}^\top \mathbf{T}_E \mathbf{V}_E \mathbf{B}_{V_E}^{-1} \mathbf{V}_E^\top \mathbf{B}_{EE^\perp},
\end{aligned}
$$

Let $\gamma \overset{\text{def}}{=} \mathcal{N}(0_{2d}, \Sigma_{\pi_E})$. γ, is well defined, since Σ_{π_E} is the covariance matrix of π_E and is thus PSD. From then, γ clearly has marginals $\mathcal{N}(0_d, \mathbf{A})$ and $\mathcal{N}(0_d, \mathbf{B})$, and is such that $(p_E, p_E)_\sharp \gamma$ is a centered Gaussian distribution with covariance matrix

$$
\begin{pmatrix} p_E & 0_{d\times d} \\ 0_{d\times d} & p_E \end{pmatrix}
\begin{pmatrix} \mathbf{A} & \mathbb{E}_\pi[XY^\top] \\ \mathbb{E}_\pi[YX^\top] & \mathbf{B} \end{pmatrix}
\begin{pmatrix} p_E & 0_{d\times d} \\ 0_{d\times d} & p_E \end{pmatrix}
= \begin{pmatrix} \mathbf{A}_E & \mathbf{A}_E \mathbf{T}_E \\ \mathbf{T}_E \mathbf{A}_E & \mathbf{B}_E \end{pmatrix},
$$

where we use that $p_E p_E = p_E$ and $p_E p_{E^\perp} = 0$. From the $k = d$ case, we recognise the covariance matrix of the optimal transport between centered Gaussians with covariance matrices $\mathbf{A}_E$ and $\mathbf{B}_E$, which proves that the marginal of γ over $E \times E$ is the optimal transport between μ_E and ν_E.

To complete the proof, there remains to show that the disintegration of γ on $E \times E$ is the product law. Denote

$$
\begin{aligned}
\mathbf{C} &\overset{\text{def}}{=} \mathbb{E}[XY^\top] \\
&= \mathbf{V}_E \mathbf{A}_E \mathbf{T}_E \left(\mathbf{V}_E^\top + (\mathbf{B}_E)^{-1} \mathbf{V}_E^\top \mathbf{B}_{EE^\perp}\right) + \mathbf{V}_{E^\perp} \mathbf{A}_{E^\perp E} \mathbf{T}_{V_E} \left(\mathbf{V}_E^\top + (\mathbf{B}_{V_E})^{-1} \mathbf{V}_E^\top \mathbf{B}_{EE^\perp}\right) \\
&= \left(\mathbf{V}_E \mathbf{A}_E + \mathbf{V}_{E^\perp} \mathbf{A}_{E^\perp E}\right) \mathbf{T}_E \left(\mathbf{V}_E^\top + (\mathbf{B}_E)^{-1} \mathbf{B}_{EE^\perp} \mathbf{V}_{E^\perp}^\top\right),
\end{aligned}
$$

and let $\Sigma_{\pi_{\text{MI}}} = \begin{pmatrix} \mathbf{A} & \mathbb{E}[XY^\top] \\ \mathbb{E}[YX^\top] & \mathbf{B} \end{pmatrix}$ as in Prop. 4. It holds that

$$\begin{aligned}
\mathbf{C}_E &\stackrel{\text{def}}{=} \mathbf{V}_E^\top \mathbf{C} \mathbf{V}_E = \mathbf{A}_E \mathbf{T}_E \\
\mathbf{C}_{E^\perp} &\stackrel{\text{def}}{=} \mathbf{V}_{E^\perp}^\top \mathbf{C} \mathbf{V}_E = \mathbf{A}_{E^\perp E} \mathbf{T}_E (\mathbf{B}_E)^{-1} \mathbf{B}_{EE^\perp} \\
\mathbf{C}_{EE^\perp} &\stackrel{\text{def}}{=} \mathbf{V}_E^\top \mathbf{C} \mathbf{V}_{E^\perp} = \mathbf{A}_E \mathbf{T}_E (\mathbf{B}_E)^{-1} \mathbf{B}_{EE^\perp} \\
\mathbf{C}_{E^\perp E} &\stackrel{\text{def}}{=} \mathbf{V}_{E^\perp}^\top \mathbf{C} \mathbf{V}_E = \mathbf{A}_{E^\perp E} \mathbf{T}_E.
\end{aligned}$$

Therefore, if $(X, Y) \sim \gamma$, then

$$\operatorname{Cov}\begin{pmatrix} X_{E^\perp} \\ Y_{E^\perp} \\ X_E \\ Y_E \end{pmatrix} = \begin{pmatrix} \mathbf{A}_{E^\perp} & \mathbf{C}_{E^\perp} & \mathbf{A}_{E^\perp E} & \mathbf{C}_{E^\perp E} \\ \mathbf{C}_{E^\perp} & \mathbf{B}_{E^\perp} & \mathbf{C}_{EE^\perp}^\top & \mathbf{B}_{E^\perp E} \\ \mathbf{A}_{EE^\perp} & \mathbf{C}_{EE^\perp} & \mathbf{A}_E & \mathbf{C}_E \\ \mathbf{C}_{E^\perp E}^\top & \mathbf{B}_{EE^\perp} & \mathbf{C}_E & \mathbf{B}_E \end{pmatrix},$$

and therefore

$$\operatorname{Cov}\begin{pmatrix} X_{E^\perp} & |X_E \\ Y_{E^\perp} & |Y_E \end{pmatrix} = \begin{pmatrix} \mathbf{A}_{E^\perp} & \mathbf{C}_{E^\perp} \\ \mathbf{C}_{E^\perp} & \mathbf{B}_{E^\perp} \end{pmatrix} - \begin{pmatrix} \mathbf{A}_{E^\perp E} & \mathbf{C}_{E^\perp E} \\ \mathbf{C}_{EE^\perp}^\top & \mathbf{B}_{E^\perp E} \end{pmatrix} \begin{pmatrix} \mathbf{A}_E & \mathbf{C}_E \\ \mathbf{C}_E & \mathbf{B}_E \end{pmatrix}^\dagger \begin{pmatrix} \mathbf{A}_{EE^\perp} & \mathbf{C}_{EE^\perp} \\ \mathbf{C}_{E^\perp E}^\top & \mathbf{B}_{EE^\perp} \end{pmatrix},$$

where $\mathbf{M}^\dagger$ denotes the Moore-Penrose pseudo-inverse of $\mathbf{M}$. In the present case, one can check that

$$\begin{pmatrix} \mathbf{A}_E & \mathbf{C}_E \\ \mathbf{C}_E & \mathbf{B}_E \end{pmatrix}^\dagger = \frac{1}{4} \begin{pmatrix} \mathbf{A}_E^{-1} & \mathbf{A}_E^{-1} \mathbf{T}_E^{-1} \\ \mathbf{T}_E^{-1} \mathbf{A}_E^{-1} & \mathbf{B}_E^{-1} \end{pmatrix},$$

which gives, after simplification

$$\begin{pmatrix} \mathbf{A}_{E^\perp E} & \mathbf{C}_{E^\perp E} \\ \mathbf{C}_{EE^\perp}^\top & \mathbf{B}_{E^\perp E} \end{pmatrix} \begin{pmatrix} \mathbf{A}_E & \mathbf{C}_E \\ \mathbf{C}_E & \mathbf{B}_E \end{pmatrix}^\dagger \begin{pmatrix} \mathbf{A}_{EE^\perp} & \mathbf{C}_{EE^\perp} \\ \mathbf{C}_{E^\perp E}^\top & \mathbf{B}_{EE^\perp} \end{pmatrix} = \begin{pmatrix} \mathbf{A}_{E^\perp E} \mathbf{A}_E^{-1} \mathbf{A}_{EE^\perp} & \mathbf{C}_{E^\perp} \\ \mathbf{C}_{E^\perp} & \mathbf{B}_{E^\perp E} \mathbf{B}_E^{-1} \mathbf{B}_{EE^\perp} \end{pmatrix},$$

and thus

$$\begin{aligned}
\operatorname{Cov}\begin{pmatrix} X_{E^\perp} & |X_E \\ Y_{E^\perp} & |Y_E \end{pmatrix} &= \begin{pmatrix} \mathbf{A}_{E^\perp} - \mathbf{A}_{E^\perp E} (\mathbf{A}_E)^{-1} \mathbf{A}_{EE^\perp} & 0_d \\ 0_d & \mathbf{B}_{E^\perp} - \mathbf{B}_{E^\perp E} (\mathbf{B}_E)^{-1} \mathbf{B}_{EE^\perp} \end{pmatrix} \\
&= \begin{pmatrix} \operatorname{Cov}(X_{E^\perp} | X_E) & 0_d \\ 0_d & \operatorname{Cov}(Y_{E^\perp} | Y_E) \end{pmatrix},
\end{aligned}$$

that is, the conditional laws of $X_{E^\perp}$ given X_E and $Y_{E^\perp}$ given Y_E are independent under γ.

■