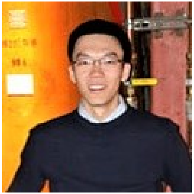
Faster width-dependent algorithm for mixed packing and covering LPs





Digvijay Boob [1] Saurabh Sawlani[2] Di Wang[3]

[1]Inudstrial and Systems Engineering, Gerogia Tech
[2]College of Computing, Georgia Tech
[3]Google Research

Oral Presentation at NeurIPS 2019
December, 2019

Outline

Mixed Packing and Covering(MPC) LP

Does there exists an $x \in \square^n := \{x \geqslant \mathbf{0}_n, \|x\|_\infty \leqslant 1\}$ such that

$$Px \leqslant \mathbf{1}_p, \qquad \text{(Packing constraints)},$$
$$Cx \geqslant \mathbf{1}_c, \qquad \text{(Covering Constraints)},$$

where $P, C \geqslant 0$.

Def: We say that x is an ε-approximate solution to the MPC problem if x satisfies $x \in \square^n, Px \leqslant (1+\varepsilon)\mathbf{1}_p, Cx \geqslant (1-\varepsilon)\mathbf{1}_c$.

Mixed Packing and Covering(MPC) LP

Does there exists an $x \in \square^n := \{x \geqslant \mathbf{0}_n, \|x\|_\infty \leqslant 1\}$ such that

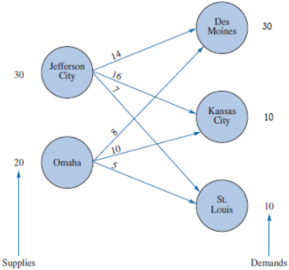
$$Px \leqslant \mathbf{1}_p, \qquad \text{(Packing constraints)},$$
$$Cx \geqslant \mathbf{1}_c, \qquad \text{(Covering Constraints)},$$

where $P, C \geqslant 0$.

Def: We say that x is an ε-approximate solution to the MPC problem if x satisfies $x \in \square^n, Px \leqslant (1+\varepsilon)\mathbf{1}_p, Cx \geqslant (1-\varepsilon)\mathbf{1}_c$.

Application: Optimal Transport Problem

Optimal transport is a problem of computing Wasserstein distance between two n-dimensional distributions.

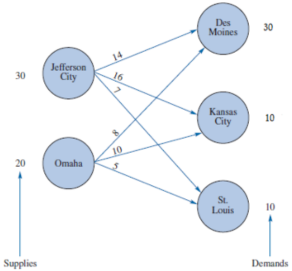


Modeled as LP:

$$\min_X \langle C, X \rangle$$
$$\text{s.t. } \{X \geq 0, X\mathbf{1} = u, X^T\mathbf{1} = v\}.$$

Application: Optimal Transport Problem

Optimal transport is a problem of computing Wasserstein distance between two n-dimensional distributions.



Modeled as LP:

$$\min_{X} \langle C, X \rangle$$
$$\text{s.t. } \{X \geqslant 0, X\mathbf{1} = u, X^T\mathbf{1} = v\}.$$

Motivation

Mixed Packing-Covering LPs

- Pure Packing
  - Bipartite matching
- Pure Covering
  - Minimum Set Cover

- Zero-sum Matrix Games
- Optimal Transport
- Multi-commodity flow
- Mechanism Design
- Positive Linear Systems
- Scheduling
- X-Ray Tomography

Previous Results

Def: Width w is maximum non-zeros in any row of P or C.

Table: Runtime for obtaining ε-approximate solution:

	Runtime	Comments
Nesterov	$\widetilde{O}(w\sqrt{n}\varepsilon^{-1})$	width-dependent
Young 2014	$\widetilde{O}(\varepsilon^{-4})$	
Mahoney *et al.*	$\widetilde{O}(\varepsilon^{-3})$	
Our work	$\widetilde{O}(w\varepsilon^{-1})$	width-dependent

Previous Results

Def: Width w is maximum non-zeros in any row of P or C.

Table: Runtime for obtaining ε-approximate solution:

	Runtime	**Comments**
Nesterov	$\widetilde{O}(w\sqrt{n}\varepsilon^{-1})$	width-dependent
Young 2014	$\widetilde{O}(\varepsilon^{-4})$	
Mahoney *et al.*	$\widetilde{O}(\varepsilon^{-3})$	
Our work	$\widetilde{O}(w\varepsilon^{-1})$	width-dependent

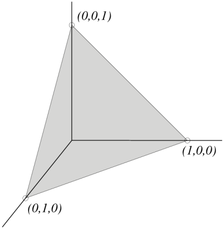
Saddle Point Problem (SPP) Reformulation

- Reformulate the MPC problem as a SPP:

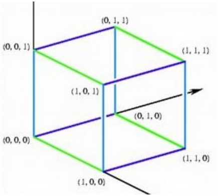
$$\min_{x \in \square^n} \max_{y \in \Delta^p, z \in \Delta^c} L(x, y, z)$$

- $u := (x, y, z)$ and $\mathcal{U} := \square^n \times \Delta^p \times \Delta^c$.
- Convergence: $u \in \mathcal{U}$ s.t. **primal-dual gap** $\mathrm{Gap}(u) := \sup_{\bar{u} \in \mathcal{U}} L(x, \bar{y}, \bar{z}) - L(\bar{x}, y, z)$ is small ($Q(u) \leqslant \varepsilon$).
- u is ε-SPP then either
  1. x is an ε-approx solution to MPC, or
  2. We obtain a certificate of infeasibility.

Δ^3



$\square^3$



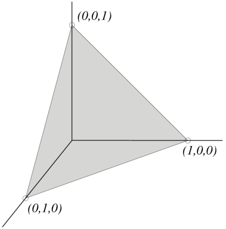
Saddle Point Problem (SPP) Reformulation

- Reformulate the MPC problem as a SPP:

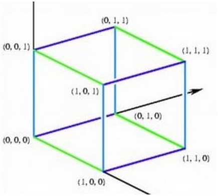
$$\min_{x \in \square^n} \max_{y \in \Delta^p, z \in \Delta^c} L(x, y, z)$$

- $u := (x, y, z)$ and $\mathcal{U} := \square^n \times \Delta^p \times \Delta^c$.
- Convergence: $u \in \mathcal{U}$ s.t. **primal-dual gap** $\mathrm{Gap}(u) := \sup_{\bar{u} \in \mathcal{U}} L(x, \bar{y}, \bar{z}) - L(\bar{x}, y, z)$ is small ($Q(u) \leqslant \varepsilon$).
- u is ε-SPP then either
  1. x is an ε-approx solution to MPC, or
  2. We obtain a certificate of infeasibility.

Δ^3



$\square^3$



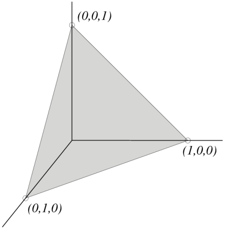
Saddle Point Problem (SPP) Reformulation

- Reformulate the MPC problem as a SPP:

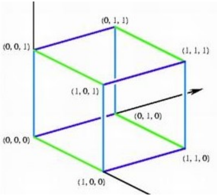
$$\min_{x \in \square^n} \max_{y \in \Delta^p, z \in \Delta^c} L(x, y, z)$$

- $u := (x, y, z)$ and $\mathcal{U} := \square^n \times \Delta^p \times \Delta^c$.
- Convergence: $u \in \mathcal{U}$ s.t. **primal-dual gap** $\mathrm{Gap}(u) := \sup_{\bar{u} \in \mathcal{U}} L(x, \bar{y}, \bar{z}) - L(\bar{x}, y, z)$ is small ($Q(u) \leqslant \varepsilon$).
- u is ε-SPP then either
  1. x is an ε-approx solution to MPC, or
  2. We obtain a certificate of infeasibility.

Δ^3



$\square^3$



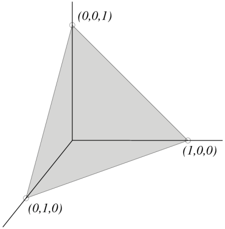
Saddle Point Problem (SPP) Reformulation

- Reformulate the MPC problem as a SPP:

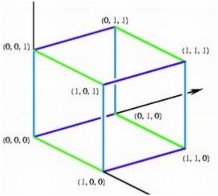
$$\min_{x \in \square^n} \max_{y \in \Delta^p, z \in \Delta^c} L(x, y, z)$$

- $u := (x, y, z)$ and $\mathcal{U} := \square^n \times \Delta^p \times \Delta^c$.
- Convergence: $u \in \mathcal{U}$ s.t. **primal-dual gap** $\mathrm{Gap}(u) := \sup_{\bar{u} \in \mathcal{U}} L(x, \bar{y}, \bar{z}) - L(\bar{x}, y, z)$ is small ($Q(u) \leqslant \varepsilon$).
- u is ε-SPP then either
  1. x is an ε-approx solution to MPC, or
  2. We obtain a certificate of infeasibility.

Δ^3



$\square^3$



Standard Methods

- General Problem: $\min_{w \in X} f(w)$
- Regularized problem: $\min_{w \in X} f(w) + \phi(w)$
- ϕ is strongly convex on X.
- Rate of convergence: e.g. Nesterov's accelerated methods: $\frac{1}{\varepsilon} \times$ range of ϕ on set X.

Standard Methods

- General Problem: $\min_{w \in X} f(w)$
- Regularized problem: $\min_{w \in X} f(w) + \phi(w)$
- ϕ is strongly convex on X.
- Rate of convergence: e.g. Nesterov's accelerated methods:

$$\frac{1}{\varepsilon} \times \text{range of } \phi \text{ on set } X.$$

Standard Methods

- General Problem: $\min_{w \in X} f(w)$
- Regularized problem: $\min_{w \in X} f(w) + \phi(w)$
- ϕ is strongly convex on X.
- Rate of convergence: e.g. Nesterov's accelerated methods:

$$\frac{1}{\varepsilon} \times \text{range of } \phi \text{ on set } X.$$

Standard Methods + MPC

- General Problem: $\min_{w \in X} f(w) + \phi(w)$
- For the case of MPC problems:
  1. f is primal-dual gap Q.
  2. w is joint primal-dual variable u.
  3. X is joint domain $\mathcal{U}(= \square^n \times \Delta^p \times \Delta^c)$.
  4. ϕ is strongly convex regularizers $\phi_1(x) + \phi_2(y) + \phi_3(z)$.
- Algorithm of choice: Nesterov's Dual Extrapolation
- Range of regularizers $\widetilde{\Theta}(w\sqrt{n})$
- Range above tight for strongly convex regularizer on $\square^n$.

Standard Methods + MPC

- General Problem: $\min_{w \in X} f(w) + \phi(w)$
- For the case of MPC problems:
  1. f is primal-dual gap Q.
  2. w is joint primal-dual variable u.
  3. X is joint domain $\mathcal{U}(= \square^n \times \Delta^p \times \Delta^c)$.
  4. ϕ is strongly convex regularizers $\phi_1(x) + \phi_2(y) + \phi_3(z)$.
- Algorithm of choice: Nesterov's Dual Extrapolation
- Range of regularizers $\widetilde{\Theta}(w\sqrt{n})$
- Range above tight for strongly convex regularizer on $\square^n$.

Standard Methods + MPC

- General Problem: $\min_{w \in X} f(w) + \phi(w)$
- For the case of MPC problems:
  1. f is primal-dual gap Q.
  2. w is joint primal-dual variable u.
  3. X is joint domain $\mathcal{U}(= \square^n \times \Delta^p \times \Delta^c)$.
  4. ϕ is strongly convex regularizers $\phi_1(x) + \phi_2(y) + \phi_3(z)$.
- Algorithm of choice: Nesterov's Dual Extrapolation
- Range of regularizers $\widetilde{\Theta}(w\sqrt{n})$
- Range above tight for strongly convex regularizer on $\square^n$.

Standard Methods + MPC

- General Problem: $\min_{w \in X} f(w) + \phi(w)$
- For the case of MPC problems:
  1. f is primal-dual gap Q.
  2. w is joint primal-dual variable u.
  3. X is joint domain $\mathcal{U}(= \square^n \times \Delta^p \times \Delta^c)$.
  4. ϕ is strongly convex regularizers $\phi_1(x) + \phi_2(y) + \phi_3(z)$.
- Algorithm of choice: Nesterov's Dual Extrapolation
- Range of regularizers $\widetilde{\Theta}(w\sqrt{n})$
- Range above tight for strongly convex regularizer on $\square^n$.

Standard Methods + MPC

- General Problem: $\min_{w \in X} f(w) + \phi(w)$
- For the case of MPC problems:
  1. f is primal-dual gap Q.
  2. w is joint primal-dual variable u.
  3. X is joint domain $\mathcal{U}(= \square^n \times \Delta^p \times \Delta^c)$.
  4. ϕ is strongly convex regularizers $\phi_1(x) + \phi_2(y) + \phi_3(z)$.
- Algorithm of choice: Nesterov's Dual Extrapolation
- Range of regularizers $\underbrace{\widetilde{\Theta}(w\sqrt{n})}_{\text{Tight}}$
- Range above tight for strongly convex regularizer on $\square^n$.

Area Convexity

Area convexity [Sherman 2017] :

1. Is weaker than strong convexity. One can obtain area convex regularizer with small range over ℓ_∞-ball.
2. Still good enough to obtain $O(\text{range of regularizer} \times \frac{1}{\varepsilon})$ convergence.

Definition

- Strong convexity: for all $t, u \in K$
  $\phi(\frac{t+u}{2}) \leqslant \frac{1}{2}(\phi(t) + \phi(u)) - \frac{1}{2}\|t - u\|^2$.
- **Def:** A function ϕ is area convex w.r.t. matrix M on convex set K iff for any $t, u, v \in K$,
  $\phi(\frac{t+u+v}{3}) \leqslant \frac{1}{3}(\phi(t) + \phi(u) + \phi(v)) - \frac{1}{3\sqrt{3}} \underbrace{(v - u)^T M(u - t)}_{\text{'area' of } \Delta(tuv)}$.

Definition

- Strong convexity: for all $t, u \in K$
  $\phi(\frac{t+u}{2}) \leqslant \frac{1}{2}(\phi(t) + \phi(u)) - \frac{1}{2}\|t-u\|^2$.
- **Def:** A function ϕ is area convex w.r.t. matrix M on convex set K iff for any $t, u, v \in K$,
  $\phi(\frac{t+u+v}{3}) \leqslant \frac{1}{3}\left(\phi(t) + \phi(u) + \phi(v)\right) - \frac{1}{3\sqrt{3}} \underbrace{(v-u)^T M (u-t)}_{\text{`area' of } \Delta(tuv)}$.

An Example

- For any $t, u \in K$, area convex ϕ requires mere convexity:$\phi(\frac{t+u}{2}) \leqslant \frac{1}{2}(\phi(t) + \phi(u))$.
- Consider $\gamma(x, y) = yx \log x + 2y \log y$.

  Area convex w.r.t. $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ on set $0 \leqslant x, y \leqslant 1$.

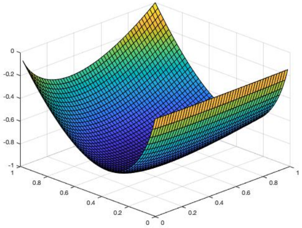


Figure: Auxiliary view

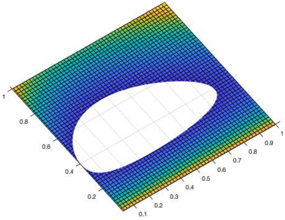


Figure: Level set $\gamma(x, y) \leqslant -0.5$

Area Convexity + MPC

- Use $\phi : \mathcal{U} \to [-\rho, 0]$ as area convex regularizer w.r.t. a matrix depending on P and C on set $\mathcal{U}$.
- Area convexity: relaxed requirement, we can show ϕ for which $\rho = O(\|P\|_\infty \log p + \|C\|_\infty \log c)$
- This ρ is of order width, w of MPC, **gets rid of the $\sqrt{n}$ factor**.

Area Convexity + MPC

- Use $\phi : \mathcal{U} \to [-\rho, 0]$ as area convex regularizer w.r.t. a matrix depending on P and C on set $\mathcal{U}$.
- Area convexity: relaxed requirement, we can show ϕ for which $\rho = O(\|P\|_\infty \log p + \|C\|_\infty \log c)$
- This ρ is of order width, w of MPC, **gets rid of the $\sqrt{n}$ factor**.

Area Convexity + MPC

- Use $\phi : \mathcal{U} \to [-\rho, 0]$ as area convex regularizer w.r.t. a matrix depending on P and C on set $\mathcal{U}$.
- Area convexity: relaxed requirement, we can show ϕ for which $\rho = O(\|P\|_\infty \log p + \|C\|_\infty \log c)$
- This ρ is of order width, w of MPC, **gets rid of the $\sqrt{n}$ factor**.

Salient Features of Our Regularizer

- Standard regularization: $\phi_1(x) + \phi_2(y)$.
- Our regularization contains terms of the following type:

$$y_j P_{ij} x_j \log x_j.$$

- Interaction of dual variable y and primal variable x.
- Standard case: separate regularization of primal and dual variable as $\phi_1(x)$ and $\phi_2(y)$.
- Depends on the problem matrix P and C. Explores the structure of the problem
- Independent to problem matrix P and C.

Salient Features of Our Regularizer

- Standard regularization: $\phi_1(x) + \phi_2(y)$.
- Our regularization contains terms of the following type:

$$y_j P_{ij} x_j \log x_j.$$

- Interaction of dual variable y and primal variable x.
- Standard case: separate regularization of primal and dual variable as $\phi_1(x)$ and $\phi_2(y)$.
- Depends on the problem matrix P and C. Explores the structure of the problem
- Independent to problem matrix P and C.

Salient Features of Our Regularizer

- Standard regularization: $\phi_1(x) + \phi_2(y)$.
- Our regularization contains terms of the following type:

$$y_j P_{ij} x_j \log x_j.$$

- Interaction of dual variable y and primal variable x.
- Standard case: separate regularization of primal and dual variable as $\phi_1(x)$ and $\phi_2(y)$.
- Depends on the problem matrix P and C. Explores the structure of the problem
- Independent to problem matrix P and C.

Summary

- Strongly convex regularizer and ℓ_∞-barrier.
- Explicit area convex regularizer for MPC which circumvents the ℓ_∞-barrier.
  1. Area convexity weaker than strong convexity. Range of the regularizer can be made $\widetilde{O}(w)$ on ℓ_∞-ball
  2. Still suffices to obtain $\widetilde{O}(\frac{w}{\varepsilon})$ convergence.
- See more details:
  Poster: #**232**
  Paper: *Faster width-dependent algorithms for mixed packing and covering LPs, arXiv 2019.*

Summary

- Strongly convex regularizer and ℓ_∞-barrier.
- Explicit area convex regularizer for MPC which circumvents the ℓ_∞-barrier.
  1. Area convexity weaker than strong convexity. Range of the regularizer can be made $\widetilde{O}(w)$ on ℓ_∞-ball
  2. Still suffices to obtain $\widetilde{O}(\frac{w}{\varepsilon})$ convergence.
- See more details:
  Poster: #**232**
  Paper: *Faster width-dependent algorithms for mixed packing and covering LPs, arXiv 2019.*

Summary

- Strongly convex regularizer and ℓ_∞-barrier.
- Explicit area convex regularizer for MPC which circumvents the ℓ_∞-barrier.
  1. Area convexity weaker than strong convexity. Range of the regularizer can be made $\widetilde{O}(w)$ on ℓ_∞-ball
  2. Still suffices to obtain $\widetilde{O}(\frac{w}{\varepsilon})$ convergence.
- See more details:

  Poster: #**232**

  Paper: *Faster width-dependent algorithms for mixed packing and covering LPs, arXiv 2019.*

Thanks!

Questions?