Online Forecasting of Total-Variation-bounded Sequences

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Abstract

We consider the problem of online forecasting of sequences of length n with total-variation at most C_n using observations contaminated by independent σ-subgaussian noise. We design an $O(n \log n)$-time algorithm that achieves a cumulative square error of $\tilde{O}(n^{1/3} C_n^{2/3} \sigma^{4/3} + C_n^2)$ with high probability. We also prove a lower bound that matches the upper bound in all parameters (up to a $\log(n)$ factor). To the best of our knowledge, this is the first *polynomial-time* algorithm that achieves the optimal $O(n^{1/3})$ rate in forecasting total variation bounded sequences and the first algorithm that *adapts to unknown* C_n. Our proof techniques leverage the special localized structure of Haar wavelet basis and the adaptivity to unknown smoothness parameters in the classical wavelet smoothing [Donoho et al., 1998]. We also compare our model to the rich literature of dynamic regret minimization and nonstationary stochastic optimization, where our problem can be treated as a special case. We show that the workhorse in those settings — online gradient descent and its variants with a fixed restarting schedule — are instances of a class of *linear forecasters* that require a suboptimal regret of $\tilde{\Omega}(\sqrt{n})$. This implies that the use of more adaptive algorithms is necessary to obtain the optimal rate.

1 Introduction

Nonparametric regression is a fundamental class of problems that has been studied for more than half a century in statistics and machine learning [Nadaraya, 1964, De Boor et al., 1978, Wahba, 1990, Donoho et al., 1998, Mallat, 1999, Scholkopf and Smola, 2001, Rasmussen and Williams, 2006]. It solves the following problem:

- Let $y_i = f(u_i) + \text{Noise}$ for $i = 1, ..., n$. How can we estimate a function f using data points $(u_1, y_1), ..., (u_n, y_n)$ and the knowledge that f belongs to a function class $\mathcal{F}$?

Function class $\mathcal{F}$ typically imposes only weak regularity assumptions on the function f such as boundedness and smoothness, which makes nonparametric regression widely applicable to many real-life applications especially those with unknown physical processes.

A recent and successful class of nonparametric regression technique called trend filtering [Steidl et al., 2006, Kim et al., 2009, Tibshirani, 2014, Wang et al., 2014] was shown to have the property of *local adaptivity* [Mammen and van de Geer, 1997] in both theory and practice. We say a nonparametric regression technique is *locally adaptive* if it can cater to local differences in smoothness, hence allowing more accurate estimation of functions with varying smoothness and abrupt changes. For example, for functions with bounded total variation (when $\mathcal{F}$ is a total variation class), standard nonparametric regression techniques such as kernel smoothing and smoothing splines have a mean square error (MSE) of $O(n^{-1/2})$ while trend filtering has the optimal $O(n^{-2/3})$.

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Trend filtering is, however, a batch learning algorithm where one observes the entire dataset ahead of the time and makes inference about the past. This makes it inapplicable to the many time series problems that motivate the study of trend filtering in the first place [Kim et al., 2009]. These include influenza forecasting, inventory planning, economic policy-making, financial market prediction and so on. In particular, it is unclear whether the advantage of trend filtering methods in estimating functions with heterogeneous smoothness (e.g., sharp changes) would carry over to the online forecasting setting. The focus of this work is in developing theory and algorithms for locally adaptive online forecasting which predicts the immediate future value of a function with heterogeneous smoothness using only noisy observations from the past.

1.1 Problem Setup

1. Fix action time intervals $1, 2, ..., n$
2. The player declares a forecasting strategy $\mathcal{A}_i : \mathbb{R}^{i-1} \to \mathbb{R}$ for $i = 1, ..., n$.
3. An adversary chooses a sequence $\theta_{1:n} = [\theta_1, \theta_2, \ldots, \theta_n]^T \in \mathbb{R}^n$.
4. For every time point $i = 1, ..., n$:
   (a) We play $x_i = \mathcal{A}_i(y_1, ..., y_{i-1})$.
   (b) We receive a feedback $y_i = \theta_i + Z_i$, where Z_i is a zero-mean, independent subgaussian noise.
5. At the end, the player suffers a cumulative error $\sum_{i=1}^n (x_i - \theta_i)^2$.

Figure 1: *Nonparametric online forecasting model. The focus of the proposed work is to design a forecasting strategy that minimizes the expected cumulative square error. Note that the problem depends a lot on the choice of the sequence θ_i. Our primary interest is on sequences with bounded total variation (TV) so that $\sum_{i=2}^n |\theta_i - \theta_{i-1}| \leq C_n$, but we will also talk about the adaptivity of our method to easier problems such as forecasting Sobolev and Holder functions.*

We propose a model for nonparametric online forecasting as described in Figure 1. This model can be re-framed in the language of the online convex optimization model with three differences.

1. We consider only quadratic loss functions of the form $\ell_t(x) = (x - \theta_t)^2$.
2. The learner receives independent *noisy* gradient feedback, rather than the exact gradient.
3. The criterion of interest is redefined as the *dynamic regret* [Zinkevich, 2003, Besbes et al., 2015]:

$$R_{\text{dynamic}}(\mathcal{A}, \ell_{1:n}) := \mathbb{E}\left[\sum_{t=1}^n \ell_t(x_t)\right] - \sum_{t=1}^n \inf_{x_t} \ell_t(x_t).$$

The new criterion is called a dynamic regret because we are now comparing to a stronger dynamic baseline that chooses an optimal x in every round. Of course in general, the dynamic regret will be linear in n [Jadbabaie et al., 2015]. To make the problem non-trivial, we restrict our attention to sequences of $\ell_1, ..., \ell_n$ that are *regular*, which makes it possible to design algorithms with *sublinear* dynamic regret. In particular, we borrow ideas from the nonparametric regression literature and consider sequences $[\theta_1, ..., \theta_n]$ that are discretizations of functions in the continuous domain. Regularity assumptions emerge naturally as we consider canonical functions classes such as the Holder class, Sobolev class and Total Variation classes [see, e.g., Tsybakov, 2008, for a review].

1.2 Assumptions

We consolidate all the assumptions used in this work and provide necessary justifications for them.

- (A1) The time horizon for the online learner is known to be n.
- (A2) The parameter σ^2 of subgaussian noise in the observations is known.
- (A3) The ground truth denoted by $\theta_{1:n} = [\theta_1, ..., \theta_n]^T$ has its total variation bounded by some positive C_n, i.e., we take $\mathcal{F}$ to be the total variation class $\mathrm{TV}(C_n) := \{\theta_{1:n} \in \mathbb{R}^n : \|D\theta_{1:n}\|_1 \leq C_n\}$ where D is the discrete difference operator. Here $D\theta_{1:n} = [\theta_2 - \theta_1, \ldots, \theta_n - \theta_{n-1}]^T$.
- (A4) $|\theta_1| \leq U$.

The knowledge of σ^2 in assumption (A2) is primarily used to get the optimal dependence of σ in minimax rate. This assumption can be relaxed in practice by using the Median Absolute Deviation estimator as described in Section 7.5 of Johnstone [2017] to estimate σ^2 robustly. Assumption (A3) features a samples from a large class of functions with spatially inhomogeneous degree of smoothness. The functions residing in this class need not even be continuous. Our goal is to propose a policy that is locally adaptive whose empirical mean squared error converges at the minimax rate for this function class. We stress that we do *not* assume that the learner knows C_n. The problem is open and nontrivial even when C_n is known. Assumption (A4) is very mild as it puts restriction only to the first value of the sequence. This assumption controls the inevitable prediction error for the first point in the sequence.

1.3 Our Results

The major contributions of this work are summarized below.

- It is known that the minimax MSE for *smoothing* sequences in the TV class is $\tilde{\Omega}(n^{-2/3})$. This implies a lowerbound of $\tilde{\Omega}(n^{1/3})$ for the dynamic regret in our setting. We present a policy ARROWS (**A**daptive **R**estarting **R**ule for **O**nline averaging using **W**avelet **S**hrinkage) with a nearly minimax dynamic regret $\tilde{O}(n^{1/3})$ and a run-time complexity of $O(n \log n)$.
- We show that a class of forecasting strategies — including the popular Online Gradient Descent (OGD) with fixed restarts [Besbes et al., 2015], moving averages (MA) [Box and Jenkins, 1970] — are fundamentally limited by $\tilde{\Omega}(\sqrt{n})$ regret.
- We also provide a more refined lower bound that characterized the dependence of U, C_n and σ, which certifies the adaptive optimality of ARROWS in all regimes. The bound also reveals a subtle price to pay when we move from the smoothing problem to the forecasting problem, which indicates the separation of the two problems when $C_n/\sigma \gg n^{1/4}$, a regime where the forecasting problem is *strictly* harder (See Figure 3).
- Lastly, we consider forecasting sequences in Sobolev classes and Holder classes and establish that ARROWS can automatically *adapt* to the optimal regret of these *simpler* function classes as well, while OGD and MA cannot, unless we change their tuning parameter (to behave suboptimally on the TV class).

2 Related Work

The topic of this paper sits well in between two amazing bodies of literature: nonparametric regression and online learning. Our results therefore contribute to both fields and hopefully will inspire more interplay between the two communities. Throughout this paper when we refer $\tilde{O}(n^{1/3})$ as the optimal regret, we assume the parameters of the problem are such that it is acheivable (see Figure 3).

Nonparametric regression. As we mentioned before, our problem — online nonparametric forecasting — is motivated by the idea of using locally adaptive nonparametric regression for time series forecasting [Mammen and van de Geer, 1997, Kim et al., 2009, Tibshirani, 2014]. It is more challenging than standard nonparametric regression because we do not have access to the data in the future. While our proof techniques make use of several components (e.g., universal shrinkage) from the seminal work in wavelet smoothing [Donoho et al., 1990, 1998], the way we use them to construct and analyze our algorithm is new and more generally applicable for converting non-parametric regression methods to forecasting methods.

Adaptive Online Learning. Our problem is also connected to a growing literature on adaptive online learning which aims at matching the performance of a stronger time-varying baseline [Zinkevich, 2003, Hall and Willett, 2013, Besbes et al., 2015, Chen et al., 2018b, Jadbabaie et al., 2015, Hazan and Seshadhri, 2007, Daniely et al., 2015, Yang et al., 2016, Zhang et al., 2018a,b, Chen et al., 2018a]. Many of these settings are highly general and we can apply their algorithms directly to our problem, but to the best of our knowledge, none of them achieves the optimal $\tilde{O}(n^{1/3})$ dynamic regret.

In the remainder of this section, we focus our discussion on how to apply the regret bounds in non-stationary stochastic optimization [Besbes et al., 2015, Chen et al., 2018b] to our problem while leaving more elaborate discussion with respect to alternative models (e.g. the constrained comparator approach [Zinkevich, 2003, Hall and Willett, 2013], adaptive regret [Jadbabaie et al., 2015, Zhang et al., 2018a], competitive ratio [Bansal et al., 2015, Chen et al., 2018a]), as well as the comparison to the classical time series models to Appendix A.

Regret from Non-Stationary Stochastic Optimization The problem of non-stationary stochastic optimization is more general than our model because instead of considering only the quadratic functions, $\ell_t(x) = (x - \theta_t)^2$, they work with the more general class of strongly convex functions and general convex functions. They also consider both noisy gradient feedbacks (stochastic first order oracle) and noisy function value feedbacks (stochastic zeroth order oracle).

In particular, Besbes et al. [2015] define a quantity V_n which captures the total amount of "variation" of the functions $\ell_{1:n}$ using $V_n := \sum_{i=1}^{n-1} \|\ell_{i+1} - \ell_i\|_\infty$. [1] Chen et al. [2018b] generalize the notion to $V_n(p, q) := \left(\sum_{i=1}^{n-1} \|\ell_{i+1} - \ell_i\|_p^q\right)^{1/q}$ for any $1 \le p, q \le +\infty$ where $\|\cdot\|_p := (\int |\cdot(x)|^p dx)^{1/p}$ is the standard L_p norm for functions[2]. Table 1 summarizes the known results under the non-stationary stochastic optimization setting.

Table 1: Summary of known minimax dynamic regret in the non-stationary stochastic optimization model. Note that the choice of q does not affect the minimax rate in any way, but the choice of p does. "-" indicates that the no upper or lower bounds are known for that setting.

	Noisy gradient feedback		Noisy function value feedback	
Assumptions on $\ell_{1:n}$	$p = +\infty$	$1 \le p < +\infty$	$p = +\infty$	$1 \le p < +\infty$
Convex & Lipschitz	$\Theta(n^{2/3} V_n^{1/3})$	$O(n^{\frac{2p+d}{3p+d}} V_n(p,q)^{\frac{p}{3p+d}})$	-	-
Strongly convex & Smooth	$\Theta(n^{1/2} V_n^{1/2})$	$\Theta(n^{\frac{2p+d}{4p+d}} V_n(p,q)^{\frac{2p}{4p+d}})$	$\Theta(n^{2/3} V_n^{1/3})$	$\Theta(n^{\frac{4p+d}{6p+d}} V_n(p,q)^{\frac{2p}{6p+d}})$

Our assumption on the underlying trend $\theta_{1:n} \in \mathcal{F}$ can be used to construct an upper bound of this quantity of variation V_n or $V_n(p, q)$. As a result, the algorithms in non-stationary stochastic optimization and their dynamic regret bounds in Table 1 will apply to our problem (modulo additional restrictions on bounded domain). However, our preliminary investigation suggests that this direct reduction does *not*, in general, lead to optimal algorithms. We illustrate this observation in the following example.

Example 1. Let $\mathcal{F}$ be the set of all bounded sequences in the total variation class $TV(1)$. It can be worked out that $V_n(p, q) = O(1)$ for all p, q. Therefore the smallest regret from [Besbes et al., 2015, Chen et al., 2018b] is obtained by taking $p \to +\infty$, which gives us a regret of $O(n^{1/2})$. Note that we expect the optimal regret to be $\tilde{O}(n^{1/3})$ according to the theory of locally adaptive nonparametric regression.

In Example 1, we have demonstrated that one cannot achieve the optimal dynamic regret using known results in non-stationary stochastic optimization. We show in section 3.1 that "Restarting OGD" algorithm has a fundamental lower bound of $\tilde{\Omega}(\sqrt{n})$ on dynamic regret in the TV class.

Online nonparametric regression. As we finalize our manuscript, it comes to our attention that our problem of interest in Figure 1 can be cast as a special case of the "online nonparametric regression" problem [Rakhlin and Sridharan, 2014, Gaillard and Gerchinovitz, 2015]. The general result of Rakhlin and Sridharan [2014] implies the *existence* of an algorithm that enjoys a $\tilde{O}(n^{1/3})$ regret for the TV class without explicitly constructing one, which shows that $n^{1/3}$ is the minimax rate when $C_n = O(1)$ (see more details in Appendix A). To the best of our knowledge, our proposed algorithm remains the first *polynomial time* algorithm with $\tilde{O}(n^{1/3})$ regret and our results reveal more precise (optimal) upper and lower bounds on all parameters of the problem (see Section 3.4).

3 Main results

We are now ready to present our main results.

[1] The V_n definition in [Besbes et al., 2015] for strongly convex functions are defined a bit differently, the $\|\cdot\|_\infty$ is taken over the convex hull of minimizers. This creates some subtle confusions regarding our results which we explain in details in Appendix I.

[2] We define $V_n(p, q)$ to be a factor of $n^{-1/q}$ times bigger than the original scaling presented in [Chen et al., 2018b] so the results become comparable to that of [Besbes et al., 2015].

3.1 Limitations of Linear Forecasters

Restarting OGD as discussed in Example 1, fails to achieve the optimal regret in our setting. A curious question to ask is whether it is the algorithm itself that fails or it is an artifact of a potentially suboptimal regret analysis. To answer this, let's consider the class of linear forecasters — estimators that outputs a fixed linear transformation of the observations $y_{1:n}$. The following preliminary result shows that Restarting OGD is a linear forecaster . By the results of Donoho et al. [1998], linear smoothers are fundamentally limited in their ability to estimate functions with heterogeneous smoothness. Since forecasting is harder than smoothing, this limitation gets directly translated to the setting of linear forecasters.

Proposition 2. *Online gradient descent with a fixed restart schedule is a linear forecaster. Therefore, it has a dynamic regret of at least $\tilde{\Omega}(\sqrt{n})$.*

Proof. First, observe that the stochastic gradient is of form $2(x_t - y_t)$ where x_t is what the agent played at time t and y_t is the noisy observation $\theta_t +$ Independent noise. By the online gradient descent strategy with the fixed restart interval and an inductive argument, x_t is a linear combination of $y_1, ..., y_{t-1}$ for any t. Therefore, the entire vector of predictions $x_{1:t}$ is a fixed linear transformation of $y_{1:t-1}$. The fundamental lower bound for linear smoothers from Donoho et al. [1998] implies that this algorithm will have a regret of at least $\tilde{\Omega}(\sqrt{n})$. □

The proposition implies that we will need fundamentally new *nonlinear* algorithmic components to achieve the optimal $O(n^{1/3})$ regret, if it is achievable at all!

3.2 Policy

In this section, we present our policy ARROWS (Adaptive Restarting Rule for Online averaging using Wavelet Shrinkage). The following notations are introduced for describing the algorithm.

- t_h denotes start time of the current bin and t be the current time point.
- $\bar{y}_{t_h:t}$ denotes the average of the y values for time steps indexed from t_h to t.
- $pad_0(y_{t_h}, ..., y_t)$ denotes the vector $(y_{t_h} - \bar{y}_{t_h:t}, ..., y_t - \bar{y}_{t_h:t})^T$ zero-padded at the end till its length is a power of 2. *i.e*, a re-centered and padded version of observations.
- $T(x)$ where x is a sequence of values, denotes the element-wise soft thresholding of the sequence with threshold $\sigma\sqrt{\beta \log(n)}$
- H denotes the orthogonal discrete Haar wavelet transform matrix of proper dimensions
- Let $Hx = \alpha = [\alpha_1, \alpha_2, ..., \alpha_k]^T$ where k being a power of 2 is the length of x. Then the vector $[\alpha_2, ..., \alpha_k]^T$ can be viewed as a concatenation of $\log_2 k$ contiguous blocks represented by $\alpha[l], l = 0, ..., \log_2(k) - 1$. Each block $\alpha[l]$ at level l contains 2^l coefficients.

ARROWS: inputs - observed y values, time horizon n, std deviation σ, $\delta \in (0, 1]$, a hyper-parameter $\beta > 24$

1. Initialize $t_h = 1$, $newBin = 1$, $y_0 = 0$
2. For t = 1 to n:
   (a) If $newBin == 1$, predict $x_t^{t_h} = y_{t-1}$, else predict $x_t^{t_h} = \bar{y}_{t_h:t-1}$
   (b) set $newBin = 0$, observe y_t and suffer loss $(x_t^{t_h} - \theta_t)^2$
   (c) Let $\tilde{y} = pad_0(y_{t_h}, ..., y_t)$ and k be the padded length.
   (d) Let $\hat{\alpha}(t_h : t) = T(H\tilde{y})$
   (e) Restart Rule: If $\frac{1}{\sqrt{k}} \sum_{l=0}^{\log_2(k)-1} 2^{l/2} \|\hat{\alpha}(t_h : t)[l]\|_1 > \frac{\sigma}{\sqrt{k}}$ then
      i. set $newBin = 1$
      ii. set $t_h = t + 1$

Our policy is the byproduct of following question: How can one lift a batch estimator that is minimax over the TV class to a minimax online algorithm?

Restarting OGD when applied to our setting with squared error losses reduces to partitioning the duration of game into fixed size chunks and outputting online averages. As described in Section 3.1, this leads to suboptimal regret. However, the notion of averaging is still a good idea to keep. If within a time interval, the Total Variation (TV) is adequately small, then outputting sample averages is reasonable for minimizing the cumulative squared error. Once we encounter a bump in the variation, a good strategy is to restart the averaging procedure. Thus we need to adaptively detect intervals with low TV. For accomplishing this, we communicate with an oracle estimator whose output can be used to construct a lowerbound of TV within an interval. The decision to restart online averaging is based on the estimate of TV computed using this oracle. Such a decision rule introduces non-linearity and hence breaks free of the suboptimal world of linear forecasters.

The oracle estimator we consider here is a slightly modified version of the soft thresholding estimator from Donoho [1995]. We capture the high level intuition behind steps (d) and (e) as follows. Computation of Haar coefficients involves smoothing adjacent regions of a signal and taking difference between them. So we can expect to construct a lowerbound of the total variation $\|D\theta_{1:n}\|_1$ from these coeffcients. The extra thresholding step $T(.)$ in (d) is done to denoise the Haar coefficients computed from noisy data. In step (e), a weighted L1 norm of denoised coefficients is used to lowerbound the total variation of the true signal. The multiplicative factors $2^{l/2}$ are introduced to make the lowerbound tighter. We restart online averaging once we detect a large enough variation. The first coefficient $\hat{\alpha}(t_h : t)_1$ is zero due to the re-centering caused by pad_0 operation. The hyper-parameter β controls the degree to which we shrink the noisy wavelet coefficients. For sufficiently small β, It is almost equivalent to the universal soft-thresholding of [Donoho, 1995]. The optimal selection of β is described in Theorem 3.

We refer to the duration between two consecutive restarts inclusive of the first restart but exclusive of the second as a bin. The policy identifies several bins across time, whose width is adaptively chosen.

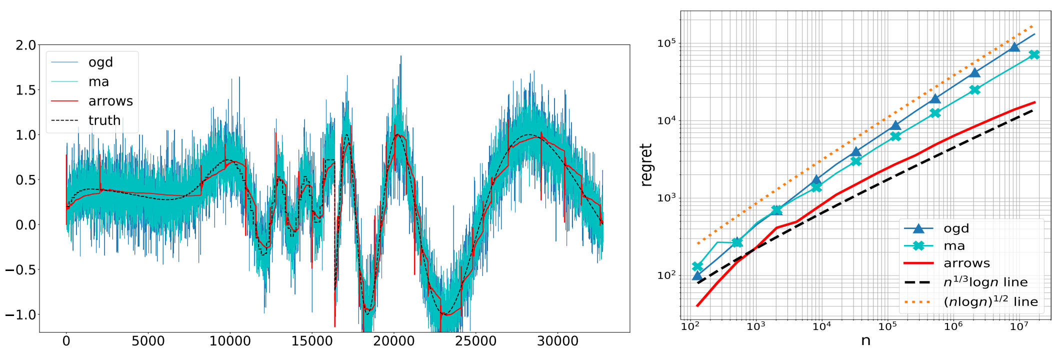


Figure 2: *An illustration of* ARROWS *on a sequence with heterogeneous smoothness. We compare qualitatively (on the left) and quantitatively (on the right) to two popular baselines: (a) restarting online gradient descent [Besbes et al., 2015]; (b) the moving averages [Box and Jenkins, 1970] with optimal parameter choices. As we can see,* ARROWS *achieves the optimal $\tilde{O}(n^{1/3})$ regret while the baselines are both suboptimal.*

3.3 Dynamic Regret of ARROWS

In this section, we provide bounds for non-stationary regret and run-time of the policy.

Theorem 3. *Let the feedback be $y_t = \theta_t + Z_t$, $t = 1, \ldots, n$ and Z_t be independent, σ-subgaussian random variables. If $\beta = 24 + \frac{8\log(8/\delta)}{\log(n)}$, then with probability at least $1-\delta$,* ARROWS *achieves a dynamic regret of $\tilde{O}(n^{1/3}\|D\theta_{1:n}\|_1^{2/3}\sigma^{4/3} + |\theta_1|^2 + \|D\theta_{1:n}\|_2^2 + \sigma^2)$ where $\tilde{O}$ hides a logarithmic factor in n and $1/\delta$.*

Proof Sketch. Our policy is similar in spirit to restarting OGD but with an adaptive restart schedule. The key idea we used is to reduce the dynamic regret of our policy in probability roughly to a sum of squared error of a soft thresholding estimator and number of restarts. This was accomplished by using a Follow The Leader (FTL) reduction. For bounding the squared error part of the sum we modified the threshold value for the estimator in Donoho [1995] and proved high probability guarantees for the convergence of its empirical mean. To bound the number of times we restart, we first establish a connection between Haar coefficients and total variation. This is intuitive since computation of Haar coefficients can be viewed as smoothing the adjacent regions of a signal and taking their difference. Then we exploit a special condition called "uniform shrinkage" of the soft-thresholding estimator which helps to optimally bound the number of restarts with high probability. □

Theorem 3 provides an upper bound of the minimax dynamic regret for forecasting the TV class.

Corollary 4. *Suppose the ground truth $\theta_{1:n} \in TV(C_n)$ and $|\theta_1| \leq U$. Then $\|D\theta_{1:n}\|_1 \leq C_n$. By noting that $\|D\theta_{1:n}\|_2 \leq \|D\theta_{1:n}\|_1$, under the setup in Theorem 3* ARROWS *achieves a dynamic regret of $\tilde{O}(n^{1/3}C_n^{2/3}\sigma^{4/3} + U^2 + C_n^2 + \sigma^2)$ with probability at-least $1-\delta$.*

Remark 5 (Adaptivity to unknown parameters.)**.** Observe that ARROWS does not require the knowledge of C_n.It adapts optimally to the unknown TV radius $C_n := \|D\theta_{1:n}\|_1$ of the ground truth $\theta_{1:n}$. The adaptivity to n can be achieved by a standard doubling trick. σ, if unknown, can be robustly estimated from the first few observations by a Median Absolute Deviation estimator (eg. Section 7.5 of Johnstone [2017]), thanks to the sparsity of wavelet coefficients of TV bounded functions.

3.4 A lower bound on the minimax regret

We now give a matching lower bound of the expected regret, which establishes that ARROWS is adaptively minimax.

Proposition 6. *Assume $\min\{U, C_n\} > 2\pi\sigma$ and $n > 3$, there is a universal constant c such that*

$$\inf_{x_{1:n}} \sup_{\theta_{1:n} \in \mathrm{TV}(C_n)} \mathbb{E}\left[\sum_{t=1}^{n} (x_t(y_{1:t-1}) - \theta_t)^2\right] \geq c(U^2 + C_n^2 + \sigma^2 \log n + n^{1/3} C_n^{2/3} \sigma^{4/3}).$$

The proof is deferred to the Appendix I. The result shows that our result in Theorem 3 is optimal up to a logarithmic term in n and $1/\delta$ for almost all regimes (modulo trivial cases of extremely small $\min\{U, C_n\}/\sigma$ and n)[3].

Remark 7 (The price of forecasting)**.** The result also shows that *forecasting is strictly harder than smoothing*. Observe that a term with C_n^2 is required even if $\sigma = 0$, whereas in the case of a one-step look-ahead oracle (or the smoothing algorithm that sees all n observations) does not have this term. This implies that the total amount of variation that *any* algorithm can handle while producing a sublinear regret has dropped from $C_n = o(n)$ to $C_n = o(\sqrt{n})$. Moreover, the regime where the $n^{1/3}C_n^{2/3}\sigma^{4/3}$ term is meaningful only when $C_n = o(n^{1/4})$. For the region where $\sigma n^{1/4} \ll C_n \ll \sigma n^{1/2}$, the minimax regret is essentially proportional to C_n^2. This is illustrated in Figure 3.

We note that in much of the online learning literature, it is conventional to consider a slightly more restrictive setting with bounded domain, which could reduce the minimax regret. The following remark summarizes a variant of our results in this setting.

Remark 8 (Minimax regret in bounded domain)**.** If we consider predicting sequences from a subset of the $TV(C_n)$ ball having an extra boundedness condition $|\theta_i| \leq B$ for $i = 1 \ldots n$, it can be shown that (see Appendix I) minimax regret is $\tilde{\Omega}\left(\min\{nB^2, n\sigma^2, n^{1/3}C_n^{2/3}\sigma^{4/3}\} + B^2 + \min\{nB^2, BC_n\} + \sigma^2\right)$. In particular, forecasting is still strictly harder than smoothing due to the $\min\{nB^2, BC_n\}$ term in the bound. The discussion in Appendix I, shows a way of using ARROWS whose regret can match this lower bound.

[3]When both U and C_n are moderately small relative to σ, the lower bound will depend on σ a little differently because the estimation error goes to 0 faster than $1/\sqrt{n}$. We know the minimax risk exactly for that case as well but it is somewhat messy [see e.g. Wasserman, 2006]. When they are both much smaller than σ, e.g., when $\min\{U, C_n\} \leq \sigma/\sqrt{n}$, then outputting 0 when we do not have enough information will be better than doing online averages.

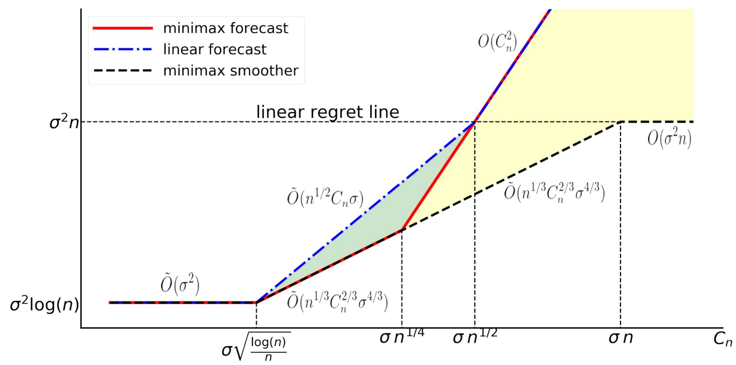


Figure 3: *An illustration of the minimax (dynamic) regret of forecasters and smoothers as a function of C_n. The non-trivial regime for forecasting is when C_n lies between $\sigma \sqrt{\frac{\log(n)}{n}}$ and $\sigma\, n^{1/4}$ where forecasting is just as hard as smoothing. When $C_n > \sigma\, n^{1/4}$, forecasting is harder than smoothing. The yellow region indicates the extra loss incurred by any minimax forecaster. The green region marks the extra loss incurred by a linear forecaster compared to minimax forecasting strategy. The figure demonstrates that linear forecasters are sub-optimal even in the non-trivial regime. When $C_n > \sigma\, n^{1/2}$, it is impossible to design a forecasting strategy with sub-linear regret. For $C_n > \sigma\, n$, identity function is optimal estimator for smoothing and when when $C_n < \sigma \sqrt{\frac{\log(n)}{n}}$, online averaging is optimal for both problems.*

3.5 The adaptivity of ARROWS to Sobolev and Holder classes

It turns out that ARROWS is also adaptively optimal in forecasting sequences in the discrete Sobolev classes and the discrete Holder classes, which are defined as

$$\mathcal{S}(C'_n) = \{\theta_{1:n} : \|D\theta_{1:n}\|_2 \le C'_n\}, \qquad \mathcal{H}(B'_n) = \{\theta_{1:n} : \|D\theta_{1:n}\|_\infty \le B'_n\}.$$

These classes feature sequences that are more spatially homogeneous than those in the TV class. The minimax cumulative error of nonparametric estimation in the discrete Sobolev class is $\Theta(n^{2/3}[C'_n]^{2/3}\sigma^{4/3})$ [see e.g., Sadhanala et al., 2016, Theorem 5 and 6].

Corollary 9. *Let the feedback be $y_t = \theta_t + Z_t$ where Z_t is an independent, σ-subgaussian random variable. Let $\theta_{1:n} \in \mathcal{S}(C'_n)$ and $|\theta_1| \le U$. If $\beta = 24 + \frac{8\log(8/\delta)}{\log(n)}$, then with probability at least $1-\delta$,* ARROWS *achieves a dynamic regret of $\tilde{O}(n^{2/3}[C'_n]^{2/3}\sigma^{4/3} + U^2 + [C'_n]^2 + \sigma^2)$ where $\tilde{O}$ hides a logarithmic factor in n and $1/\delta$.*

Thus despite the fact that ARROWS is designed for total variation class, it adapts to the optimal rates of forecasting sequences that are spatially regular. To gain some intuition, let's minimally expand the Sobolev ball to a TV ball of radius $C_n = \sqrt{n} C'_n$. The chosen scaling of C_n activates the embedding $\mathcal{S}(C'_n) \subset TV(C_n)$ (see the illustration in Table 2) with both classes having same minimax rate in the batch setting. This implies that dynamic regret of ARROWS is simultaneously minimax optimal over $\mathcal{S}(C'_n)$ and $TV(C_n)$ wrt the term containing n. It can be shown that ARROWS is optimal wrt to the additive $[C'_n]^2, U^2, \sigma^2$ terms as well. Minimaxity in Sobolev class implies minimaxity in Holder class since it is known that a Holder ball is sandwiched between two Sobolev balls having the same minimax rate [see e.g., Tibshirani, 2015]. A proof of the Corollary and related experiments are presented in Appendix F and J.

3.6 Fast computation

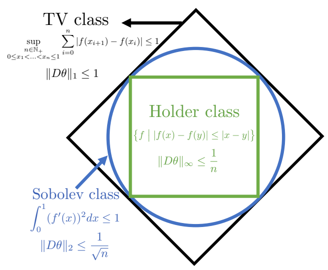
Last but not least, we remark that there is a fast implementation of ARROWS that reduces the overall time-complexity for n step from $O(n^2)$ to $O(n \log n)$.

Proposition 10. *The run time of* ARROWS *is $O(n\log(n))$, where n is the time horizon.*

The proof exploits the sequential structure of our policy and sparsity in wavelet transforms, which allows us to have $O(\log n)$ incremental updates in all but $O(\log n)$ steps. See Appendix G for details.

Table 2: *Minimax rates for cumulative error $\sum_{i=1}^{n}(\hat{\theta}_i - \theta)^2$ in various settings and policies that achieve those rates.* ARROWS *is adaptively minimax across all of the described function classes while linear forecasters fail to perform optimally over the TV class. For simplicity, we assume U is small and hide a $\log n$ factors in all the forecasting rates.*

Class		Minimax rate for Forecasting	Minimax rate for Smoothing	Minimax rate for Linear Forecasting
TV	$\|D\theta_{1:n}\|_1 \leq C_n$	$n^{1/3}C_n^{2/3}\sigma^{4/3} + C_n^2 + \sigma^2$	$n^{1/3}C_n^{2/3}\sigma^{4/3} + \sigma^2$	$n^{1/2}C_n\sigma + C_n^2 + \sigma^2$
Sobolev	$\|D\theta_{1:n}\|_2 \leq C_n'$	$n^{2/3}[C_n']^{2/3}\sigma^{4/3} + [C_n']^2 + \sigma^2$	$n^{2/3}[C_n']^{2/3}\sigma^{4/3} + \sigma^2$	$n^{2/3}[C_n']^{2/3}\sigma^{4/3} + [C_n']^2 + \sigma^2$
Holder	$\|D\theta_{1:n}\|_\infty \leq L_n$	$nL_n^{2/3}\sigma^{4/3} + nL_n^2 + \sigma^2$	$nL_n^{2/3}\sigma^{4/3} + \sigma^2$	$nL_n^{2/3}\sigma^{4/3} + nL_n^2 + \sigma^2$
Minimax Algorithm		ARROWS	Wavelet Smoothing Trend Filtering	Restarting OGD Moving Averages



Canonical Scaling[a]		Forecasting	Smoothing	Linear Forecasting
TV	$C_n \asymp 1$	$n^{1/3}$	$n^{1/3}$	$n^{1/2}$
Sobolev	$C_n' \asymp 1/\sqrt{n}$	$n^{1/3}$	$n^{1/3}$	$n^{1/3}$
Holder	$L_n \asymp 1/n$	$n^{1/3}$	$n^{1/3}$	$n^{1/3}$

[a] The "canonical scaling" are obtained by discretizing functions in canonical function classes. Under the canonical scaling, Holder class $\subset$ Sobolev class $\subset$ TV class, as shown in the figure on the left. ARROWS is optimal for the Sobolev and Holder classes inscribed in the TV class. MA and Restarting OGD on the other hand require different parameters and prior knowledge of variational budget (i.e C_n or C_n') to achieve the minimax linear rates for the TV class and the Sobolev/Holder class.

3.7 Experimental Results

To empirically validate our results, we conducted a number of numerical simulations that compares the regret of ARROWS, (Restarting) OGD and MA. Figure 2 shows the results on a function with heterogeneous smoothness (see the exact details and more experiments in Appendix B) with the hyper-parameters selected according to their theoretical optimal choice for the TV class (See Theorem 11, 12 for OGD and MA in Appendix C). The left panel illustrates that ARROWS is locally adaptive to heterogeneous smoothness of the ground truth. Red peaks in the figure signifies restarts. During the initial and final duration, the signal varies smoothly and ARROWS chooses a larger window size for online averaging. In the middle, signal varies rather abruptly. Consequently ARROWS chooses a smaller window size. On the other hand, the linear smoothers OGD and MA use a constant width and cannot adapt to the different regions of the space. This differences are also reflected in the quantitative evaluation on the right, which clearly shows that OGD and MA has a suboptimal $\tilde{O}(\sqrt{n})$ regret while ARROWS attains the $\tilde{O}(n^{1/3})$ minimax regret!

4 Concluding Discussion

In this paper, we studied the problem of online nonparametric forecasting of bounded variation sequences. We proposed a new forecasting policy ARROWS and proved that it achieves a cumulative square error (or dynamic regret) of $\tilde{O}(n^{1/3}C_n^{2/3}\sigma^{4/3}+\sigma^2+U^2+C_n^2)$ with total runtime of $O(n\log n)$. We also derived a lower bound for forecasting sequences with bounded total variation which matches the upper bound up to a logarithmic term which certifies the optimality of ARROWS in all parameters. Through connection to linear estimation theory, we assert that no linear forecaster can achieve the optimal rate. ARROWS is highly adaptive and has essentially no tuning parameters. We show that it is adaptively minimax (up to a logarithmic factor) simultaneously for all discrete TV classes, Sobolev classes and Holder classes with unknown radius. Future directions include generalizing to higher order TV class and other convex loss functions.

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DB and YW were supported by a start-up grant from UCSB CS department and a gift from Amazon Web Services. The authors thank Yining Wang for a preliminary discussion that inspires the work, and Akshay Krishnamurthy and Ryan Tibshirani for helpful comments to an earlier version of the paper.

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A Discussion on other related works

Regret from Adaptive Optimistic Mirror Descent. In Jadbabaie et al. [2015], the authors propose Adaptive Optimistic Mirror Descent (AOMD) algorithm that minimizes the dynamic regret against a comparator sequence $\{u_t\}_{t=1}^n$. Their learning framework is the full information setting where learner predict $x_t \in \mathcal{X}$ for a convex set $\mathcal{X} \subseteq R^d$. Then a loss function $f_t(x)$ is revealed to the learner. To capture the regularity of the comparator, they define a quantity $C_n(u_1, u_2, ..., u_n) := \sum_{t=1}^n \|u_t - u_{t-1}\|$. They capture the regularity of loss functions by incorporating some external knowledge about their gradients via a predictable sequence $\{M_t\}_{t=1}^n$. They define: $D_n := \sum_{t=1}^n \|\nabla f_t(x_t) - M_t\|_*^2$. Finally to account for the temporal variability of f_t, they introduce V_n as discussed earlier. The final regret bound is expressed in terms of these three quantities. However, their algorithm is adaptive and requires no prior knowledge about them.

We note that our problem can be reduced to their framework if one considers loss functions $f_t(x) = (x - y_t)^2$. Then the expected dynamic regret against the comparator sequence $\{\theta_t\}_{t=1}^n$ is given by

$$\sum_{t=1}^n E[(x - y_t)^2 - (\theta_t - y_t)^2] = E[\sum_{t=1}^n (x - \theta_t)^2], \tag{1}$$

where the expectation at right hand side is over the randomness of forecasting strategy. Hence we observe that their regret bound can be directly applied to bound the dynamic regret of our problem. It can be shown that (see Appendix H) given a fixed total variation bound $C_n = O(1)$, then V_n and D_n can be proved to be $O(n)$ with high probability. Plugging this into their regret bound yields an $\tilde{O}(\sqrt{n})$ rate in probability. However, it is unclear that whether AOMD is fundamentally limited by this rate or is there a potential suboptimality in their analysis of regret on our particular problem.

Other Dynamic Regret minimizing policies. [Yang et al., 2016] defines a path variation budget that is equivalent to our C_n to characterize the sequence of convex loss functions. However, under the noisy gradient feedback structure, they use a version of restarting OGD to get $C^{1/2}n^{1/2}$ regret rate. This is very similar to the policy in [Besbes et al., 2015]. Since OGD is a linear forecaster, it is sub-optimal for predicting bounded variation sequences under the squared error metric.

In [Koolen et al., 2015], they consider minimizing the dynamic regret wrt to a comparator class that obeys $\|D\theta_{1:n}\|_2 \le C'_n$. This is basically the discrete Sobolev class. As shown in appendix E, our policy is minimax for forecasting such sequences as well when the observed values are noisy versions of the ground truth. However it should be noted that [Koolen et al., 2015] does not have this distributional assumption on the observations.

[Chen et al., 2018a] considers the Smoothed Online Convex Optimization framework where they simultaneously minimize the hitting loss f_t and a switching cost. They provide dynamic regret bounds on this composite cost in the setting that f_t is known to the learner before making the prediction. If we consider $f_t(x) = (x - y_t)^2$, then the baseline they compare against reduces to the offline Trend Filtering (TF) estimate when $\sum_{i=2}^n |x_t - x_{t-1}| \le L = C_n$. Then Theorem 10 of [Chen et al., 2018a] states that the cumulative composite cost incurred by their proposed policy differs from that of the TF estimate by a term that is $O(\sqrt{nC_n})$. However, this doesn't translate to a meaningful regret bound in our setting.

[Hall and Willett, 2013] proposes the Dynamic Mirror Descent (DMD) algorithm that make use of a family of dynamical models for making prediction at each time step. They achieve a dynamic regret bound of $O(\sqrt{n}(1 + V_{\phi_t}(\boldsymbol{\theta}_T)))$ where the second term measures the quality of the dynamical models in predicting ground truth.

Comparison to Online Isotonic Regression. [Kotłowski et al., 2016] considers the dynamic regret minimization,

$$\sum_{t=1}^n (x_t - y_t)^2 - \min_{(\theta_1,...,\theta_n)} \sum_{t=1}^n (\theta_t - y_t)^2,$$

where $y_t \le B$ is a label revealed by the environment, $x_t \le B$ is the prediction of the learner, and the comparator sequence should obey $0 \le \theta_1 \le ... \le \theta_n \le B$. Here B is a fixed positive number. Note that their setting and our framework are mutually reducible to each other in terms of regret guarantees via 1. They propose a minimax policy that achieves a dynamic regret of $\tilde{O}(n^{1/3})$ which translates to an $\tilde{O}(n^{1/3})$ in probability in our setting under the isotonic ground truth restriction.

We note that the isotonic comparator sequence belong to a TV class of variational budget $C_n = B$. By using an argument similar to that in appendix H which involves converting to deterministic noise setting and conditioning on a high probability event, it can be shown that our policy is out of the box minimax with high probability in isotonic framework when observations are noisy versions of an isotonic sequence.

Comparison to Online Non-Parametric regression methods. We note that our problem falls into the more general framework of online non-parametric regression setting studied in [Rakhlin and Sridharan, 2015]. We can reduce our dynamic regret minimization to their framework by using a similar argument as above through (1). Since the bounded TV class is sandwiched between Besov spaces $B^1_{1,q}$ for the range $1 \leq q \leq \infty$, the discussion in section 5.8 of [Rakhlin and Sridharan, 2015] establishes that minimax growth w.r.t n as $O(n^{1/3})$ in the online setting for TV class. Thus our bounds, modulo logarithmic factor, matches with theirs though we give the precise dependence on C_n and σ as well. It is worthwhile to point out that while the bound in [Rakhlin and Sridharan, 2015] is non-constructive, we achieve the same bound via an efficient algorithm.

[Gaillard and Gerchinovitz, 2015] proposes a minimax policy wrt to comaparator functions that are Holder smooth. In particular, for the Holder class H_1 that satisfy $|f(x) - f(y)| \leq \lambda|x - y|$, their algorithm yields a regret of $\tilde{O}(n^{1/3})$. It is known ([Tibshirani, 2015]) that H_1 is sandwiched between two Sobolev balls having the same minimax rate in the iid batch statistical learning setting. Since our policy is optimal for Sobolev space (appendix F), it is also optimal over Holder ball H_1 when the observations are noisy versions of a Holder smooth functions. Though the framework of [Gaillard and Gerchinovitz, 2015] doesn't impose this distributional assumption. The runtime of their policy for H_1 class is $O(n^{7/3} \log n)$. It should be noted that Sobolev and Holder classes are arguably easier to tackle than the TV class since both of them can be embedded inside a TV class.

Strongly Adaptive Regret. Daniely et al. [2015] introduced the notion of Strongly Adaptive (SA) regret where the online learner is guaranteed to have low static regret for any interval within the duration of the game. They also propose a meta algorithm which can convert an algorithm of good static regret to one with good SA regret. However low static regret for any interval doesn't help in our setting because in each interval we are competing with a stronger dynamic adversary. A notion of SA dynamic regret would an interesting topic to explore.

For minimizing dynamic regret, Zhang et al. [2018b] proposed a meta policy that uses an algorithm with good SA regret as its subroutine. Hence we can use their framework with squared error loss functions as discussed above. They show that OGD has an SA regret of $O(\log(n))$ for strongly convex loss functions. Using OGD as the subroutine and applying corollary 7 of their paper yields a bound $\tilde{O}(n)$. By a similar argument one gets the same linear regret rate when online newton step is used as the subroutine. However, we should note that their algorithm works without the knowledge of radius of the TV ball C_n.

Classical time series forecasting models. Finally, we note that our work is complementary to much of the classical work in time-series forecasting (e.g., Box-Jenkins method/ARIMA Box and Jenkins [1970], Hidden Markov Models [Baum and Petrie, 1966]). These methods aim at using dynamical systems to capture the recurrent patterns under a stationary stochastic process, while we focus on harnessing the nonstationarity. Our work is closer to the "trend smoothing" literature (e.g., the celebrated Hodrick-Prescott filter [Hodrick and Prescott, 1997], trend filtering [Kim et al., 2009, Tibshirani, 2014, Hutter and Rigollet, 2016]).

B Additional Experiments

The function that we generated in Figure 2 is a hybrid function which in the first half is a "discretized cubic spline" with more knots closely placed towards the end. In the second half it is a Doppler function $f(t) = \sin\left(\frac{2\pi(1+\epsilon)}{t/n+0.38}\right)$ with n being the time horizon. We observe noisy data $y_i =$

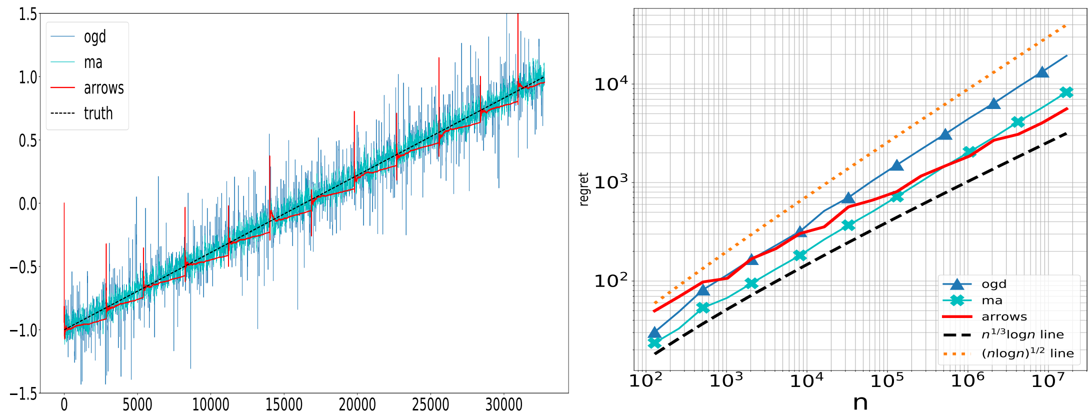

Figure 4: *An illustration of* ARROWS *on a linear trend which has homogeneous smoothness*

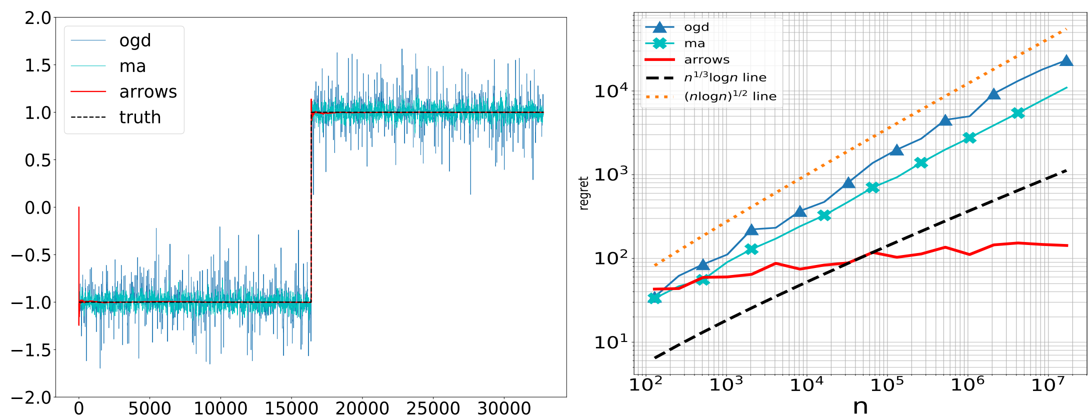

Figure 5: *An illustration of* ARROWS *on a step trend with abrupt inhomogeneity.*

$f(i/n) + z_i, i = 1, ..., n$ and z_i are iid normal variables with $\sigma = 1$. The value of C_n for $n > 60K$ is around 17. Hence for all $n > 83521$, we are under the $n^{1/3}$ regime of $\sigma\sqrt{\log(n)/n} < C_n < \sigma n^{1/4}$.

The window size for moving averages and partition width of OGD were tuned optimally for the TV class (see Appendix C for details). Figure 2 depicts the estimated signals and dynamic regret averaged across 5 runs in a log log plot. The left panel illustrates that ARROWS is locally adaptive to heterogeneous smoothness of the ground truth. Red peaks in the figure signifies restarts. During the initial and final duration, the signal varies smoothly and ARROWS chooses a larger window size for online averaging. In the middle, signal varies rather abruptly. Consequently ARROWS chooses a smaller window size. On the other hand, the linear smoothers OGD and MA attains a suboptimal $\tilde{O}(\sqrt{n})$ regret.

In Figure 4 and 5 we plot the estimates and log-log regret for two more functions: A linear function that is homogeneously smooth and less challenging and a step function which has an abrupt discontinuity making it more inhomogeneous than linear but have lesser inhomogeneity w.r.t hybrid signal considered in 3.7. Both OGD and MA were optimally tuned for the TV class as in Appendix C.

The red peaks corresponds to restarts by ARROWS. For linear functions we can see that ARROWS chooses inter-restart duration/bin-widths that are constant throughout. This is expected as a linear trend is spatially homogeneous. For the step function, we see that ARROWS restart only once since the start. Further, notice that it quickly restarts once the bump is hit. For both of these functions, necessary scaling is done so that we are in the $n^{1/3}$ regime quite early.

C Upper bounds of linear forecasters

In this section we compute the optimal batch size for Restarting OGD and optimal window size for moving averages to yield the $\tilde{O}(\sqrt{n})$ regret rate.

Theorem 11. *Let the feedback be $y_t = \theta_t + Z_t$ where Z_t is an independent, σ-subgaussian random variable. Let $\theta_{1:n} \in \mathrm{TV}(C_n)$. Restarting OGD with batch size of $\sqrt{n \log n}\frac{\sigma}{C_n}$ achieves an expected dynamic regret of $\tilde{O}(U^2 + C_n^2 + \sigma C_n \sqrt{n})$.*

Proof. Note that in our setting with squared error losses $f_t(x) = (x - \theta_t)^2$, the update rule of restarting OGD reduces to computing online averages. Thus OGD essentially divides the time horizon n into fixed size batches and output online averages within each batch. Our objective here is to compute the optimal batch size that minimizes the dynamic regret.

We will bound the expected regret. Let x_t be the estimate of OGD at time t. Let batches be numbered as $1, ..., \lceil n/L \rceil$ where L is the fixed batch size. Let the total variation of ground truth within batch i be C_i. Time interval of batch i is denoted by $[t_h^{(i)}, t_l^{(i)}]$. Due to bias variance decomposition within a batch we have,

$$R_i = \sum_{t=t_h^{(i)}}^{t_l^{(i)}} E[(x_t - \theta_t)^2] = (\theta_{t_h^{(i)}-1} - \theta_{t_h^{(i)}})^2 + \sum_{t=t_h^{(i)}+1}^{t_l^{(i)}} (\theta_t - \bar{\theta}_{t_h^{(i)}:t-1})^2 + \frac{\sigma^2}{t - t_h^{(i)}}, \tag{2}$$

$$\leq (\theta_{t_h^{(i)}-1} - \theta_{t_h^{(i)}})^2 + LC_i^2 + \sigma^2(2 + \log L),$$

with the convention $\theta_0 = 0$ and at start of bin our prediction is just the noisy realization of the previous data point.

Summing across all bins gives,

$$\sum_{i=1}^{\lceil n/L \rceil} R_i \leq LC_n^2 + 2\sigma^2 \frac{n(2 + \log L)}{L} + U^2 + C_n^2.$$

where we have used assumption (A4) to bound the bias of the first prediction. The above expression can be minimized by setting $L = \sqrt{n \log n}\frac{\sigma}{C_n}$ to yield a regret bound of $O(U^2 + C^2 + \sigma C_n \sqrt{n \log n})$ $\square$

Theorem 12. *Under the same setup as in Theorem 11, moving averages with window size $\frac{\sigma\sqrt{n}}{C_n}$ yields a dynamic regret of $O(\sigma C_n \sqrt{n} + U^2 + C_n^2)$*

Proof. Let the window size of moving averages be denoted by m. Consider the prediction at a time $x_t, t \geq m$. By bias variance decomposition we have,

$$E[(x_t - \theta_t)^2] = \left(\theta_i - \frac{\sum_{j=i-m}^{i-1} \theta_j}{m}\right)^2 + \frac{\sigma^2}{m}.$$

By Jensen's inequality,

$$\left(\theta_i - \frac{\sum_{j=i-m}^{i-1} \theta_j}{m}\right)^2 \leq \frac{\sum_{j=i-m}^{i-1} (\theta_j - \theta_i)^2}{m},$$

$$\leq \frac{2\sum_{j=i-m}^{i-1} (j - i + 1 + m)(\theta_{j+1} - \theta_j)^2}{m} \text{ ,by } (a+b)^2 \leq 2a^2 + 2b^2.$$

Notice that the term $(\theta_i - \theta_{i-1})^2$ will be multiplied by a factor m in the above bias bound at time point i, $m - 1$ times in the next time point $i + 1$ and so on. By summing this bias bound across the times points, we obtain

$$\sum_{i=m}^{n} \frac{2\sum_{j=i-m}^{i-1} (j - i + 1 + m)(\theta_{j+1} - \theta_j)^2}{m} \leq 4m \sum_{i=1}^{n-1} (\theta_i - \theta_{i+1})^2 + U^2,$$

$$\leq 4mC_n^2 + U^2.$$

The squared bias for the initial points can be bounded by.

$$\sum_{i=1}^{m-1} (\theta_i - \hat{\theta}_{(1:i-1)})^2 \leq U^2 + C_n^2.$$

Summing the variance terms yields,

$$\begin{aligned}\sum_{t=1}^{n} \mathrm{Var}(x_t) &= \sum_{t=1}^{m-1} \frac{\sigma^2}{t} + \sum_{t=m}^{n} \frac{\sigma^2}{m}, \\ &\leq \frac{(1+\log m + n)\sigma^2}{m}.\end{aligned}$$

Thus the total MSE can be minimized by setting $m = \frac{\sigma\sqrt{n}}{C_n}$, we obtain a dynamic regret bound of $O(\sigma C_n \sqrt{n} + U^2 + C_n^2)$

□

D Proof of useful lemmas

We begin by recording an observation that follows directly from the policy.

Lemma 13. *For m^{th} bin that spans the interval $[t_h^{(m)}, t_l^{(m)}]$, discovered by the policy, let the lengths of $\hat{\alpha}(t_h^{(m)} : t_l^{(m)} - 1)$ and $\hat{\alpha}(t_h^{(m)} : t_l^{(m)})$ be k and k^+ respectively. Then $\sum_{l=0}^{\log_2(k)-1} 2^{l/2} \|\hat{\alpha}(t_h^{(m)} : t_l^{(m)} - 1)[l]\|_1 \leq \sigma$ and $\sum_{l=0}^{\log_2(k^+)-1} 2^{l/2} \|\hat{\alpha}(t_h^{(m)} : t_l^{(m)})[l]\|_1 > \sigma$*

Next we prove the marginal sub-gaussianity of the wavelet coefficients.

Lemma 14. *Consider the observation model $y_i = \theta_i + \sigma z_i$, where z_i is iid sub-gaussian with parameter 1, $i = 1, .., n$. Let α_i denote the wavelet coefficients of the sequence $z = pad_0(y_1, ...y_n)$. Then each α_i is sub-gaussian with parameter 2σ.*

Proof. Without loss of generaility let's charecterize α_1. Let $\boldsymbol{u} = [u_1, ...u_n, u_{n+1}, \ldots, u_{|z|}]^T$ denote the first row of the orthonormal wavelet transform matrix. Then,

$$\alpha_1 = \sum_{i=1}^{n} y_i \left(u_i (1 - \frac{1}{n}) - \sum_{j=1, j\neq i}^{n} \frac{u_j}{n} \right).$$

Thus α_1 is a differentiable function of iid sub-gaussian noise z_i. We can find its Lipschtiz constant by bounding the gradient w.r.t z_i as follows,

$$\begin{aligned}\|\nabla \alpha_1(z_1, ..., z_n)\|_2 &\leq \sigma \left(\sum_{i=1}^{n} 2u_i^2 (1 - \frac{1}{n})^2 + \frac{2}{n} \sum_{j=1, j\neq i}^{n} u_j^2 \right)^{\frac{1}{2}}, \\ &\leq \sigma \left(2 + 2\right)^{\frac{1}{2}}, \\ &= 2\sigma.\end{aligned}$$

By proposition 2.12 in Johnstone [2017] we conclude that α_1 sub-gaussian with parameter 2σ. □

In the next lemma, we record the uniform shrinkage property of soft-thresholding estimator.

Lemma 15. *For any interval $[t_h, t_l]$, let $Y = pad_0(y_{t_h}, ..., y_{t_l})$ and $\Theta = pad_0(\theta_{t_h}, ..., \theta_{t_l})$. Then $|(T(HY))_i| \leq |(H\Theta)_i|$ with probability at-least $1 - 2n^{3-\beta/8}$ for each co-ordinate i.*

Proof. Consider a fixed bin $[\underline{l}, \bar{l}]$ with zero padded vector $Y \in R^k$. Due to sub-gaussian tail inequality, we have $|(HY)_i - (H\Theta)_i| \leq \sigma\sqrt{\beta \log(n)}$ with probability at-least $1 - 2/n^{\beta/8}$. Consider the case $(H\Theta)_i \geq \sigma\sqrt{\beta\log(n)}$. Then both the scenarios $(HY)_i \leq \sigma\sqrt{\beta\log(n)}$ and $(HY)_i > \sigma\sqrt{\beta\log(n)}$

leads to shrinkage to a value that is smaller than $|(H\Theta)_i|$ in magnitude due to soft-thresholding with threshold set to $\sigma\sqrt{\beta\log(n)}$. Now consider the case when $0 \leq (H\Theta)_i \leq \sigma\sqrt{\beta\log(n)}$. Again, soft-thresholding in both scenarios $(HY)_i \leq \sigma\sqrt{\beta\log(n)}$ and $\sigma\sqrt{\beta\log(n)} \leq (HY)_i \leq (H\Theta)_i + \sigma\sqrt{\beta\log(n)}$ leads to shrinkage to a value that is smaller than $|(H\Theta)_i|$ in magnitude. One can come up with a similar argument for the case where $(H\Theta)_i \leq 0$. Now applying a union bound across all $O(n)$ co-ordinates and all $O(n^2)$ bins, we get the statement of the lemma. □

Lemma 16. *The number of bins, M, discovered by the policy is at-most $\max\{1, 2n^{1/3}C_n^{2/3}\sigma^{-2/3}\log(n)\}$ with probability at-least $1-2n^{3-\beta/2}$.*

Proof. Let $\Theta_m = [\theta_1^{(m)}, \theta_2^{(m)}, ..., \theta_p^{(m)}]^T$ be the mean subtracted and zero padded ground truth sequence values in m^{th} bin $[\underline{l}, \bar{l}]$ discovered by our policy. $y^{(m)} = [y_1^{(m)}, y_2^{(m)}, ..., y_p^{(m)}]^T$ be the corresponding mean subtracted and zero padded observations. Note that due to zero padding $p \leq 2(\bar{l} - \underline{l})$ and some of the last values in the vector can be zeroes. Let $\alpha_m(\underline{l} : \bar{l}) = H\Theta$ denotes the discrete wavelet coefficient vector. We can view the computation of the Haar coefficients as a recursion. At each level l of the recursion, the entire length p, is divided into 2^l intervals. Let the sample averages of elements of Θ_m in these intervals be denoted by the sequence $\{\tilde{\theta}_1, \tilde{\theta}_2, ..., \tilde{\theta}_{2^l}\}$. Let $\alpha_m^{(l)} \in \mathbb{R}^{2^l}$ denotes the vector of Haar coefficients at level l.

First note that the Haar coefficient $\alpha_m^{(l)}(i) = \frac{1}{2}\sqrt{\frac{p}{2^l}}(\tilde{\theta}_{2i} - \tilde{\theta}_{2i-1})$ with $i = 1, ..., 2^l$.

$$
\begin{aligned}
\|\alpha_m^{(l)}\|_1^2 &\leq \frac{p}{2^{l+2}}\left(\sum_{i=1}^{2^l}|\tilde{\theta}_{2i} - \tilde{\theta}_{2i-1}|\right)^2, \\
&\leq \frac{pTV^2[\underline{l}-1:\bar{l}]}{2^l},
\end{aligned}
$$

where $TV[a,b]$ denotes the total variation of the true sequence in the interval $[a,b]$. The last inequality holds because the total variation of the smoothed sequence must be at-most four times the entire total variation of true sequence. The factor 4 is due to the fact that total variation when we pad a mean zero sequence with zeroes is at-most twice the total variation before zero padding.

We have,

$$
\frac{1}{\sqrt{p}}\sum_{l=0}^{\log_2(p)-1} 2^{l/2}\|\alpha_m^{(l)}\|_1 \leq \log p\, TV[\underline{l}-1:\bar{l}].
$$

In the policy we compute $\hat{\alpha}_m(\underline{l} : \bar{l}) = T(Hy^{(m)})$ with the soft thresholding factor of $\sigma\sqrt{\beta\log(n)}$. From lemma 15, we have $|(T(Y))_i| \leq |(H\Theta)_i|\ \forall i \in [1,p]$ with probability at-least $1-2n^{3-\beta/8}$. Since $[\underline{l}, \bar{l}]$ is a bin discovered by policy, lemma 13 gives a lowerbound on $\|\alpha_m(\underline{l} : \bar{l})\|$. Putting it all together yields the relation,

$$
\frac{\sigma}{\sqrt{p}} < \frac{1}{\sqrt{p}}\sum_{l=0}^{\log_2(p)-1} 2^{l/2}\|\hat{\alpha}_m^{(l)}(\underline{l}:\bar{l})\|_1 \leq \frac{1}{\sqrt{p}}\sum_{l=0}^{\log_2(p)-1} 2^{l/2}\|\alpha_m^{(l)}(\underline{l}:\bar{l})\|_1 \leq \log(p)TV[\underline{l}-1:\bar{l}], \quad (3)
$$

with probability at-least $1-2n^{3-\beta/8}$.

Thus the total variation in the time interval $[\underline{l}-1, \bar{l}]$ can be lower bounded in probability as

$$
TV[\underline{l}-1:\bar{l}] > \frac{\sigma}{\sqrt{p}\log n}.
$$

Due to assumption $(A3)$ we have,

$$
\sum_{i=1}^{M} TV[\underline{l}^i - 1 : \bar{l}^i] = C_n,
$$

where $[\underline{l}^i : \bar{l}^i]$ are the bins discovered by the policy.

Let p_i be the padded width of bin i discovered by the policy. Then,

$$
\begin{aligned}
C_n \log n &\geq \sum_{i=1}^{M} \frac{\sigma}{\sqrt{p_i}}, \\
&\geq \frac{M^2 \sigma}{\sum_{i=1}^{M} \sqrt{p_i}},
\end{aligned}
$$

where the last line is obtained via Jensen's inequality. Now using Holder's inequality $\|x\|_\beta \leq d^{\frac{1}{\beta}-\frac{1}{\alpha}} \|x\|_\alpha$ for $0 < \beta \leq \alpha$, $x \in \mathbb{R}^d$ with $\alpha = 1/2$, $\beta = 1$ and noting that $\sum_{i=1}^{M} p_i \leq 2T$ gives,

$$
\begin{aligned}
\sigma M^2 &\leq C_n \log n \sum_{i=1}^{M} \sqrt{p_i}, \\
&\leq C_n \log n \sqrt{Mn}.
\end{aligned}
$$

Hence we get $M \leq (2n)^{1/3} (C_n \log n)^{2/3} \sigma^{-2/3} \leq 2n^{1/3} C_n^{2/3} \sigma^{-2/3} \log(n)$.

When $C_n = 0$, (3) implies that our policy will not restart with probability at-least $1 - 2n^{3-\beta/8}$ making $M = 1$. □

We restate two useful results from Donoho [1995].

Lemma 17. *Consider the observation model $y = \alpha + Z$, where $y \in R^k$ and $|Z_i| \leq \delta \forall\, i \in [1, k]$. Let $\hat{\alpha}_\delta$ be the soft thresholding estimator with input y and threshold δ, then*

$$
\|\hat{\alpha}_\delta - \alpha\|^2 \leq \sum_{i=1}^{k} min\{\alpha_i^2, 4\delta^2\}.
$$

Lemma 18. *Consider the observation model $y = \alpha + Z$, where $y \in R^k$, $\alpha \in A$ and each Z_i is sub-gaussian with parameter σ^2. If A is solid and orthosymmetric, then*

$$
\inf_{\hat{\alpha}} \sup_{\alpha \in A} E[\|\hat{\alpha} - \alpha\|^2] \geq \frac{1}{2.22} \sup_{A} \sum_{i=1}^{k} min\{\alpha_i^2, \sigma^2\}.
$$

Let's pause a moment to ponder how remarkable the above lemma is. The observations need not be even iid. Given A is solid and orthosymmetric, all that is required is the marginal sub-gaussianity as the soft-thresholding operation works co-ordinate wise. Now we reprove theorem 4.2 from Donoho [1995] with a slight modification of threshold value in the estimator.

Theorem 19. *With probability at-least $1 - 2n^{-\beta/2}$, under the model in lemma 18, the soft thresholding estimator $\hat{\alpha}_\delta$ with $\delta = \sigma\sqrt{\beta \log(n)}$ obeys*

$$
\|\hat{\alpha}_\delta - \alpha\|^2 \leq 8.88\beta(1 + \log(n)) \inf_{\hat{\alpha}} \sup_{\alpha \in A} E[\|\hat{\alpha} - \alpha\|^2]. \tag{4}
$$

Proof. Consider the soft thresholding estimator $\hat{\alpha}_\delta$. By Gaussian tail inequality we have $P(\sup_i |Z_i| \geq \delta) \leq 2n^{-\beta/2}$. Conditioning on the event $\sup_i |Z_i| \leq \delta$ and applying lemma 17,

$$\begin{aligned}
\|\hat{\alpha}_\delta - \alpha\|^2 &\leq \sum_{i=1}^{k} min\{\alpha_i^2, 4\delta^2\}, \\
&= \sum_{i=1}^{k} min\{\alpha_i^2, 4\beta\sigma^2 \log(n)\}, \\
&\leq max\{1, 4\beta \log(n)\} \sum_{i=1}^{k} min\{\alpha_i^2, \sigma^2\}, \\
&\leq (1 + 4\beta \log(n)) \sup_{\alpha \in A} \sum_{i=1}^{k} min\{\alpha_i^2, \sigma^2\}, \\
&\leq 4\beta(1 + \log(n))\, 2.22 \inf_{\hat{\alpha}} \sup_{\alpha \in A} E[\|\hat{\alpha} - \alpha\|^2],
\end{aligned}$$

where the last line follows from lemma 18. □

It can be shown that wavelet coefficients of functions residing in the TV class is solid and orthosymmetric. As shown in lemma 14, the noisy wavelet coefficients are marginally sub-gaussian. Thus in the coefficient space, we are under the same observation model as in lemma 18. Using a uniform bound argument across all $O(n^2)$ bins and all $O(n)$ points within a bin along with lemma 14 leads to the following corollary.

Corollary 20. *The soft-thresholded wavelet coefficients of re-centered and zero padded noisy data within any interval $[t_h, t_l]$ satisfy relation (4) with probability atleast $1 - 2n^{3-\beta/8}$.*

Next, we record an important preliminary bound that will be used in proving the main result.

Lemma 21. *With probability at-least $1 - \frac{\delta}{2}$, the total squared error for online averaging between two arbitrarily chosen time points t_h and t_l satisfies*

$$\sum_{t=t_h}^{t_l} (x_t^{t_h} - \theta_t)^2 \leq 4\sigma^2 \log(4n^3/\delta)(2 + \log(t_l - t_h + 1)) + 2(\theta_{t_h - 1} - \theta_{t_h})^2 + 2 \sum_{t=t_h+1}^{t_l} (\bar{\theta}_{t_h:t-1} - \theta_t)^2. \tag{5}$$

Proof. Throughout this lemma we assume the notation $\theta_0 = 0$. For proving this, first we bound the squared error for online sample averages within a bin, $b[\underline{l}, \bar{l}]$, that starts and ends at fixed times $\underline{l}$ and $\bar{l}$ respectively. Then a uniform bound argument will be used for bounding the squared error within any arbitrarily chosen bin. Note that $b[\underline{l}, \bar{l}]$ represents any fixed time interval and may not be even chosen by the policy. For $t \in [\underline{l}, \bar{l}]$, consider the prediction $x_t^{\underline{l}}$, with same notation as used in the policy. Define a random variable Z_t as

$$Z_t = \frac{(x_t^{\underline{l}} - \theta_t) - (\lambda_t - \theta_t)}{\sigma\sqrt{1/[t - \underline{l}]_{1+}}},$$

where $[x]_{1+} = max\{1, x\}$, $\lambda_l = \theta_{\underline{l}-1}$ and $\lambda_t = \bar{\theta}_{\underline{l}:t-1}, \forall t > \underline{l}$. Z_t is subgaussian with variance parameter 1 and mean 0. Hence by sub-gaussian tail inequality, we have $P(|Z_t| \geq \sqrt{2\log(4/\delta)}) \leq \delta/2$. By noting that length of a bin is $O(n)$ and applying uniform bound across all time points within the current bin we have

$$P\left(\sup_{\underline{l} \leq t \leq \bar{l}} |Z_t| \geq \sqrt{2\log(4n^3/\delta)}\right) \leq \delta/2n^2.$$

Hence with probability at-least $1 - \delta/2n^2$,

$$|x_t^{\underline{l}} - \theta_t| \leq |\lambda_t - \theta_t| + \sigma\sqrt{\frac{2\log(4n^3/\delta)}{[t - \underline{l}]_{1+}}}, \forall t \in [\underline{l}, \bar{l}]. \tag{6}$$

So the squared error within a bin can be bounded in probability as

$$\sum_{t=\underline{l}}^{\bar{l}}(x_t^{\underline{l}} - \theta_t)^2 \leq 2(\theta_{\underline{l}-1} - \theta_{\underline{l}})^2 + 2\sum_{t=\underline{l}+1}^{\bar{l}}(\bar{\theta}_{\underline{l}:t-1} - \theta_t)^2 + 2\sum_{t=\underline{l}}^{\bar{l}}\sigma^2\frac{2\log(4n^3/\delta)}{[t-\underline{l}]_{1+}}.$$

Here we applied the inequality $(a+b)^2 \leq 2a^2 + 2b^2$ on (6). Ultimately we are interested in analyzing the MSE within a bin detected by the policy. However the observations within a bin satisfies the restarting criterion of the policy and cannot be regarded independent. To break free of this constraint, we uniformly bound the quantity of interest — MSE here — across all possible bins. Noting that number of bins is $O(n^2)$ and applying uniform bound across all bins gives the following single sided tail bound.

Let E denote the event:
$\sup_{b[\underline{l}:\bar{l}]}(x_t^{\underline{l}} - \theta_t)^2 - 2(\theta_{\underline{l}-1} - \theta_{\underline{l}})^2 - 2\sum_{t=\underline{l}+1}^{\bar{l}}(\bar{\theta}_{\underline{l}:t-1} - \theta_t)^2 - 2\sum_{t=\underline{l}}^{\bar{l}}\sigma^2\frac{2\log(4n^3/\delta)}{[t-\underline{l}]_{1+}} \geq 0.$

Then,

$$P(E) \leq \delta/2.$$

Hence with probability at-least $1 - \delta/2$, any bin $b[t_h : t_l]$ satisfies (5). □

Since (5) holds for any arbitrary interval of the time axis, it is particularly true for the bins discovered by the policy. Therefore the total squared error T of the policy is upper bounded in probability by the sum of bin bounds of the form,

$$T \leq \sum_{m=1}^{M} 4\sigma^2\log(4n^3/\delta)(2+\log(t_l^{(m)} - t_h^{(m)}+1)) + 2(\theta_{t_h^{(m)}-1} - \theta_{t_h^{(m)}})^2 + 2\sum_{t=t_h^{(m)}+1}^{t_l^{(m)}}(\bar{\theta}_{t_h^{(m)}:t-1} - \theta_t)^2, \quad (7)$$

where the outer sum iterates over the bins and M is the total number of bins. The first term inside the outer summation can be controlled if we can upper bound M. Now we set out to prove our main theorem.

E Proof of Theorem 1

From the discussion in section 1.1, the goal of bounding dynamic regret of the policy can be achieved by bounding the total squared error of its predictions. Our solution proceeds in two steps. First we upper bound the total squared error within a bin. Then we construct an upper bound for the number of bins spawned by the policy. With these two bounds in place, we bound the total squared error of the policy (7).

Let's first proceed to get a bound on the last summation term in (7). We use a reduction towards Follow The Leader (FTL) strategy. The term is basically the regret incurred by an FTL game with quadratic loss function for the duration $[t_h, t_l]$.

Let $\Theta(t_h : t_l - 1) = pad_0(\theta_{t_h}, ..., \theta_{t_l-1}) = [\Theta_{t_h}, ..., \Theta_{t_h+k-1}]^T$ denotes mean subtracted the zero padded true sequence in the interval $[t_h, t_l - 1]$. Then,

$$\begin{aligned}
\sum_{t=t_h}^{t_l}(\bar{\theta}_{t_h:t-1} - \theta_t)^2 &= (\bar{\theta}_{t_h:t_l-1} - \theta_{t_l})^2 + \sum_{t=t_h}^{t_l-1}(\bar{\theta}_{t_h:t-1} - \theta_t)^2, \\
&\leq (\bar{\theta}_{t_h:t_l-1} - \theta_{t_l})^2 + \sum_{t=t_h}^{t_l-1}\frac{(\bar{\theta}_{t_h:t-1} - \theta_t)^2}{(t - t_h + 1)} + \sum_{t=t_h}^{t_l-1}(\bar{\theta}_{t_h:t_l-1} - \theta_t)^2, \\
&= (\bar{\theta}_{t_h:t_l-1} - \theta_{t_l})^2 + \sum_{t=t_h}^{t_l-1}\frac{(\bar{\theta}_{t_h:t-1} - \theta_t)^2}{(t - t_h + 1)} + \|\Theta(t_h : t_l - 1)\|^2. \qquad (8)
\end{aligned}$$

We have applied FTL reduction for online game of predicting the true sequence $\theta_{t_h}, ..., \theta_{t_l-1}$ to get (8).

In the discussion below we assume that $\|D\theta_{1:n}\|_1 \leq C_n$ and $|\theta_1| \leq U$.

Now let's try to bound the term $\|\Theta(t_h : t_l - 1)\|_2^2$. This is basically the regret of the best expert. By triangle inequality,

$$
\begin{aligned}
\|\Theta(t_h : t_l - 1)\|^2 \leq \|\hat{\alpha}(t_h : t_l - 1)\|_1^2 &+ \|\hat{\alpha}(t_h : t_l - 1) - \alpha(t_h : t_l - 1)\|_2^2, \\
&\leq \left(\sum_{l=0}^{\log_2(p)-1} 2^{l/2} \|\hat{\alpha}(t_h : t_l - 1)[l]\|_1 \right)^2 \\
&\quad + \|\hat{\alpha}(t_h : t_l - 1) - \alpha(t_h : t_l - 1)\|_2^2, \qquad (9)
\end{aligned}
$$

where p is the padded length.

We can base our online averaging restart rule on the output of wavelet smoother. Suppose we decide to restart when $\|\hat{\alpha}(t_h : t_l)\|_1 \geq K n^{-1/3}$ for a constant K. Then the first term of (9) gives the optimal rate of $O(n^{1/3})$ when summed across all bins. But the estimation error term $\|\hat{\alpha}(t_h : t_l - 1) - \Theta(t_h : t_l - 1)\|^2$ should also be controlled. If the smoother is minimax over any bin $[t_h, t_l]$, then we can hope to get minimaxity over the entire horizon. However, the total variation inside the bin is not known to the smoother. This is where the adaptive minimaxity of wavelet smoother comes to rescue.

Suppose $\mathcal{F}$ denotes the class of functions f with total variation $TV(f) \leq C_n$. Let $\mathcal{A}$ denote the set of all coefficients of the continuous wavelet transform of functions $f \in \mathcal{F}$. Then $\mathcal{A} \subset \Theta^{1/2}_{1,\infty}(C_n)$, where $\Theta^{1/2}_{1,\infty}(C_n)$ is a Besov body as defined in Donoho et al. [1998]. The minimax rate of estimation in this Besov body is $O(n^{-2/3} C_n^{2/3} \sigma^{4/3})$ where n is the number of observations. However, this is the rate of convergence of the L_2 function norm instead of the discrete (input-averaged) norm that we consider here. Over the Besov spaces, these two norms are close enough that the rates do not change (see section 15.5 of Johnstone [2017]). Hence Corollary 20 can be used to control the bias.

Let $\hat{y}(t_h : t)$ denotes the soft-thresholding estimates of the vector $pad_0(y_{t_h:t})$.
i.e $\hat{y}(t_h : t) = H^T T(H\, pad_0(y(t_h : t)))$.

$$
\begin{aligned}
(\bar{\theta}_{t_h:t_l-1} - \theta_{t_l})^2 &\leq 2(\theta_{t_l-1} - \theta_{t_l})^2 + 2(\bar{\theta}_{t_h:t_l-1} - \theta_{t_l-1})^2, \\
&\leq 2(\theta_{t_l-1} - \theta_{t_l})^2 + 4(\hat{y}(t_h : t_l - 1)[t_l - 1] - (\bar{\theta}_{t_h:t_l-1} - \theta_{t_l-1}))^2 \\
&\quad + 4(\hat{y}(t_h : t_l - 1)[t_l - 1])^2. \qquad (10)
\end{aligned}
$$

Since L1 norm is greater than L2 norm, the policy's restart rule implies that

$$
(\hat{y}(t_h : t_l - 1)[t_l - 1])^2 \leq \sigma^2 \qquad (11)
$$

Combining (10) and (11), we get

$$
(\bar{\theta}_{t_h:t_l-1} - \theta_{t_l})^2 \leq 2(\theta_{t_l} - \theta_{t_l-1})^2 + \gamma_1 (t_l - t_h)^{1/3}\, TV^{2/3}[t_h : t_l]\, \sigma^{4/3} + \sigma^2, \qquad (12)
$$

where last line holds with probablity atleast $1 - 2n^{3-\beta/8}$ due to Corollary 20. Here γ_1 is a constant which can depend logarithmically on the width $t_l - t_h$.

Now let's bound the second term in (8). For any $t \in [t_h, t_l - 1]$ we have,

$$
\begin{aligned}
\sum_{t=t_h}^{t_l-1} \frac{(\bar{\theta}_{t_h:t-1} - \theta_t)^2}{(t - t_h + 1)} &\leq \sum_{t=t_h}^{t_l-1} \frac{2(\theta_t - \theta_{t-1})^2 + 2(\bar{\theta}_{t_h:t-1} - \theta_{t-1})^2}{t - t_h + 1}, \\
&\leq \sum_{t=t_h}^{t_l-1} 2(\theta_t - \theta_{t-1})^2 \\
&\quad + \sum_{t=t_h}^{t_l-1} \frac{4(\hat{y}(t_h : t-1)[t-1] - (\bar{\theta}_{t_h:t-1} - \theta_{t-1}))^2 + 4(\hat{y}(t_h : t-1)[t-1])^2}{t - t_h + 1}, \\
&\leq \sum_{t=t_h}^{t_l-1} 2(\theta_t - \theta_{t-1})^2 + (\gamma_2 (t_l - t_h)^{1/3}\, TV^{2/3}[t_h : t_l]\, \sigma^{4/3} + 4\sigma^2)(1 + \log n),
\end{aligned} \tag{13}
$$

where the last line holds with probability at-least $1 - 2n^{3-\beta/8}$.

$$
\begin{aligned}
\|\Theta(t_h : t_l - 1)\|_2^2 &\leq \left(\sum_{l=0}^{\log_2(p)-1} 2^{l/2} \|\hat{\alpha}(t_h : t_l - 1)[l]\|_1 \right)^2, \\
&\quad + \gamma_3 (t_l - t_h)^{1/3}\, TV^{2/3}[t_h : t_l]\, \sigma^{4/3}, \\
&\leq \sigma^2 + \gamma_3 (t_l - t_h)^{1/3}\, TV^{2/3}[t_h : t_l]\, \sigma^{4/3},
\end{aligned} \tag{14}
$$

with probability at-least $1 - 2n^{3-\beta/8}$ for some constant γ_3 which can depend logarithmically on the width $t_l - t_h$.

Due to Corollary 20 the bounds (12), (13), (14) all simultaneously holds with probability at-least $1 - 2n^{3-\beta/8}$. Combining these bounds, we get

$$
\sum_{t=t_h}^{t_l} (\bar{\theta}_{t_h:t-1} - \theta_t)^2 \leq 2\|D\theta_{t_h:t_l}\|_2^2 + \gamma (t_l - t_h)^{1/3}\, TV^{2/3}[t_h : t_l]\, \sigma^{4/3} + 6\sigma^2(1 + \log(n)),
$$

with probability at-least $1 - 2n^{3-\beta/8}$ and $\gamma = \gamma_1 + \gamma_2(1 + \log(n)) + \gamma_3$.

When summed across all bins as in (7), with probability at-least $1 - 2n^{3-\beta/8}$ we have,

$$
\begin{aligned}
\sum_{m=1}^{M} \sum_{t=t_h^{(m)}}^{t_l^{(m)}} (\bar{\theta}_{t_h^{(m)}:t-1} - \theta_t)^2 &\leq U^2 + 2\|D\theta_{1:n}\|_2^2 + 6M\sigma^2(1 + \log n) \\
&\quad + \sum_{m=1}^{M} \gamma\, (k^{(m)})^{1/3}\, TV^{2/3}[t_h^{(m)} : t_l^{(m)}]\, \sigma^{4/3}, \\
&\leq U^2 + 2\|D\theta_{1:n}\|_2^2 + 6M\sigma^2(1 + \log n) \\
&\quad + \gamma \sigma^{4/3} n^{1/3} \left(\sum_{m=1}^{M} \frac{k^{(m)}}{n} \right)^{\frac{1}{3}} \left(\sum_{m=1}^{M} TV[t_h^{(m)} : t_l^{(m)}] \right)^{\frac{2}{3}}, \qquad (15) \\
&\leq U^2 + 2\|D\theta_{1:n}\|_2^2 + 6M\sigma^2(1 + \log n) \\
&\quad + 2\gamma \sigma^{4/3} n^{1/3} C_n^{2/3}. \qquad (16)
\end{aligned}
$$

Here $k^{(m)}$ is the length of $\Theta(t_h^{(m)} : t_l^{(m)} - 1)$. The term $(\theta_{t_h^{(m)}-1} - \theta_{t_h^{(m)}})^2$ is at-most U^2 for the first bin. We arrive at (15) by applying Holder's inequality $x^T y \le \|x\|_p \|y\|_q$ with $p = 3$ and $q = 3/2$. For both (15) and (16) we use the fact that $\sum_{m=1}^{M} k^{(m)} \le 2n$ where the factor 2 is an artifact of zero-padding.

By appealing to lemma 16, we have with probability at-least $1 - 4n^{3-\beta/8}$,

$$\sum_{m=1}^{M} \sum_{t=t_h^{(m)}}^{t_l^{(m)}} (\bar{\theta}_{t_h^{(m)}:t-1} - \theta_t)^2 \le U^2 + 2\|D\theta_{1:n}\|_2^2 + 12\sigma^2 \log n \\ + 24(\log(n))^2 n^{1/3} C_n^{2/3} \sigma^{4/3} + \gamma\sigma^{4/3} n^{1/3} C_n^{2/3}. \tag{17}$$

Next, we proceed to bound the first summation terms in (7). For this, we upperbound the number of bins to control the concentration terms in (7) when summed across all bins. Essentially our decision rule should not lead to over binning. Observe that the sum of total variations across all bins is C_n. If the decision rule guarantees (at-least in probability) that total variation inside any detected bin is $\tilde{\Omega}(n^{-1/3})$, then the number of bins is optimally $O(n^{1/3})$. Such a TV lower bounding property is satisfied by wavelet soft-thresholding as described in lemma 16. This is facilitated by the uniform shrinkage property of soft-thresholding estimator. More precisely,

Let's denote

$$V_m = 4\sigma^2 \log(2n^3/\delta)(2 + \log(t_l^{(m)} - t_h^{(m)} + 1)).$$

Then,

$$\begin{aligned}\sum_{m=1}^{M} V_m &\le 4\sigma^2 \log(4n^3/\delta)(2 + \log(n)) \max\{1, 2n^{1/3} C_n^{2/3} \sigma^{-2/3} \log(n)\}, \\ &\le 4\sigma^2 \log(4n^3/\delta)(2 + \log(n)) \\ &\quad + 8n^{1/3} C_n^{2/3} \sigma^{4/3} \log(n) \log(4n^3/\delta)(2 + \log(n)),\end{aligned} \tag{18}$$

with probability at-least $1 - 2n^{3-\beta/8}$. Here $[t_h^m, t_l^m]$ corresponds to the m^{th} bin discovered by the policy. This relation follows due to Lemma 16.

Combining (18) and (17) we have with probability at-least $1 - 4n^{3-\beta/8} - \delta/2$

$$\begin{aligned}T &\le 8n^{1/3} C_n^{2/3} \sigma^{4/3} (2 + \log(n)) \log(n) \\ &\quad + 4\sigma^2 \log(4n^3/\delta)(2 + \log(n)) \\ &\quad + U^2 + 2\|D\theta_{1:n}\|_2^2 + 12\sigma^2 \log n \\ &\qquad + 24(\log(n))^2 n^{1/3} C_n^{2/3} \sigma^{4/3} + 2\gamma\sigma^{4/3} n^{1/3} C_n^{2/3}.\end{aligned} \tag{19}$$

By observing that $\|D\theta_{1:n}\|_2 \le \|D\theta_{1:n}\|_1 = C_n$ we get the bound,

$$\begin{aligned}T &\le 8n^{1/3} C_n^{2/3} \sigma^{4/3} (2 + \log(n)) \log(n) \\ &\quad + 4\sigma^2 \log(4n^3/\delta)(2 + \log(n)) \\ &\quad + U^2 + 2C_n^2 + 12\sigma^2 \log n \\ &\qquad + 24(\log(n))^2 n^{1/3} C_n^{2/3} \sigma^{4/3} + 2\gamma\sigma^{4/3} n^{1/3} C_n^{2/3}.\end{aligned}$$

The above bounds holds with probability at-least $1 - \delta$, if we set $\beta = 24 + \frac{8\log(8/\delta)}{\log(n)}$.

We conclude our proof by observing that the above arguments can be readily extended to any batch smoother that satisfy the following criteria.

- Adaptive minimaxity over any interval within the time horizon.
- The restart decision rule optimally lowerbounds the total variation of any spawned bin.

Thus our policy can be viewed as a meta-algorithm that lifts a "well behaved smoother" to an optimal forecaster in the online setting.

Next we remark how the proof can be adapted to the setting where an extra boundedness constraint is put on the ground truth. i.e, $\theta_{1:n} \in TV(C_n)$ and $|\theta_i| \leq B, i = 1, \ldots, n$. Then the U^2 term in (19) becomes B^2. The additive $\|D\theta_{1:n}\|_2^2$ term can be bounded as,

$$
\begin{aligned}
\|D\theta_{1:n}\|_2^2 &= \sum_{i=2}^{n} (\theta_i - \theta_{i-1})^2, \\
&\leq \sum_{i=2}^{n} (|\theta_i| + |\theta_{i-1}|)(|\theta_i - \theta_{i-1}|), \\
&\leq 2BC_n.
\end{aligned}
$$

With the boundedness constraint, we also have $\|D\theta_{1:n}\|_2^2 \leq 4nB^2$. This essentially implies that $\|D\theta_{1:n}\|_2^2 \leq \min\{4nB^2, 2BC_n\}$.

Thus when $\|\theta_{1:n}\|_\infty \leq B$ and if we set $\beta = 24 + \frac{8\log(8/\delta)}{\log(n)}$ then with probability at-least $1-\delta$,

$$
\begin{aligned}
T \leq\ & 8n^{1/3} C_n^{2/3} \sigma^{4/3} (2 + \log(n)) \log(n) \\
&+ 4\sigma^2 \log(4n^3/\delta)(2 + \log(n)) \\
&+ B^2 + 2\min\{4nB^2, 2BC_n\} + 12\sigma^2 \log n \\
&\quad + 24(\log(n))^2 n^{1/3} C_n^{2/3} \sigma^{4/3} + 2\gamma\sigma^{4/3} n^{1/3} C_n^{2/3}.
\end{aligned}
\tag{20}
$$

F Adaptive Optimality in Discrete Sobolev class

In this section, we establish that despite the fact that ARROWS is designed for the total variation class, it adapts to the optimal rates forecasting sequences that are more regular.

The discrete Sobelov class is defined as

$$\mathcal{S}(C_n') = \{\theta_{1:n} : \|D\theta_{1:n}\|_2 \leq C_n'\}.$$

The minimax cumulative error of nonparametric estimation in the discrete Sobolev class is $\theta_{1:n}(n^{2/3}[C_n']^{2/3}\sigma^{4/3})$ [see e.g., Sadhanala et al., 2016, Theorem 5 and 6].

Recall that the discrete Total Variation class that we considered in this paper is defined as

$$\mathcal{T}(C_n) = \{\theta_{1:n} : \|D\theta_{1:n}\|_1 \leq C_n\}.$$

By the norm inequalities, we know that

$$\mathcal{T}(C_n') \subset \mathcal{S}(C_n') \subset \mathcal{T}(C_n'\sqrt{n}).$$

The following refinement of our main theorem establishes that ARROWS also achieves the minimax rate in discrete Sobolev classes.

Theorem 22. *Let the feedback be $y_t = \theta_t + Z_t$ where Z_t is an independent, σ-subgaussian random variable. Let $\theta_{1:n} \in \mathcal{S}(C_n')$. If $\beta = 24 + \frac{8\log(8/\delta)}{\log(n)}$, then with probability at least $1-\delta$,* ARROWS *achieves a dynamic regret of $\tilde{O}(n^{2/3}[C_n']^{2/3}\sigma^{4/3} + U^2 + [C_n']^2 + \sigma^2)$ where $\tilde{O}$ hides a logarithmic factor in n and $1/\delta$.*

Proof. Let's minimally expand the Sobolev ball to a TV ball of radius $C_n = \sqrt{n}C_n'$. This chosen radius of the TV ball is in accordance with the canonical scaling introduced in [Sadhanala et al., 2016]. This activates the following embedding:

$$\mathcal{S}_1(C_n') \subseteq TV(C_n).$$

We can rewrite (19) as

$$
\begin{aligned}
T \leq\ & 8n^{1/3}\|D\theta_{1:n}\|_1^{2/3}\sigma^{4/3}(2+\log(n))\log(n) \\
& + 4\sigma^2\log(4n^3/\delta)(2+\log(n)) \\
& + U^2 + 2\|D\theta_{1:n}\|_2^2 + 12\sigma^2\log n \\
& \quad + 24(\log(n))^2 n^{1/3}\|D\theta_{1:n}\|_1^{2/3}\sigma^{4/3} + 2\gamma\sigma^{4/3}n^{1/3}\|D\theta_{1:n}\|_1^{2/3}.
\end{aligned} \tag{21}
$$

The above representation reveals the optimality of our policy over Sobolev class $S_1(C'_n)$. Enlarging the Sobolev class to the TV class that contains it does not change the minimax rate in the smoothing setting. See, e.g., Theorem 5 and 6 of [Sadhanala et al., 2016] and take $d = 1$, and $C'_n = n^{-1/2}C_n$. By using $\|x\|_1 \leq n^{1/2}\|x\|_2$ for $x \in \mathbb{R}^n$,

$$\frac{\|D\theta_{1:n}\|_1}{n^{1/2}} \leq \|D\theta_{1:n}\|_2 \leq C'_n = \frac{C_n}{n^{1/2}}.$$

Plugging this bound on $\|D\theta_{1:n}\|_1$ in (21) recovers the minimax regret for the Sobolev class of radius C'_n. The additional term of $\|D\theta_{1:n}\|_2^2$ — similar to as shown in in appendix I — is unavoidable in the online setting for predicting discrete Sobolev sequences.

□

Remark 23. Note that $\mathcal{T}(C'_n) \subset \mathcal{S}(C'_n)$, therefore our lower bound from Proposition 6 still applies, which suggests that the additional $[C'_n]^2 + \sigma^2$ is required and that ARROWS is an optimal forecaster for sequences in Sobolev classes as well.

G Fast Computation

We describe the proof of $O(n\log n)$ runtime guarantee below.

We use an inductive argument. Without loss of generality let the start of current bin be at time 1. Suppose we know the wavelet transform of points upto time t. Let the next highest power of 2 for both t and $t+1$ be p. We identify this value as a pivot for time t and $t+1$. Zero padding is done to hit this pivot. We can view the pad_0 operation at time $t+1$ as the difference between the padded original data and and a step signal. This step signal assume the value $\bar{y}_{1:t+1}$ in time $[1, t+1]$ and 0 in $[t+2, p]$. For computing wavelet transform of the step, we need to update only $O(\log(p))$ coefficients. Inputs to the Haar transform of the padded data at times t and $t+1$ differs by just one co-ordinate. Hence coefficients of only $\log(p)$ wavelets need to be changed. Each such change can be performed in $O(1)$ time in an incremental fashion.

Now let's consider the case when the pivot for time $t+1$ is $2t$. Suppose we know the Haar wavelet coefficients upto time t. In this case, we need to compute the coefficients of $\log(t)$ newly introduced wavelets that span the interval $[t, 2t]$ since the zero padding will force most of the new wavelet coefficients to be zero. The computation of each of those new coefficients can be done in $O(1)$ due to sparsity of signal in interval $[t, 2t]$. We also need to change the first two wavelet coefficients which can be done again in in $O(1)$ time. In all these cases, we only need to do soft-thresholding to the newly updated coefficients. At the base case, when the pivot is just 2, then the computation can be in $O(1)$ time. Thus within a pivot p, the number of computations required is $O(p\log(p))$ which translates to $O(k^{(m)}\log(k^{(m)}))$ computations within the m^{th} bin. Summing across all the bins yields a runtime complexity of $O(n\log(n))$.

H Regret of AOMD

In this section we prove that for any predictable sequence $\{M_t\}_{t=1}^n$, the AOMD algorithm has a dynamic regret of $\tilde{O}(\sqrt{n})$ when applied to our problem. As discussed in Section 2, consider loss functions $f_t(x) = (x - y_t)^2$ and comparator sequence $\{u_t\}_{t=1}^n$. First let's consider a deterministic noise setting [Donoho, 1995]:

$$y_t = \theta_t + \delta\,\sigma\sqrt{20\log(n)},$$

where $|\delta|\leq 1$ is chosen by a clever adversary. Let's proceed to get a bound on the quantity D_n. The gradient of our loss function is $2(x-y_t)$. So after observing the values of x_t and M_t, an adversary can pick a suitable δ such that each term of D_n

$$D_n = \sum_{t=1}^{n} \|\nabla f_t(x_t) - M_t\|_*^2.$$

can be made $O(1)$. This gives an $O(n)$ bound for D_n.

We can show that V_n is $O(n)$ if we assume that $\mathcal{X}$ is compact and all of the y_t is bounded. Boundedness of y_t follows from the assumptions (A3) and (A4). By appealing to assumption (A3) we see that

$$C_n(u_1, u_2, ..., u_n) = \sum_{t=1}^{n} \|u_t - u_{t-1}\|.$$

$C_n(\theta_1, ..., \theta_n)$ is $O(1)$. Plugging this into the regret bound specified in Jadbabaie et al. [2015] bounds the dynamic regret in our setting as $\tilde{O}(\sqrt{n})$.

We now relate this deterministic noise setting to the guassian setting where the observations are produced according to $y_t = \theta_t + Z_t$, where Z_t is a zero mean sub gaussian with parameter σ^2. As described in proof of theorem 19, $P(\sup_i|Z_i|\geq \sigma\sqrt{20\log(n)}) \leq 2n^{-9}$. Hence by conditioning on the event that $\sup_i|Z_i|\leq \sigma\sqrt{20\log(n)}$, the regret bound of the deterministic noise setting applies to gaussian setting with high probability.

I Lower bound proof

Proof of Proposition 6. First, a lower bound of $\Omega(n^{1/3}C_n^{2/3}\sigma^{4/3})$ is given by [Donoho et al., 1998] for the smoothing estimator $x_{1:n}$ that has more information than we do. The argument uses the fact that the TV-ball is sandwiched between two Besov-bodies with identical minimax rate. To the best of our knowledge, the dependence on C_n and σ is first made explicit in, e.g., [Birge and Massart, 2001].

By the fact that "the max is larger than the mean", we have that for any prior distribution $\mathcal{P}$,

$$\sup_{\theta_{1:n}\in \mathrm{TV}(C_n)} \mathbb{E}\left[\sum_{t=1}^{n}(x_t-\theta_t)^2\right] \geq \mathbb{E}_{\theta_{1:n}\sim\mathcal{P}}\left[\mathbb{E}[\sum_{t=1}^{n}(x_t-\theta_t)^2|\theta_{1:n}]\right].$$

Take $\mathcal{P}$ such that

1. $\theta_1 = U$ with probability 0.5 and $-U$ otherwise.
2. $\theta_2 = \theta_1 + C_n$ with probability 0.5 and $\theta_1 - C_n$ otherwise.
3. $\theta_t = \theta_2$ for $t = 3, 4, ..., n$.

Note that x_1 does not observe anything yet, therefore $x_1 = 0$ is the Bayes optimal decision rule. This gives a trivial lower bound of $\mathbb{E}\left[(x_1-\theta_1)^2\right] \geq U^2$. Now, let's reveal θ_1 to x_2 an additional information, then by the same argument, we have that $\mathbb{E}\left[(x_2-\theta_2)^2\right] \geq C_n^2$.

Consider an alternative $\mathcal{P}$ when $\theta_1 = ... = \theta_n = \theta$. Let the noise be iid Gaussian with variance σ^2. In this case the problem reduces to a naive statistical estimation problem with $\theta \in [-U, U]$. For each t which observes $t-1$ iid samples from $\mathcal{N}(\theta, \sigma^2)$, then by Bickel et al. [1981], the minimax risk for this problem is

$$\inf_{\hat{\theta}} \sup_{\theta\in[-U,U]} \mathbb{E}(\hat{\theta}-\theta)^2 = \frac{\sigma^2}{t} - \frac{\pi^2\sigma^4}{tU^2} + o(\frac{\sigma^4}{tU^2}).$$

Summing over $t = 2, 3, ..., n$, and apply the upper/lower bounds of the harmonic series, we have a lower bound of

$$\mathbb{E}\left[\sum_{t=1}^{n}(x_t-\theta_t)^2\right] \geq \max\{0, \sigma^2\log(n+1) - \frac{\pi^2\sigma^4}{U^2}(1+\log(n))(1+o(1))\}.$$

Take the condition that $U > 2\pi\sigma$ and $n > 3$, the above expression can be further lower bounded by $0.5\sigma^2 \log(n)$. Note that this bound applies even if $C_n = 0$.

Finally, we can similarly apply the same argument to the case when $\theta_1 = 0$ and $\theta_2 = ... = \theta_n = \theta$ and where the constraint is that $-C_n \leq \theta \leq C_n$. This gives us a lower bound of

$$\mathbb{E}\left[\sum_{t=2}^{n}(x_t - \theta_t)^2\right] \geq \max\{0, \sigma^2 \log(n) - \frac{\pi^2\sigma^4}{C_n^2}(1 + \log(n-1))(1 + o(1))\}.$$

If $C_n > 2\pi\sigma$ and $n > 3$, we can again bound it below by $0.5\sigma^2 \log(n)$. In other word, we get the $\sigma^2 \log(n)$ lower bound provided that either C_n or U is greater than $2\pi\sigma$.

The proof is complete by taking the average of lower bounds above. We can take $c = 1/6$. $\square$

I.1 Lower bound with extra boundedness constraint on ground truth

Suppose we assume $|\theta_i| \leq B, i = 1, \ldots, n$. Then we can adapt the proof presented above by considering a prior $\mathcal{P}$ such that $\theta_i = \epsilon_i B, i = 1, \ldots, \min\{n, 1 + \lfloor C_n/2B \rfloor\}$. $\theta_i = \theta_{1+\lfloor C_n/2B \rfloor}, \forall i > \min\{n, 1 + \lfloor C_n/2B \rfloor\}$. Here ϵ_i are independent random variables assuming value $+1$ with probability 0.5 and -1 with probability 0.5. Assume that we reveal to learner the probability law of observations θ_i. Under this setting we can see that $\mathbb{E}\left[\sum_{t=1}^{n}(x_t - \theta_t)^2\right] \geq B^2 + \min\{(n-1)B^2, BC_n/2\}$.

Under the setting of $y_i = \theta_i + \epsilon_i$ for iid σ^2 sub-gaussian ϵ_i, $|\theta_i| \leq B$ and $i = 1, \ldots, n$,[Donoho et al., 1990] establishes that minimax total squared error scales as $n \min\{B^2, \sigma^2\}$. This along with previous discussions imply that in the bounded ground truth setting the minimax risk is $\tilde{\Omega}\left(\min\{nB^2, n\sigma^2, n^{1/3}C_n^{2/3}\sigma^{4/3}\} + B^2 + \min\{nB^2, BC_n\} + \sigma^2\right)$.

I.2 Minimax regret using ARROWS for bounded ground truth

From (20) the regret of ARROWS T_{ARROWS} satisfy

$$T_{\text{ARROWS}} = \tilde{O}(n^{1/3}C_n^{2/3}\sigma^{4/3} + \min\{nB^2, BC_n\} + \sigma^2).$$

Let T_1 be the regret of an algorithm, say $\mathcal{A}_1$, that predicts $p \sim N(0, \sigma^2)$ at time step 1 and zero for remaining times. Then it can seen that

$$\begin{aligned} T_1 &= O(nB^2 + \sigma^2), \\ &= O(nB^2 + \sigma^2 + \min\{nB^2, BC_n\}). \end{aligned}$$

Let T_2 be the regret of an algorithm, say $\mathcal{A}_2$, that predicts y_{t-1} at time t. Then,

$$T_2 = O(n\sigma^2 + \min\{nB^2, BC_n\}).$$

Now consider running exponentially weighted average forecaster [Cesa-Bianchi and Lugosi, 2006] with three experts: ARROWS, $\mathcal{A}_1$ and $\mathcal{A}_2$. Since squared error is exponentially concave, by Proposition 3.1 of [Cesa-Bianchi and Lugosi, 2006] such a forecaster when run with $\eta = 2$ gives a regret T that satisfy,

$$\begin{aligned} T &= O\left(\min\{T_{\text{ARROWS}}, T_1, T_2\} + \log 3\right), \\ &= \tilde{O}\left(\min\{nB^2, n\sigma^2, n^{1/3}C_n^{2/3}\sigma^{4/3}\} + B^2 + \min\{nB^2, BC_n\} + \sigma^2 + \log 3\right). \end{aligned}$$

Thus we acheive the optimal cumulative squared error upto a small additive term of $\log 3$. If we look at the per round regret this additive term contributes to a small $O(1/n)$ quantity.

I.3 Connections to other lower bounds in literature

[Besbes et al., 2015] derived a lower bound of $O(n^{1/2}V_n^{1/2})$ by packing a sequence of quadratic loss functions. Note that this is larger than the upper bound that we attain with quadratic losses.

Though this observation seems confusing, a careful study reveals that there is no contradiction. For constructing the lowerbound, [Besbes et al., 2015] used a variational budget V_n as , $V_n = \sum_{t=2}^{n} \sup_{x \in conv(\theta_1,\ldots\theta_n)} |f_t(x) - f_{t-1}(x)| = \sum_{t=2}^{n} \sup_{x\in[\theta_{min},\theta_{max}]} |(x-\theta_t)^2 - (x-\theta_{t-1})^2|$, where $conv(.)$ denotes the convex hull of a sequence of points. This is different from the variational budget they use in section 2 of their paper and is also different from C_n that we use for the TV class. When applied to our setting this V_n is no longer proportional to our C_n, instead, it is proportional to $(\theta_{max} - \theta_{min})C_n$.

The packing set constructed through the functions defined in equation (A-12) of [Besbes et al., 2015] obeys $(\theta_{max} - \theta_{min}) = \frac{1}{2} V_n^{1/4} n^{-1/4}$. So we have $C_n = \frac{V_n}{V_n^{1/4} n^{-1/4}} = V_n^{3/4} n^{1/4}$, where we have subsumed proportionality constants. Thus we see that $V_n = \frac{C_n^{4/3}}{n^{1/3}}$. Putting this into their lowerbound recovers exactly our $n^{1/3}C^{2/3}$ bound.

The additional C_n^2 term that appears in our upper bound is required for any methods that do online forecasting of sequences in the TV class. The reason why OGD appears to not require C_n^2 according to [Besbes et al., 2015] is because they require the θ_t to be bounded for all t, while we only require θ_1 to be bounded by U (see Theorem 11).

The lowerbound discussed in [Yang et al., 2016] considers a more general setting of smooth non-strongly convex sequence of loss functions. Such a lowerbound will not apply in our more restrictive setting.

J Optimality of linear forecasters in Discrete Sobolev class

In this section we first establish that just like ARROWS, linear strategies such as OGD and MA are also optimal forecasters for sequences in Discrete Sobolev class. Then we substantiate it using experiments.

Theorem 24. *Let the feedback be $y_t = \theta_t + Z_t$ where Z_t is an independent, σ-subgaussian random variable. Let $\theta_{1:n} \in \mathcal{S}(C_n')$. Restarting OGD with batch size of $\frac{\sigma^{2/3}(n\log n)^{1/3}}{[C_n']^{2/3}}$ achieves an expected dynamic regret of $\tilde{O}(U^2 + [C_n']^2 + n^{2/3}[C_n']^{2/3}\sigma^{4/3})$.*

Proof. We stick to the same notations as in Appendix C. Let's start the analysis from (2). Let $t' = t - t_h^{(i)}$.

$$
\begin{aligned}
(\theta_t - \bar{\theta}_{t_h^{(i)}:t-1})^2 &\leq \frac{\left(\sum_{i=t_h^{(i)}}^{t-1} (\theta_t - \theta_i)\right)^2}{[t']^2}, \\
&\leq \frac{t'}{[t']^2} \sum_{i=t_h^{(i)}}^{t-1} (\theta_t - \theta_i)^2, \\
&\lesssim L[C_i']^2.
\end{aligned}
$$

Hence summing across all points yields,

$$R_i \lesssim L^2[C_i']^2 + \sigma^2 \log L.$$

So the total expected regret becomes,

$$\sum_{i=1}^{\lceil n/L \rceil} R_i \lesssim L^2[C_n']^2 + \frac{n}{L}\sigma^2 \log L.$$

By choosing $L = \frac{\sigma^{2/3}(n\log n)^{1/3}}{[C_n']^{2/3}}$ we get the theorem. The additive term $[C_n']^2$ arises similarly as in proof of Theorem 11 $\square$

The optimality of Moving Averages can be proved similarly.

Remark 25. Thus from Theorems 3, 9, 11, 24 we see that ARROWS is minimax over both the classes $TV(C_n)$ and $\mathcal{S}(C_n/\sqrt{n})$ while linear forecasters such as OGD and MA require different tuning parameters to perform optimally in each class.

Next, we give numerical experiments substantiating the claims.

Experimental results: Here we consider a doppler function $f(t) = \sin\left(\frac{2\pi(1+\epsilon)}{t/n+0.01}\right)$ with n

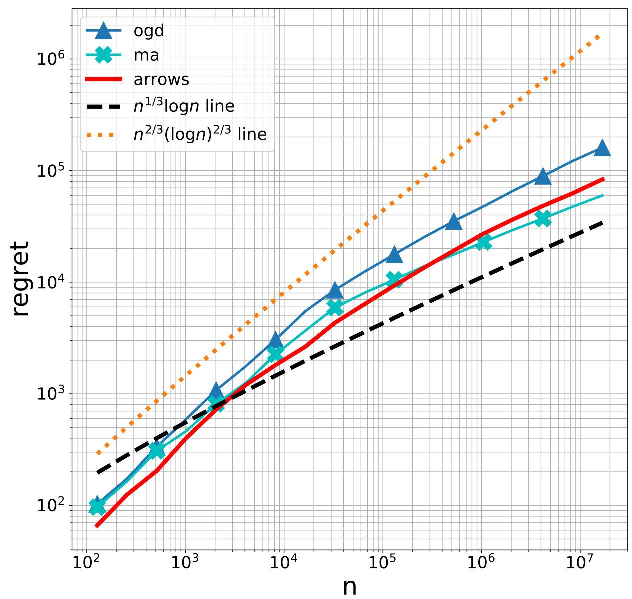

Figure 6: Regret plot for policies calibrated according to Sobolev radius for a Doppler function

being the time horizon. For this function $C'_n = \|D\theta\|_2 = O(C_n/\sqrt{n})$ when n is sufficiently large and $\|D\theta\|_2 = O(C_n)$ for small n for a TV bound $C_n = O(1)$. Thus for sufficiently large n, this sequence belong to a small Sobolev ball with radius $O(1/\sqrt{n})$ while the TV class that encloses that Sobolev ball as per Theorem 22 has radius $O(1)$.

We observe noisy data $y_i = f(i/n) + z_i, i = 1, ..., n$ and z_i are iid normal variables with $\sigma = 1$.Figure 6 plots the regret averaged across 5 runs in a log log scale. The necessary input calibration was made as per Remark 23 while running ARROWS. We can see that in this case all the algorithms perform in an optimal manner.

Specifically we identify two regimes one for small n and other for larger n. When n is large, we obtain the minimax regret rate $\tilde{O}(n^{1/3})$ due to small C'_n which can be considered as $O(1/\sqrt{n})$. Numerically for $n > 10^5$, C'_n is less than 0.1% of C_n. For smaller values of n where C'_n can be not too small, we attain a regret in accordance with the $\tilde{O}(n^{2/3})$ minimax rate. Numerically when $n < 10^4$, C'_n is atleast 8.5% of C_n which can be considered as $O(C_n) = O(1)$.