Appendix

Organization of the Appendix

In Appendix A we review classical results on the Taylor expansions for functions on Riemannian manifold. In Appendix B we provide the proof of Lemma 2 which requires to expand the iterates on the tangent space in the the saddle point. Finally, in Appendix C, we provide the proofs of Lemma 7 and Lemma 8 which enable to prove the main theorem of the paper.

Throughout the paper we assume that the objective function and the manifold are smooth. Here we list the assumptions that are used in the following lemmas.

Assumption 1 (Lipschitz gradient)**.** *There is a finite constant β such that*

$$\|\mathrm{grad} f(y) - \Gamma_x^y \mathrm{grad} f(x)\| \leq \beta d(x, y) \quad \textit{for all } x, y \in \mathcal{M}.$$

Assumption 2 (Lipschitz Hessian)**.** *There is a finite constant ρ such that*

$$\|H(y) - \Gamma_x^y H(x) \Gamma_y^x\|_2 \leq \rho d(x, y) \quad \textit{for all } x, y \in \mathcal{M}.$$

Assumption 3 (Bounded sectional curvature)**.** *There is a finite constant K such that*

$$|K(x)[u, v]| \leq K \quad \textit{for all } x \in \mathcal{M} \textit{ and } u, v \in \mathcal{T}_x\mathcal{M}$$

A Taylor expansions on Riemannian manifold

We provide here the Taylor expansion for functions and gradients of functions defined on a Riemannian manifold.

A.1 Taylor expansion for the gradient

For any point $x \in \mathcal{M}$ and $z \in \mathcal{M}$ be a point in the neighborhood of x where the geodesic $\gamma_{x\to z}$ is defined.

$$\begin{aligned}\Gamma_z^x(\mathrm{grad} f(z)) &= \mathrm{grad} f(x) + \nabla_{\gamma'_{x\to z}(0)} \mathrm{grad} f + \textstyle\int_0^1 (\Gamma^x_{\gamma_{x\to z}(\tau)} \nabla_{\gamma'_{x\to z}(\tau)} \mathrm{grad} f - \nabla_{\gamma'_{x\to z}(0)} \mathrm{grad} f) dx_\tau \\ &= \mathrm{grad} f(x) + \nabla_{\gamma'_{x\to z}(0)} \mathrm{grad} f + \Delta(z), \end{aligned} \tag{1}$$

where $\Delta(z) := \int_0^1 (\Gamma^x_{\gamma_{x\to z}(\tau)} \nabla_{\gamma'_{x\to z}(\tau)} \mathrm{grad} f - \nabla_{\gamma'_{x\to z}(0)} \mathrm{grad} f) d\tau$. The Taylor approximation in Eq. (1) is proven by Absil et al. (2009, Lemma 7.4.7).

A.2 Taylor expansion for the function

Taylor expansion of the gradient enables us to approximate the iterations of the main algorithm, but obtaining the convergence rate of the algorithm requires proving that the function value decreases following the iterations. We need to give the Taylor expansion of f with the parallel translated gradient on LHS of Eq. (1). To simplify the notation, let γ denote the $\gamma_{x\to z}$.

$$f(z) - f(x) = \int_0^1 \frac{d}{d\tau} f(\gamma(\tau)) d\tau \tag{2a}$$

$$= \int_0^1 \langle \gamma'(\tau), \mathrm{grad} f(\gamma(\tau)) \rangle d\tau \tag{2b}$$

$$= \int_0^1 \langle \Gamma^x_{\gamma(\tau)} \gamma'(\tau), \Gamma^x_{\gamma(\tau)} \mathrm{grad} f(\gamma(\tau)) \rangle d\tau \tag{2c}$$

$$= \int_0^1 \langle \gamma'(0), \Gamma^0_{\gamma(\tau)} \mathrm{grad} f(\gamma(\tau)) \rangle d\tau \tag{2d}$$

$$= \int_0^1 \langle \gamma'(0), \mathrm{grad} f(x) + \nabla_{\tau\gamma'(0)} \mathrm{grad} f + \Delta(\gamma(\tau)) \rangle d\tau \tag{2e}$$

$$= \langle \gamma'(0), \mathrm{grad} f(x) + \tfrac{1}{2} \nabla_{\gamma'(0)} \mathrm{grad} f + \bar{\Delta}(z) \rangle. \tag{2f}$$

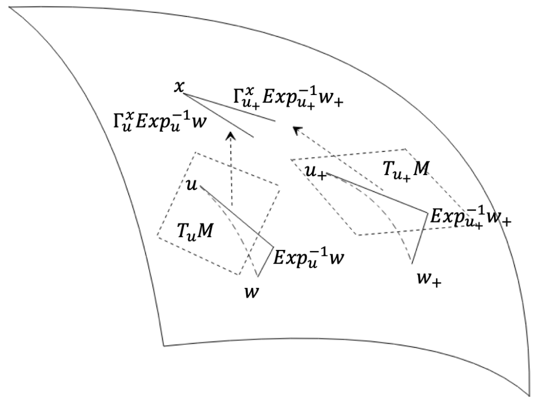


Figure 1: Lemma 1. First map w and w_+ to $\mathcal{T}_u\mathcal{M}$ and $\mathcal{T}_{u_+}\mathcal{M}$, and transport the two vectors to $\mathcal{T}_x\mathcal{M}$, and get their relation.

$\Delta(z)$ is defined in Eq. (1). $\bar{\Delta}(z) = \int_0^1 \Delta(\gamma(\tau))d\tau$. The second line is just rewriting by definition. Eq. (2c) means the parallel translation preserves the inner product (Tu, 2017, Prop. 14.16). Eq. (2d) uses $\Gamma_{\gamma(t)}^x \gamma'(t) = \gamma'(0)$, meaning that the velocity stays constant along a geodesic (Absil et al., 2009, (5.23)). Eq. (2e) uses Eq. (1). In Euclidean space, the Taylor expansion is

$$f(z) - f(x) = \langle z, \nabla f(x) + \nabla^2 f(x) z + \int_0^1 (\nabla^2 f(\tau z) - \nabla^2 f(x)) z d\tau \rangle. \tag{3}$$

Compare Eq. (2) and Eq. (3), z is replaced by $\gamma'(0) := \gamma'_{x\to z}(0)$ and τz is replaced by $\tau\gamma'_{x\to z}(0)$ or $\gamma_{x\to z}(\tau)$.

Now we have

$$f(u_t) = f(x) + \langle \gamma'(0), \mathrm{grad} f(x)\rangle + \frac{1}{2} H(x)[\gamma'(0), \gamma'(0)] + \langle \gamma'(0), \bar{\Delta}(u_t)\rangle.$$

B Linearization of the iterates in a fixed tangent space

In this section we linearize the progress of the iterates of our algorithm in a fixed tangent space $\mathcal{T}_x\mathcal{M}$. We always assume here that all points are within a region of diameter $R := 12\mathscr{S} \leq \mathfrak{I}$. In the course of the proof we need several auxilliary lemmas which are stated in the last two subsections of this section.

B.1 Evolution of $\mathrm{Exp}_u^{-1}(w)$

We first consider the evolution of $\mathrm{Exp}_u^{-1}(w)$ in a fixed tangent space $\mathcal{T}_x\mathcal{M}$. We show in the following lemma that it approximately follows a linear reccursion.

Lemma 1. *Define $\gamma = \sqrt{\hat{\rho}\epsilon}$, $\kappa = \frac{\beta}{\gamma}$, and $\mathscr{S} = \sqrt{\eta\beta}\frac{\gamma}{\hat{\rho}}\log^{-1}(\frac{d\kappa}{\delta})$. Let us consider x be a $(\epsilon, -\sqrt{\hat{\rho}\epsilon})$ saddle point, and define $u^+ = \mathrm{Exp}_u(-\eta\,\mathrm{grad} f(u))$ and $w^+ = \mathrm{Exp}_w(-\eta\,\mathrm{grad} f(w))$. Under Assumptions 1, 2, 3, if all pairwise distances between u, w, u^+, w^+, x are less than $12\mathscr{S}$, then for some explicit constant $C_1(K, \rho, \beta)$ depending only on K, ρ, β, there is*

$$\begin{aligned}&\|\Gamma_{u^+}^x \mathrm{Exp}_{u^+}^{-1}(w^+) - (I - \eta H(x))\Gamma_u^x \mathrm{Exp}_u^{-1}(w)\| \\ &\leq C_1(K, \rho, \beta) d(u, w) \left(d(u, w) + d(u, x) + d(w, x)\right).\end{aligned}$$

for some explicit function C_1.

This lemma is illustrated in Fig. 1.

Proof. Denote $-\eta\,\mathrm{grad} f(u) = v_u$, $-\eta\,\mathrm{grad} f(w) = v_w$. v is a smooth map. We first prove the following claim.

Claim 1.

$$d(u_+, w_+) \leq c_6(K) d(u, w),$$

where $c_6(K) = c_4(K) + 1 + c_2(K)R^2$.

To show this, note that

$$d(u_+, w_+) \leq d(u_+, \tilde{w}_+) + d(\tilde{w}_+, w_+),$$

and using Lemma 5 with $\tilde{w}_+ = \mathrm{Exp}_w(\Gamma_u^w v_u)$,

$$\begin{aligned} d(\tilde{w}_+, w_+) &= d(\mathrm{Exp}_w(v_w), \mathrm{Exp}_w(\Gamma_u^w v_u)) \\ &\leq (1 + c_2(K)R^2)\|v_w - \Gamma_u^w v_u\| \\ &\leq \beta(1 + c_2(K)R^2) d(u, w). \end{aligned}$$

Using Lemma 5,

$$d(\tilde{w}_+, u_+) \leq c_4(K) d(u, w). \tag{4}$$

Adding the two inequalities proves the claim.

We use now Lemma 3 between (u, w, u_+, w_+) in two different ways. First let us use it for $a = \mathrm{Exp}_u^{-1}(w)$ and $y = \Gamma_w^u v_w$. We obtain:

$$d(w_+, \mathrm{Exp}_u(\mathrm{Exp}_u^{-1}(w) + \Gamma_w^u v_w)) \leq c_1(K) d(u, w)(d(u, w)^2 + \|v_w\|^2). \tag{5}$$

Then we use it for $a = \mathrm{Exp}_u^{-1}(v_u)$ and $y = \Gamma_{u_+}^u \mathrm{Exp}_{u_+}^{-1}(w_+)$ which yields

$$\begin{aligned} & d(w_+, \mathrm{Exp}_u(v_u + \Gamma_{u_+}^u \mathrm{Exp}_{u_+}^{-1}(w_+))) \\ \leq\ & c_1(K) d(u_+, w_+)(d(u_+, w_+)^2 + \|v_u\|^2) \\ \leq\ & c_1(K) c_5(K, \|v_u\|, \|v_w\|) d(u, w) \cdot \Big[c_5(K, \|v_u\|, \|v_w\|)^2 d(u, w)^2 + \|v_u\|^2\Big]. \end{aligned}$$

Using the triangular inequality we have

$$\begin{aligned} & d(\mathrm{Exp}_u(\mathrm{Exp}_u^{-1}(w) + \Gamma_w^u v_w), \mathrm{Exp}_u(v_u + \Gamma_{u_+}^u \mathrm{Exp}_{u_+}^{-1}(w_+))) \\ \leq\ & d(w_+, \mathrm{Exp}_u(\mathrm{Exp}_u^{-1}(w) + \Gamma_w^u v_w)) + d(w_+, \mathrm{Exp}_u(v_u + \Gamma_{u_+}^u \mathrm{Exp}_{u_+}^{-1}(w_+))) \\ \leq\ & c_7 d(u, w) \end{aligned}$$

with c_7 defined as

$$c_7 = c_1(K) c_6(K) \cdot [c_5(K, \|v_u\|, \|v_w\|)^2 d(u, w)^2 + \|v_u\|^2 + \|v_w\|^2\Big].$$

We use again Lemma 4,

$$\|\Gamma_{u_+}^u \mathrm{Exp}_{u_+}^{-1}(w_+)) - \mathrm{Exp}_u^{-1}(w) - [v_u - \Gamma_w^u v_w]\| \leq (1 + c_3(K)R^2) \cdot c_7 d(u, w).$$

Therefore we have linearized the iterate in $T_u\mathcal{M}$. We should see how to transport it back to $T_x\mathcal{M}$. With Lemma 6 we have

$$\|[\Gamma_u^x \Gamma_{u_+}^u - \Gamma_{u_+}^x]\mathrm{Exp}_{u_+}^{-1}(w_+))\| = c_5(K) d(u, x) d(u_+, w_+)\|v_u\|.$$

Note v_u and v_w are $-\eta \mathrm{grad} f(u)$ and $-\eta \mathrm{grad} f(w)$, we define $\nabla v(x)$ the gradient of v, i.e., $-\eta H$. Using Hessian Lipschitz,

$$\begin{aligned} & \|v_u - \Gamma_w^u v_w + \eta H(u)\mathrm{Exp}_u^{-1}(w)\| \\ =\ & \|v_u - \Gamma_w^u v_w - \nabla v(u)\mathrm{Exp}_u^{-1}(w)\| \\ \leq\ & \rho d(u, w)^2, \end{aligned}$$

and

$$\|\nabla v(u)\mathrm{Exp}_u^{-1}(w) - \Gamma_x^u \nabla v(x) \Gamma_u^x \mathrm{Exp}_u^{-1}(w)\| \leq \rho d(u, w) d(u, x).$$

So we have

$$\begin{aligned} & \|\Gamma_{u_+}^x \mathrm{Exp}_{u_+}^{-1}(w_+) - (I + \nabla v(x))\Gamma_u^x \mathrm{Exp}_u^{-1}(w)\| \\ \leq\ & c_7 d(u, w) + \rho d(u, w)(d(u, w) + d(u, x)) + c_5(K) d(u, x) d(u_+, w_+)\|v_u\| := D_1 \end{aligned} \tag{6}$$

$\square$

B.2 Evolution of $\mathrm{Exp}_x^{-1}(w) - \mathrm{Exp}_x^{-1}(u)$

We consider now the evolution of $\mathrm{Exp}_x^{-1}(w) - \mathrm{Exp}_x^{-1}(u)$ in the fixed tangent space $\mathcal{T}_x\mathcal{M}$. We show in the following lemma that it also approximately follows a linear iteration.

Lemma 2. *Define $\gamma = \sqrt{\hat{\rho}\epsilon}$, $\kappa = \frac{\beta}{\gamma}$, and $\mathscr{S} = \sqrt{\eta\beta}\frac{\gamma}{\hat{\rho}}\log^{-1}(\frac{d\kappa}{\delta})$. Let us consider x be a $(\epsilon, -\sqrt{\hat{\rho}\epsilon})$ saddle point, and define $u^+ = \mathrm{Exp}_u(-\eta \mathrm{grad} f(u))$ and $w^+ = \mathrm{Exp}_w(-\eta \mathrm{grad} f(w))$. Under Assumptions 1, 2, 3, if all pairwise distances between u, w, u^+, w^+, x are less than $12\mathscr{S}$, then for some explicit constant $C(K, \rho, \beta)$ depending only on K, ρ, β, there is*

$$\begin{aligned}
&\|\mathrm{Exp}_x^{-1}(w^+) - \mathrm{Exp}_x^{-1}(u^+) - (I - \eta H(x))(\mathrm{Exp}_x^{-1}(w) - \mathrm{Exp}_x^{-1}(u))\| \\
&\leq C(K, \rho, \beta) d(u, w) \left(d(u, w) + d(u, x) + d(w, x)\right).
\end{aligned} \tag{7}$$

This lemma controls the error of the linear approximation of the iterates hen mapped in $\mathcal{T}_x\mathcal{M}$ and largely follows from Lemma 1.

Proof. We have that

$$\begin{aligned}
w &= \mathrm{Exp}_x(\mathrm{Exp}_x^{-1}(w)) &(8)\\
&= \mathrm{Exp}_u(\mathrm{Exp}_u^{-1}(w)). &(9)
\end{aligned}$$

Use Eq. (9), let $a = \mathrm{Exp}_x^{-1}(u)$ and $v = \Gamma_u^x \mathrm{Exp}_u^{-1}(w)$, Lemma 3 suggests that

$$\begin{aligned}
&d(\mathrm{Exp}_u(\mathrm{Exp}_u^{-1}(w)), \mathrm{Exp}_x(\mathrm{Exp}_x^{-1}(u) + \Gamma_u^x \mathrm{Exp}_u^{-1}(w))) \\
&\leq c_1(K)\|\mathrm{Exp}_u^{-1}(w)\|(\|\mathrm{Exp}_u^{-1}(w)\| + \|\mathrm{Exp}_x^{-1}(u)\|)^2.
\end{aligned}$$

Compare with Eq. (8), we have

$$\begin{aligned}
&d(\mathrm{Exp}_x(\mathrm{Exp}_x^{-1}(w)), \mathrm{Exp}_x(\mathrm{Exp}_x^{-1}(u) + \Gamma_u^x \mathrm{Exp}_u^{-1}(w))) \\
&\leq c_1(K)\|\mathrm{Exp}_u^{-1}(w)\|(\|\mathrm{Exp}_u^{-1}(w)\| + \|\mathrm{Exp}_x^{-1}(u)\|)^2 \\
&:= D.
\end{aligned} \tag{10}$$

Denote the quantity above by D. Now use Lemma 4

$$\|\mathrm{Exp}_x^{-1}(w) - (\mathrm{Exp}_x^{-1}(u) + \Gamma_u^x \mathrm{Exp}_u^{-1}(w))\| \leq (1 + c_3(K)R^2)D.$$

Analogously

$$\|\mathrm{Exp}_x^{-1}(w_+) - (\mathrm{Exp}_x^{-1}(u_+) + \Gamma_{u_+}^x \mathrm{Exp}_{u_+}^{-1}(w_+))\| \leq (1 + c_3(K)R^2)D_+$$

where

$$D_+ = c_1(K)\|\mathrm{Exp}_{u_+}^{-1}(w_+)\|(\|\mathrm{Exp}_{u_+}^{-1}(w_+)\| + \|\mathrm{Exp}_x^{-1}(u_+)\|)^2 \tag{11}$$

And we can compare $\Gamma_u^x \mathrm{Exp}_u^{-1}(w)$ and $\Gamma_{u_+}^x \mathrm{Exp}_{u_+}^{-1}(w_+)$ using Eq. (6). In the end we have

$$\begin{aligned}
&\|\mathrm{Exp}_x^{-1}(w^+) - \mathrm{Exp}_x^{-1}(u^+) - (I - \eta H(x))(\mathrm{Exp}_x^{-1}(w) - \mathrm{Exp}_x^{-1}(u))\| \\
&\leq \|\mathrm{Exp}_x^{-1}(w_+) - (\mathrm{Exp}_x^{-1}(u_+) + \Gamma_{u_+}^x \mathrm{Exp}_{u_+}^{-1}(w_+))\| \\
&\quad + \|\mathrm{Exp}_x^{-1}(w) - (\mathrm{Exp}_x^{-1}(u) + \Gamma_u^x \mathrm{Exp}_u^{-1}(w))\| \\
&\quad + \|\Gamma_{u_+}^x \mathrm{Exp}_{u_+}^{-1}(w_+) - \Gamma_u^x \mathrm{Exp}_u^{-1}(w) - \nabla v(x)\Gamma_u^x \mathrm{Exp}_u^{-1}(w)\| \\
&\quad + \|\nabla v(x)(\Gamma_u^x \mathrm{Exp}_u^{-1}(w) - (\mathrm{Exp}_x^{-1}(w) - \mathrm{Exp}_x^{-1}(u)))\| \\
&\leq (1 + c_3(K)R^2)(D_+ + D) + D_1 + \eta\|H(x)\|D.
\end{aligned}$$

D, D_+ and D_1 are defined in Eq. (10), Eq. (11) and Eq. (6), they are all order $d(u, w)\big(d(u, w) + d(u, x) + d(w, x)\big)$ so we get the correct order in Eq. (7). $\square$

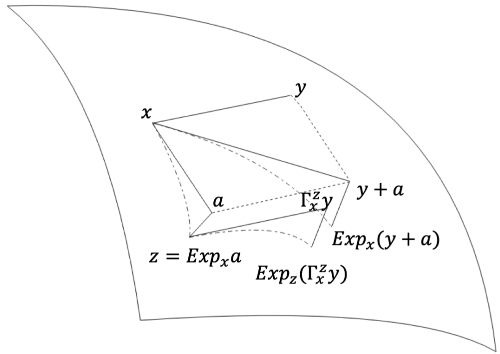


Figure 2: Lemma 3 bounds the difference of two steps starting from x: (1) take $y + a$ step in $\mathcal{T}_x\mathcal{M}$ and map it to manifold, and (2) take a step in $\mathcal{T}_x\mathcal{M}$, map to manifold, call it z, and take $\Gamma_x^z y$ step in $\mathcal{T}_x\mathcal{M}$, and map to manifold. $\mathrm{Exp}_z(\Gamma_x^z y)$ is close to $\mathrm{Exp}_x(y + a)$.

B.3 Control of two-steps iteration

In the following lemma we control the distance between the point obtained after moving along the sum of two vectors in the tangent space, and the point obtained after moving a first time along the first vector and then a second time along the transport of the second vector. This is illustrated in Fig. 2.

Lemma 3. *Let $x \in \mathcal{M}$ and $y, a \in T_x\mathcal{M}$. Let us denote by $z = \mathrm{Exp}_x(a)$ then under Assumption 3*

$$d(\mathrm{Exp}_x(y + a), \mathrm{Exp}_z(\Gamma_x^z y)) \leq c_1(K) \min\{\|a\|, \|y\|\}(\|a\| + \|y\|)^2. \tag{12}$$

This lemma which is crucial in the proofs of Lemma 2 and Lemma 1 tightens the result of Karcher (1977, C2.3), which only shows an upper-bound $O(\|a\|(\|a\| + \|y\|)^2)$.

Proof. We adapt the proof of Karcher (1977, Eq. (C2.3) in App C2.2), the only difference being that we bound more carefully the initial normal component. We restate here the whole proof for completeness.

Let $x \in \mathcal{M}$ and $y, a \in T_x\mathcal{M}$. We denote by $\gamma(t) = \mathrm{Exp}_x(ta)$. We want to compare the point $\mathrm{Exp}_x(r(y + a))$ and $\mathrm{Exp}_\gamma(1)(\Gamma_x^{\gamma(1)y})$. These two points , for a fixed r are joined by the curve

$$t \mapsto c(r, t) = \mathrm{Exp}_{\gamma(t)}(r\Gamma_x^{\gamma(t)}(y + (1 - t)a)).$$

We note that $\frac{d}{dt}c(r, t)$ is a Jacobi field along the geodesic $r \mapsto c(r, t)$, which we denote by $J_t(r)$. We importantly remark that the length of the geodesic $r \mapsto c(r, t)$ is bounded as $\|\frac{d}{dr}c(r, t)\| \leq \|y + (1 - t)a\|$. We denote this quantity by $\rho_t = \|y + (1 - t)a\|$. The initial condition of the Jacobi field J_t are given by:

$$J_t(0) = \frac{d}{dt}\gamma(t) = \Gamma_x^{\gamma(t)} a$$
$$\frac{D}{dr}J_t(0) = \frac{D}{dr}\Gamma_x^{\gamma(t)}(y + (1 - t)a) = -\Gamma_x^{\gamma(t)} a.$$

These two vectors are linearly dependent and it is therefore possible to apply Karcher (1977, Proposition A6) to bound J_t^{norm}. Moreover, following Karcher (1977, App A0.3), the tangential component of the Jacobi field is known explicitly, independent of the metric, by

$$J_t^{\tan}(r) = \left(J_t^{\tan}(0) + r\frac{D}{dr}J_t^{\tan}(0)\right)\frac{d}{dr}c(r, t)$$

where the initial conditions of the tangential component of the Jacobi fields are given by $J_t^{\tan}(0) = \langle J_t(0), \frac{\frac{d}{dr}c(r,t)}{\|\frac{d}{dr}c(r,t)\|}\rangle$ and $\frac{D}{dr}J_t^{\tan}(0) = \langle \frac{D}{dr}J_t(0), \frac{\frac{d}{dr}c(r,t)}{\|\frac{d}{dr}c(r,t)\|}\rangle = -J_t^{\tan}(0)$. Therefore

$$J_t^{\tan}(r) = (1 - r)J_t^{\tan}(0)\frac{d}{dr}c(r, t),$$

and $J_t^{\tan}(1) = 0$.

We estimate now the distance $d(\mathrm{Exp}_x(y+a), \mathrm{Exp}_z(\Gamma_x^z y))$ by the length of the curve $t \mapsto c(r,t)$ as follows:

$$d(\mathrm{Exp}_x(y+a), \mathrm{Exp}_z(\Gamma_x^z y)) \leq \int_0^1 \|\frac{d}{dt}c(1,t)\|dt = \int_0^1 \|J_t^{\mathrm{norm}}(1)\|dt,$$

where we use crucially that $J_t^{tan}(1) = 0$.

We utilize (Karcher, 1977, Proposition A.6) to bound $\|J_t^{\mathrm{norm}}(1)\|$ as

$$\|J_t^{\mathrm{norm}}(1)\| \leq \|J_t^{\mathrm{norm}}(0)\|(\cosh(\sqrt{K}\rho_t) - \frac{\sinh(\sqrt{K}\rho_t)}{\sqrt{K}\rho_t})$$

using (Karcher, 1977, Equation (A6.3)) with $\kappa = 0$, $f_\kappa(1) = 0$ and recalling that the geodesics $r \mapsto c(r,t)$ have length ρ_t.

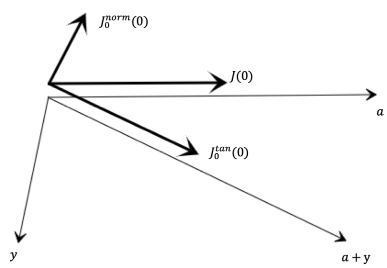


Figure 3: Figure for Lemma 3.

In particular for small value $\|a\| + \|y\|$ we have for some constant $c_1(K)$,

$$\|J_t^{\mathrm{norm}}(1)\| \leq \|J_t^{\mathrm{norm}}(0)\|c_1(K)\rho_t^2.$$

We bound $\|J_t^{\mathrm{norm}}(0)\|$ now. This is the main difference with the original proof of Karcher (1977) who directly bounded $\|J_t^{\mathrm{norm}}(0)\| \leq \|J_t(0)\| = \|a\|$ and $\rho_t \leq \|a\| + \|y\|$. Therefore his proof does not lead to the correct dependence in $\|y\|$.

We have $J_t^0 = \Gamma_x^{\gamma(t)}a$, and the tangential component (velocity of $r \to c(r,t)$) is in the $\Gamma_x^{\gamma(t)}(y + (1-t)a)$ direction. Let $\tilde{z} = \Gamma_x^{\gamma(t)}(y + (1-t)a)$ and $\mathcal{P}_{\tilde{z}^\perp}$ and $\mathcal{P}_{a^\perp}$ denote the projection onto orthogonal complement of $\tilde{z}$ and a.

$$\begin{aligned}
\|J_t^{\mathrm{norm}}(0)\|^2 &= \|\mathcal{P}_{\tilde{z}^\perp}(a)\|^2 \\
&= \|a\|^2 - \frac{(a^T\tilde{z})^2}{\|\tilde{z}\|^2} \\
&= \frac{\|a\|^2}{\|\tilde{z}\|^2}\left(\|\tilde{z}\|^2 - \frac{(a^T\tilde{z})^2}{\|\tilde{z}\|^2}\right) \\
&\leq \frac{\|a\|^2}{\|\tilde{z}\|^2}\|\mathcal{P}_{a^\perp}(\Gamma_x^{\gamma(t)}(y + (1-t)a))\|^2 \\
&\leq \frac{\|a\|^2}{\|\tilde{z}\|^2}\|\mathcal{P}_{a^\perp}(\Gamma_x^{\gamma(t)}((1-t)a)) + \mathcal{P}_{a^\perp}(\Gamma_x^{\gamma(t)}y)\|^2 \\
&= \frac{\|a\|^2}{\|\tilde{z}\|^2}\|\mathcal{P}_{a^\perp}(\Gamma_x^{\gamma(t)}y)\|^2 \\
&\leq \frac{\|a\|^2\|y\|^2}{\|\tilde{z}\|^2}.
\end{aligned}$$

So

$$\begin{aligned}\|J_t^{\text{norm}}(1)\| &\le \|J_t^{\text{norm}}(0)\| c_1(K)\rho_t^2\\ &\le \frac{\|a\|\cdot\|y\|}{\|\tilde z\|} c_1(K)\|\tilde z\|^2\\ &\le c_1(K)\|a\|\cdot\|y\|(\|a\|+\|y\|),\end{aligned}$$

and

$$d(\mathrm{Exp}_x(y+a), \mathrm{Exp}_z(\Gamma_x^z y)) \le c_1(K)\|a\|\cdot\|y\|(\|a\|+\|y\|).$$

$\square$

B.4 Auxilliary lemmas

In the proofs of Lemma 1 and Lemma 2 we needed numerous auxiliary lemmas we are stating here.

We needed the following lemma which shows that both the exponential map and its inverse are Lipschitz.

Lemma 4. *Let $x, y, z \in M$, and the distance of each two points is no bigger than R. Then under Assumption 3*

$$(1+c_2(K)R^2)^{-1} d(y,z) \le \|\mathrm{Exp}_x^{-1}(y) - \mathrm{Exp}_x^{-1}(z)\| \le (1+c_3(K)R^2) d(y,z).$$

Intuitively this lemma relates the norm of the difference of two vectors of $\mathcal{T}_x\mathcal{M}$ to the distance between the corresponding points on the manifold $\mathcal{M}$ and follows from bounds on the Hessian of the square-distance function (Sakai, 1996, Ex. 4 p. 154).

Proof. The upper-bound is directly proven in Karcher (1977, Proof of Cor. 1.6), and we prove the lower-bound via Lemma 3 in the supplement. Let $b = \mathrm{Exp}_y(\Gamma_x^y(\mathrm{Exp}_x^{-1}(z) - \mathrm{Exp}_x^{-1}(y)))$. Using $d(y,b) = \|\mathrm{Exp}_y^{-1}(b)\|$ and Lemma 3,

$$\begin{aligned}d(y,z) &\le d(y,b) + d(b, \mathrm{Exp}_x(\mathrm{Exp}_x^{-1}(z)))\\ &\le \|\mathrm{Exp}_x^{-1}(y) - \mathrm{Exp}_x^{-1}(z)\|\\ &\quad + c_1(K)\|\mathrm{Exp}_x^{-1}(y) - \mathrm{Exp}_x^{-1}(z)\|(\|\mathrm{Exp}_x^{-1}(y) - \mathrm{Exp}_x^{-1}(z)\| + \|\mathrm{Exp}_x^{-1}(y)\|)^2\end{aligned}$$

$\square$

The following contraction result is fairly classical and is proven using the Rauch comparison theorem from differential geometry (Cheeger & Ebin, 2008).

Lemma 5. *(Mangoubi et al., 2018, Lemma 1) Under Assumption 3, for $x, y \in \mathcal{M}$ and $w \in T_x\mathcal{M}$,*

$$d(\mathrm{Exp}_x(w), \mathrm{Exp}_y(\Gamma_x^y w)) \le c_4(K) d(x,y).$$

Eventually we need the following corollary of the famous Ambrose-Singer holonomy theorem (Ambrose & Singer, 1953).

Lemma 6. *(Karcher, 1977, Section 6) Under Assumption 3, for $x, y, z \in \mathcal{M}$ and $w \in T_x\mathcal{M}$,*

$$\|\Gamma_y^z\Gamma_x^y w - \Gamma_x^z w\| \le c_5(K) d(x,y) d(y,z)\|w\|.$$

C Proof of Lemma 7 and 8

In this section we prove two important lemmas from which the proof of our main result mainly comes out. Then we show, in the last subsection, how to combine them to prove this main result.

Lemma 7. *Assume Assumptions 1, 2, 3 hold, and*

$$\epsilon \le \min\left\{\frac{\hat\rho}{56\max\{c_2(K), c_3(K)\}\eta\beta}\log\left(\frac{d\beta}{\sqrt{\hat\rho}\epsilon\delta}\right), \left(\frac{\Im\hat\rho}{12\hat c\sqrt{\eta\beta}}\log\left(\frac{d\beta}{\sqrt{\hat\rho}\epsilon\delta}\right)\right)^2\right\} \tag{14}$$

from the main theorem. There exists a constant $c_{\max}$*,* $\forall \hat{c} > 3, \delta \in (0, \frac{d\kappa}{e}]$*, for any* u_0 *with* $d(\tilde{x}, u_0) \leq 2\mathscr{S}/(\kappa \log(\frac{d\kappa}{\delta}))$*,* $\kappa = \beta/\gamma$*.*

$$T = \min\left\{\inf_t \left\{t | \tilde{f}_{u_0}(u_t) - f(u_0) \leq -3\mathscr{F}\right\}, \hat{c}\mathscr{T}\right\},$$

then $\forall \eta \leq c_{\max}/\beta$*, we have* $\forall 0 < t < T$*,* $d(u_0, u_t) \leq 3(\hat{c}\mathscr{S})$*.*

Lemma 8. *Assume Assumptions 1, 2, 3 and Eq. (14) hold. Take two points* u_0 *and* w_0 *which are perturbed from approximate saddle point, where* $d(\tilde{x}, u_0) \leq 2\mathscr{S}/(\kappa \log(\frac{d\kappa}{\delta}))$*,* $\mathrm{Exp}_{\tilde{x}}^{-1}(w_0) - \mathrm{Exp}_{\tilde{x}}^{-1}(u_0) = \mu r e_1$*,* e_1 *is the smallest eigenvector*[1] *of* $H(\tilde{x})$*,* $\mu \in [\delta/(2\sqrt{d}), 1]$*, and the algorithm runs two sequences* $\{u_t\}$ *and* $\{w_t\}$ *starting from* u_0 *and* w_0*. Denote*

$$T = \min\left\{\inf_t \left\{t | \tilde{f}_{w_0}(w_t) - f(w_0) \leq -3\mathscr{F}\right\}, \hat{c}\mathscr{T}\right\},$$

then $\forall \eta \leq c_{\max}/l$*, if* $\forall 0 < t < T$*,* $d(\tilde{x}, u_t) \leq 3(\hat{c}\mathscr{S})$*, we have* $T < \hat{c}\mathscr{T}$*.*

C.1 Proof of Lemma 7

Suppose $f(u_{t+1}) - f(u_t) \leq -\frac{\eta}{2}\|\mathrm{grad} f(u_t)\|^2$.

$$\begin{aligned}
d(u_{\hat{c}\mathscr{T}}, u_0)^2 &\leq (\sum_0^{\hat{c}\mathscr{T}-1} d(u_{t+1}, u_t))^2 \\
&\leq \hat{c}\mathscr{T} \sum_0^{\hat{c}\mathscr{T}-1} d(u_{t+1}, u_t)^2 \\
&\leq \eta^2 \hat{c}\mathscr{T} \sum_0^{\hat{c}\mathscr{T}-1} \|\mathrm{grad} f(u_t)\|^2 \\
&\leq 2\eta \hat{c}\mathscr{T} \sum_0^{\hat{c}\mathscr{T}-1} f(u_t) - f(u_{t+1}) \\
&= 2\eta \hat{c}\mathscr{T}(f(u_0) - f(u_{\hat{c}\mathscr{T}})) \\
&\leq 6\eta \hat{c}\mathscr{T}\mathscr{F} = 6\hat{c}\mathscr{S}^2.
\end{aligned}$$

C.2 Proof of Lemma 8

Note that, for any points inside a region with diameter R, under the assumption of Lemma 8, we have $\max\{c_2(K), c_3(K)\}R^2 \leq 1/2$.

Define $v_t = \mathrm{Exp}_{\tilde{x}}^{-1}(w_t) - \mathrm{Exp}_{\tilde{x}}^{-1}(u_t)$, let $v_0 = e_1$ be the smallest eigenvector of $H(\tilde{x})$, then let $\hat{y}_{2,t}$ be a unit vector, we have

$$\begin{aligned}
v_{t+1} = (I - \eta H(\tilde{x}))v_t &+ C(K, \rho, \beta) d(u_t, w_t) \\
&\cdot (d(u_t, \tilde{x}) + d(w_t, \tilde{x}) + d(\tilde{x}, u_0))\hat{y}_{2,t}.
\end{aligned} \tag{16}$$

Let $C := C(K, \rho, \beta)$. Suppose lemma 8 is false, then $0 \leq t \leq T$, $d(u_t, \tilde{x}) \leq 3\hat{c}\mathscr{S}$, $d(w_t, \tilde{x}) \leq 3\hat{c}\mathscr{S}$, so $d(u_t, w_t) \leq 6\hat{c}\mathscr{S}$, and the norm of the last term in Eq. (16) is smaller than $14\eta C\hat{c}\mathscr{S}\|v_t\|$.

Lemma 4 in the main paper indicates that

$$\|v_t\| \in [1/2, 2] \cdot d(u_t, w_t) = [3/2, 6] \cdot \hat{c}\mathscr{S}. \tag{17}$$

Let ψ_t be the norm of v_t projected onto e_1, the smallest eigenvector of $H(0)$, and φ_t be the norm of v_t projected onto the remaining subspace. Then Eq. (16) is

$$\psi_{t+1} \geq (1 + \eta\gamma)\psi_t - \mu\sqrt{\psi_t^2 + \phi_t^2},$$

$$\phi_{t+1} \leq (1 + \eta\gamma)\phi_t + \mu\sqrt{\psi_t^2 + \phi_t^2}.$$

[1] "smallest eigenvector" means the eigenvector corresponding to the smallest eigenvalue.

Prove that for all $t \leq T$, $\phi_t \leq 4\mu t\psi_t$. Assume it is true for t, we have

$$4\mu(t+1)\psi_{t+1} \geq 4\mu(t+1)\cdot\left((1+\eta\gamma)\psi_t - \mu\sqrt{\psi_t^2+\phi_t^2}\right),$$

$$\phi_{t+1} \leq 4\mu t(1+\eta\gamma)\phi_t + \mu\sqrt{\psi_t^2+\phi_t^2}.$$

So we only need to show that

$$(1+4\mu(t+1))\sqrt{\psi_t^2+\phi_t^2} \leq (1+\eta\gamma)\psi_t.$$

By choosing $\sqrt{c_{\max}} \leq \frac{1}{56\hat{c}^2}$ and $\eta \leq c_{\max}/\beta$, we have

$$4\mu(t+1) \leq 4\mu T \leq 4\eta C\mathscr{S}\cdot 14\hat{c}^2\mathscr{T} = 56\hat{c}^2\frac{C}{\hat{\rho}}\sqrt{\eta\beta} \leq 1.$$

This gives

$$4(1+\eta\gamma)\psi_t \geq 2\sqrt{2\psi_t^2} \geq (1+4\mu(t+1))\sqrt{\psi_t^2+\phi_t^2}.$$

Now we know $\phi_t \leq 4\mu t\psi_t \leq \psi_t$, so $\psi_{t+1} \geq (1+\eta\gamma)\psi_t - \sqrt{2}\mu\psi_t$, and

$$\mu = 14\hat{c}\eta C\mathscr{S} \leq 14\hat{c}\sqrt{c_{\max}}\eta\gamma C\log^{-1}(\frac{d\kappa}{\delta})/\hat{\rho} \leq \eta\gamma/2,$$

so $\psi_{t+1} \geq (1+\eta\gamma/2)\psi_t$.

We also know that $\|v_t\| \leq 6\hat{c}\mathscr{S}$ for all $t \leq T$ from Eq. (17), so

$$\begin{aligned}
6\hat{c}\mathscr{S} &\geq \|v_t\| \geq \psi_t \geq (1+\eta\gamma/2)^t\psi_0 \\
&= (1+\eta\gamma/2)^t\frac{\mathscr{S}}{\kappa}\log^{-1}(\frac{d\kappa}{\delta}) \\
&\geq (1+\eta\gamma/2)^t\frac{\delta\mathscr{S}}{2\sqrt{d}\kappa}\log^{-1}(\frac{d\kappa}{\delta}).
\end{aligned}$$

This implies

$$\begin{aligned}
T &< \frac{\log(12\frac{\kappa\sqrt{d}}{\delta}\hat{c}\log(\frac{d\kappa}{\delta}))}{2\log(1+\eta\gamma/2)} \\
&\leq \frac{\log(12\frac{\kappa\sqrt{d}}{\delta}\hat{c}\log(\frac{d\kappa}{\delta}))}{\eta\gamma} \\
&\leq (2+\log(12\hat{c}))\mathscr{T}.
\end{aligned}$$

By choosing $\hat{c}$ such that $2+\log(12\hat{c}) < \hat{c}$, we have $T \leq \hat{c}\mathscr{T}$, which finishes the proof.

C.3 Proof of function value decrease at an approximate saddle point

With Lemma 7 and 8 proved, we can lower bound the function value in $O(\mathscr{T})$ iterations decrease by $\Omega(\mathscr{F})$, thus match the convergence rate in the main theorem. Let $T' := \inf_t\left\{t|\tilde{f}_{u_0}(u_t) - f(u_0) \leq -3\mathscr{F}\right\}$. Let $\breve{}$ denote the operator $\mathrm{Exp}_{u_0}^{-1}(\cdot)$. If $T' \leq T$,

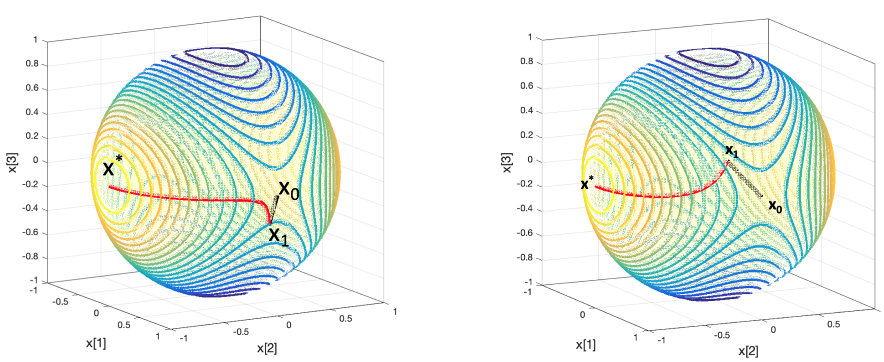
$$\begin{aligned}
&f(u_{T'}) - f(u_0) \\
\leq\ &\nabla f(u_0)^T(u_{T'} - u_0) + \frac{1}{2}H(u_0)[\breve{u}_{T'} - u_0, \breve{u}_{T'} - u_0] \\
&+ \frac{1}{2}(\Gamma_{\tilde{x}}^{u_0}H(\tilde{x})\Gamma_{u_0}^{\tilde{x}} - H(u_0))[\breve{u}_{T'} - u_0, \breve{u}_{T'} - u_0] \\
&+ \frac{\rho}{6}\|\breve{u}_{T'} - u_0\|^3 \\
\leq\ &\tilde{f}_{u_0}(u_t) - f(u_0) + \rho d(u_0,\tilde{x})\|\breve{u}_{T'} - u_0\|^2 \\
\leq\ &-3\mathscr{F} + O(\rho\mathscr{S}^3) \leq -2.5\mathscr{F}.
\end{aligned}$$

If $T' > T$, then $\inf_t\left\{t|\tilde{f}_{w_0}(w_t) - f(w_0) \leq -3\mathscr{F}\right\} \leq T$, and we know $f(w_T) - f(w_0) \leq -2.5\mathscr{F}$.

Remark 1. *What is left is bounding the volume of the stuck region, to get the probability of getting out of the stuck region by the perturbation. The procedure is the same as in Jin et al. (2017). We sample from a unit ball in $\mathcal{T}_x\mathcal{M}$, where x is the approximate saddle point. In Lemma 7 and 8, we study the inverse exponential map at the approximate saddle point x, and the coupling difference between $\mathrm{Exp}_x^{-1}(w)$ and $\mathrm{Exp}_x^{-1}(u)$. The iterates we study and the noise are all in the tangent space $\mathcal{T}_x\mathcal{M}$ which is a Euclidean space, so the probability bound is same as the one in Jin et al. (2017).*

D Experiment with retraction

In the main algorithm and its proof, we use the exponential map in the algorithm. The exponential map is easy to compute for many manifolds, but one may also use *retraction* as a first order approximation of exponential map. We do not theoretically study retraction, but the experiment below shows that replacing exponential by a smooth retraction works well practically.



(a) (b)

Figure 4: (a) Function f with saddle point on a sphere. $f(x) = x_1^2 - x_2^2 + 4x_3^2$. We plot the contour of this function on unit sphere. The main algorithm initializes at $x_0 = [1, 0, 0]$ (a saddle point), perturbs it towards x_1 and runs Riemannian gradient descent, and terminates at $x^* = [0, -1, 0]$ (a local minimum). We amplify the first iteration to make saddle perturbation visible. (b) We replace exponential map by retraction $R_x(v) = (x + v)/\|x + v\|_2$ and do the same experiment, which addresses the generality of the result. We do not provide in this paper proof for algorithm with retraction, but practically the iterates converge to an approximate saddle point.

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