(Probably) Concave Graph Matching
Supplementary material

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1 Frank-Wolfe with concave search

An orthogonal basis to $\lin(\mathcal{F})$ is computed similarly to Lemma 1 in the paper:

Lemma 1 (orthonormal basis for one-sided permutations). *If the columns of $F \in \mathbb{R}^{n_0 \times (n_0-1)}$ form an orthonormal basis for $\mathbf{1}^{\perp}$ in $\mathbb{R}^{n_0}$ then the columns of $F \otimes I_n$ are an orthonormal basis for $\lin(\mathcal{F})$.*

The energy $E_2(X)$ in this case does not model the matching problem well since it gives rise to trivial solutions. Instead, we chose to optimize a similar energy (Solomon et al., 2016): $E(X) = \sum_{ijkl} X_{ij} X_{kl} (A_{ik} - B_{jl})^2$. This energy can also be written in matrix form: $[X]^T M [X]$ where $M = -2B \otimes A + 11^T \otimes A.^2 + B.^2 \otimes 11^T$ (where $C.^2$ implies entry-wise operation) and after restricting it to $\lin(\mathcal{F})$ its Hessian is of the form $-2FBF \otimes A + FB.^2 F \otimes 11^T$. Assuming A, B are Euclidean distance matrices, the right summand is negative semidefinite, but the left summand is not. This is because that A is not conjugated by F: it has a large positive eigenvalue as a result of the Perron-Frobenius Theorem.

The linear program solved in each iteration of the algorithm takes a surprisingly simple form: it amounts to solving $\min_{X \in \hull(\mathcal{F})} \tr(\nabla E(X_0)^T X)$ which can be solved simply by assigning the value 1 to the index of the minimal value in each row of $\nabla E(X_0)$. This procedure always outputs solutions in $\mathcal{F}$.

The convex energies we subtract from the objective during the concave search should be constant on $\mathcal{F}$ so a reduction in the subtracted energy is the same as in the original energy $E(X)$. We use the quadratic form defined by $\lambda * \Lambda$ where Λ is a $nn_0 \times nn_0$ diagonal matrix defined by $D_{ijij} = \max_j \{\sum_{kl} |M_{ijkl}|\}$. D is a positive definite matrix and for $\lambda = 1$, $W - D$ is guaranteed to be negative semidefinite. The values of λ need not be discretized since there are only n different critical values - the ones that change the minimum calculation mentioned in the previous paragraph.

References

Solomon, J., Peyré, G., Kim, V. G., and Sra, S. (2016). Entropic metric alignment for correspondence problems. *ACM Transactions on Graphics (TOG)*, 35(4):72.