Supplementary Material for Generalized Inverse Optimization through Online Learning

Chaosheng Dong, Yiran Chen, Bo Zeng

A Omitted mathematical reformulations

A.1 Single level reformulation for the Inverse Linear Optimization Problem

When the objective function is linear, namely, the optimization problem has the following form

$$\begin{aligned} \min_{\mathbf{x}\in\mathbb{R}^n_+} \quad & \mathbf{c}^T\mathbf{x} \\ s.t. \quad & A\mathbf{x} \geq \mathbf{b}. \end{aligned} \tag{LP}$$

Suppose that the right hand side $\mathbf{b}$ changes over time t. That is, $\mathbf{b} = \mathbf{b}_t$ at time t. When trying to learn $\mathbf{c}$, the single level reformulation the inverse problem is

$$\begin{aligned} \min_{\mathbf{c}\in\Theta} \quad & \tfrac{1}{2}\|\mathbf{c}-\mathbf{c}_t\|_2^2 + \eta_t\|\mathbf{y}_t-\mathbf{x}\|_2^2 \\ s.t. \quad & \mathbf{A}\mathbf{x} \geq \mathbf{b}_t,\ \mathbf{x} \geq \mathbf{0}, \\ & \mathbf{A}^T\mathbf{u} \leq \mathbf{c}, \\ & \mathbf{x} \leq M_1\mathbf{z}_1, \\ & \mathbf{c} - \mathbf{A}^T\mathbf{u} \leq M_1(1-\mathbf{z}_1), \\ & \mathbf{u} \leq M_2\mathbf{z}_2, \\ & \mathbf{A}\mathbf{x} - \mathbf{b}_t \leq M_2(1-\mathbf{z}_2), \\ & \mathbf{x}\in\mathbb{R}^n_+,\ \ \mathbf{u}\in\mathbb{R}^m_+,\ \ \mathbf{z}_1\in\{0,1\}^n,\ \ \mathbf{z}_2\in\{0,1\}^m, \end{aligned}$$

where M_1 and M_2 are appropriate numbers used to bound $\mathbf{x}$ and $\mathbf{c} - \mathbf{A}^T\mathbf{u}$, $\mathbf{u}$ and $\mathbf{A}\mathbf{x} - \mathbf{b}_t$ respectively.

We have a similar single level reformulation when learning the Right-hand side $\mathbf{b}$. Clearly, this is a Mixed Integer Second Order Cone program(MISOCP) when learning either $\mathbf{c}$ or $\mathbf{b}$.

A.2 Single level reformulation for the Inverse Quadratic Optimization Problem

When the objective functions are quadratic, namely, the optimization problem has the following form

$$\begin{aligned} \min_{\mathbf{x}\in\mathbb{R}^n} \quad & \tfrac{1}{2}\mathbf{x}^TQ\mathbf{x} + \mathbf{c}^T\mathbf{x} \\ s.t. \quad & A\mathbf{x} \geq \mathbf{b}. \end{aligned} \tag{QP}$$

Suppose that $\mathbf{c}$ changes over time t. That is, $\mathbf{c} = \mathbf{c}_t$ at time t. When trying to learn $\mathbf{b}$, the single level reformulation for the inverse problem is

$$\begin{aligned} \min_{\mathbf{b}\in\Theta} \quad & \tfrac{1}{2}\|\mathbf{b}-\mathbf{b}_t\|_2^2 + \eta_t\|\mathbf{y}_t-\mathbf{x}\|_2^2 \\ \text{s.t.} \quad & \mathbf{A}\mathbf{x} \geq \mathbf{b}, \\ & \mathbf{u} \leq M\mathbf{z}, \\ & \mathbf{A}\mathbf{x} - \mathbf{b} \leq M(1-\mathbf{z}), \\ & Q\mathbf{x} + \mathbf{c}_t - \mathbf{A}^T\mathbf{u} = 0, \\ & \mathbf{b}\in\mathbb{R}^m,\ \ \mathbf{x}\in\mathbb{R}^n,\ \ \mathbf{u}\in\mathbb{R}^m_+,\ \ \mathbf{z}\in\{0,1\}^m, \end{aligned}$$

where M is an appropriate number used to bound $\mathbf{u}$ and $\mathbf{A}\mathbf{x} - \mathbf{b}$.

We have a similar single level reformulation when learning the objective $\mathbf{c}$. Clearly, this is a Mixed Integer Second Order Cone program(MISOCP) when learning either $\mathbf{c}$ or $\mathbf{b}$.

B Omitted Proofs

B.1 Proof of Lemma 3.1

Proof. By Assumption 3.1(b), we know that $S(u,\theta)$ is a single-valued set for each $u \in \mathcal{U}$.

$\forall \mathbf{y} \in \mathcal{Y}$, $\forall u \in \mathcal{U}$, $\forall \theta_1, \theta_2 \in \Theta$, without of loss of generality, let $l(\mathbf{y},u,\theta_1) \geq l(\mathbf{y},u,\theta_2)$. Then,

$$\begin{aligned} |l(\mathbf{y},u,\theta_1) - l(\mathbf{y},u,\theta_2)| &= l(\mathbf{y},u,\theta_1) - l(\mathbf{y},u,\theta_2) \\ &= \|\mathbf{y} - S(u,\theta_1)\|_2^2 - \|\mathbf{y} - S(u,\theta_2)\|_2^2 \\ &= \langle S(u,\theta_2) - S(u,\theta_1), 2\mathbf{y} - S(u,\theta_1) - S(u,\theta_2)\rangle \\ &\leq 2(B+R)\|S(u,\theta_2) - S(u,\theta_1)\|_2. \end{aligned} \tag{1}$$

The last inequality is due to Cauchy-Schwartz inequality and the Assumptions 3.1(a), that is

$$\|2\mathbf{y} - S(u,\theta_1) - S(u,\theta_2)\|_2 \leq 2(B+R). \tag{2}$$

Next, we will apply Proposition 6.1 in Bonnans and Shapiro [1998] to bound $\|S(u,\theta_2) - S(u,\theta_1)\|_2$.

Under Assumptions 3.1 - 3.2, the conditions of Proposition 6.1 in Bonnans and Shapiro [1998] are satisfied. Therefore,

$$\|S(u,\theta_2) - S(u,\theta_1)\|_2 \leq \frac{2\kappa}{\lambda}\|\theta_1 - \theta_2\|_2. \tag{3}$$

Plugging (2) and (3) in (1) yields the claim. □

B.2 Proof of Theorem 3.2

Proof. we will use Theorem 3.2 in Kulis and Bartlett [2010] to prove our theorem.

Let $G_t(\theta) = \frac{1}{2}\|\theta - \theta_t\|_2^2 + \eta_t l(\mathbf{y}_t, u_t, \theta)$.

We will now show the loss function is convex. The first step is to show that if Assumption 3.3 holds, then the loss function $l(\mathbf{y},u,\theta)$ is convex in θ. $\forall \mathbf{y} \in \mathcal{Y}$, $\forall u \in \mathcal{U}$, $\forall \theta_1, \theta_2 \in \Theta$, we have

$$\begin{aligned} & \alpha l(\mathbf{y},u,\theta_1) + \beta l(\mathbf{y},u,\theta_2) - l(\mathbf{y},u,\alpha\theta_1 + \beta\theta_2) \\ = \; & \alpha\|\mathbf{y} - S(u,\theta_1)\|_2^2 + \beta\|\mathbf{y} - S(u,\theta_2)\|_2^2 - \|\mathbf{y} - S(u,\alpha\theta_1 + \beta\theta_2)\|_2^2 \\ = \; & \alpha\|\mathbf{y} - S(u,\theta_1)\|_2^2 + \beta\|\mathbf{y} - S(u,\theta_2)\|_2^2 - \|\mathbf{y} - \alpha S(u,\theta_1) - \beta S(u,\theta_2)\|_2^2 \\ & + \|\mathbf{y} - \alpha S(u,\theta_1) - \beta S(u,\theta_2)\|_2^2 - \|\mathbf{y} - S(u,\alpha\theta_1 + \beta\theta_2)\|_2^2 \\ = \; & \alpha\beta\|S(u,\theta_1) - S(u,\theta_2)\|_2^2 + \|\mathbf{y} - \alpha S(u,\theta_1) - \beta S(u,\theta_2)\|_2^2 - \|\mathbf{y} - S(u,\alpha\theta_1 + \beta\theta_2)\|_2^2 \\ = \; & \alpha\beta\|S(u,\theta_1) - S(u,\theta_2)\|_2^2 \\ & - \langle \alpha S(u,\theta_1) + \beta S(u,\theta_2) - S(u,\alpha\theta_1 + \beta\theta_2), 2\mathbf{y} - S(u,\alpha\theta_1 + \beta\theta_2) - \alpha S(u,\theta_1) - \beta S(u,\theta_2)\rangle \\ \geq \; & \alpha\beta\|S(u,\theta_1) - S(u,\theta_2)\|_2^2 - \|\alpha S(u,\theta_1) + \beta S(u,\theta_2) - S(u,\alpha\theta_1 + \beta\theta_2)\|_2 \|2\mathbf{y} - S(u,\alpha\theta_1 \\ & + \beta\theta_2) - \alpha S(u,\theta_1) - \beta S(u,\theta_2)\|_2. \end{aligned} \tag{4}$$

The last inequality is by Cauchy-Schwartz inequality. Note that

$$\begin{aligned} & \|\alpha S(u,\theta_1) + \beta S(u,\theta_2) - S(u,\alpha\theta_1 + \beta\theta_2)\|_2 \|2\mathbf{y} - S(u,\alpha\theta_1 + \beta\theta_2) - \alpha S(u,\theta_1) - \beta S(u,\theta_2)\|_2 \\ & \leq 2(B+R)\|\alpha S(u,\theta_1) + \beta S(u,\theta_2) - S(u,\alpha\theta_1 + \beta\theta_2)\|_2 \\ & \leq \alpha\beta\|S(u,\theta_1) - S(u,\theta_2)\|_2 \qquad \text{(By Assumption 3.3)}. \end{aligned} \tag{5}$$

Plugging (5) in (4) yields the result.

Using Theorem 3.2 in Kulis and Bartlett [2010], for $\alpha_t \leq \frac{G_t(\theta_{t+1})}{G_t(\theta_t)}$, we have

$$R_T \leq \; \sum_{t=1}^{T} \frac{1}{\eta_t}(1 - \alpha_t)\eta_t l(\mathbf{y}_t, u_t, \theta_t) + \frac{1}{2\eta_t}(\|\theta_t - \theta^*\|_2^2 - \|\theta_{t+1} - \theta^*\|_2^2). \tag{6}$$

Notice that

$$
\begin{aligned}
G_t(\theta_t) - G_t(\theta_{t+1}) & = \eta_t(l(\mathbf{y}_t, u_t, \theta_t) - l(\mathbf{y}_t, u_t, \theta_{t+1})) - \tfrac{1}{2}\|\theta_t - \theta_{t+1}\|_2^2 \\
& \leq \tfrac{4(B+R)\kappa\eta_t}{\lambda}\|\theta_t - \theta_{t+1}\|_2 - \tfrac{1}{2}\|\theta_t - \theta_{t+1}\|_2^2 \\
& \leq \tfrac{8(B+R)^2\kappa^2\eta_t^2}{\lambda^2}.
\end{aligned} \tag{7}
$$

The first inequality follows by applying Lemma 3.1.
Let $\alpha_t = \frac{G_t(\theta_{t+1})}{G_t(\theta_t)}$. Using (7), we have

$$
(1-\alpha_t)\eta_t l(\mathbf{y}_t, u_t, \theta_t) = (1-\alpha_t)G_t(\theta_t) = G_t(\theta_t) - G_t(\theta_{t+1}) \leq \tfrac{8(B+R)^2\kappa^2\eta_t^2}{\lambda^2}. \tag{8}
$$

Plug (8) in (6), and note the telescoping sum,

$$
R_T \leq \sum_{t=1}^{T} \frac{8(B+R)^2\kappa^2\eta_t}{\lambda^2} + \sum_{t=1}^{T} \frac{1}{2\eta_t}(\|\theta_t - \theta^*\|_2^2 - \|\theta_{t+1} - \theta^*\|_2^2).
$$

Setting $\eta_t = \frac{D\lambda}{2(B+R)\kappa\sqrt{2t}}$, we can upper bound the second summation by $\frac{4\sqrt{2}(B+R)D\kappa}{\lambda}\sqrt{T}$ since $\|\theta_1 - \theta^*\|_2 \leq 2D$, $\sqrt{t} \leq \sqrt{T}$, and then the sum telescopes. The first sum simplifies using $\sum_{t=1}^{T} \frac{1}{\sqrt{t}} \leq 2\sqrt{T} - 1$ to obtain the result

$$
R_T \leq \frac{8\sqrt{2}(B+R)D\kappa}{\lambda}\sqrt{T}.
$$

Note that choosing $\eta_t = \frac{1}{\sqrt{t}}$ also yields $\mathcal{O}(\sqrt{T})$ regret, but the result above is tighter. □

B.3 Proof of Theorem 3.3

Proof. Since $f(\mathbf{x}, u, \theta)$ is strongly convex in $\mathbf{x}$ on $\mathbb{R}^n$ by Assumption 3.1, it is also strictly convex in $\mathbf{x}$ on $\mathbb{R}^n$. Then, all the conditions required in Theorem 3. of Aswani et al. [2018] are naturally satisfied under our assumptions. Applying that theorem yields

$$
\frac{1}{T}\sum_{t\in[T]} l(\mathbf{y}_t, u_t, \theta^T) \xrightarrow{p} \mathbb{E}\left[l(\mathbf{y}, u, \theta^*)\right], \tag{9}
$$

where $\theta^T = \arg\min_{\theta\in\Theta}\{\sum_{t\in[T]} l(\mathbf{y}_t, u_t, \theta)\}$ is the estimation of the parameter in batch setting.

From Theorem 3.2 we have

$$
\frac{1}{T}\sum_{t\in[T]} l(\mathbf{y}_t, u_t, \theta_t) - \frac{1}{T}\sum_{t\in[T]} l(\mathbf{y}_t, u_t, \theta^T) \leq \frac{8\sqrt{2}(B+R)D\kappa}{\lambda\sqrt{T}} \xrightarrow{p} 0. \tag{10}
$$

Adding (9) and (10) up, we have the risk consistency result

$$
\frac{1}{T}\sum_{t\in[T]} l(\mathbf{y}_t, u_t, \theta_t) \xrightarrow{p} \mathbb{E}\left[l(\mathbf{y}, u, \theta^*)\right].
$$

□

B.4 Proof of Corollary 3.3.1

Proof. Note that $\forall\theta \in \Theta$,

$$
\mathbb{E}\left[l(\mathbf{y}, u, \theta)\right] = \mathbb{E}\left[\min_{\tilde{\mathbf{x}}\in S(u,\theta)} \|\mathbf{x} + \epsilon - \tilde{\mathbf{x}}\|_2^2\right] = \mathbb{E}\left[\min_{\tilde{\mathbf{x}}\in S(u,\theta)} \|\mathbf{x} - \tilde{\mathbf{x}}\|_2^2\right] + \mathbb{E}[\epsilon^T\epsilon] \geq \mathbb{E}[\epsilon^T\epsilon].
$$

We further notice that $\mathbb{E}\left[\min_{\tilde{\mathbf{x}}\in S(u,\theta_0)} \|\mathbf{x} - \tilde{\mathbf{x}}\|_2^2\right] = 0$, since $\mathbf{x} \in S(u, \theta_0)$. Therefore, we have

$$
\mathbb{E}\left[l(\mathbf{y}, u, \theta^*)\right] = \mathbb{E}\left[l(\mathbf{y}, u, \theta_0)\right] = \mathbb{E}[\epsilon^T\epsilon].
$$

Then, applying Theorem 3.3 yields the result, since we have shown $\mathbb{E}\left[l(\mathbf{y}, u, \theta^*)\right] = \mathbb{E}[\epsilon^T\epsilon]$. □

C Omitted Examples

C.1 Examples for which Assumption 3.3 holds

Consider for example the following quadratic program

$$\begin{aligned} \min_{\mathbf{x}\in\mathbb{R}^n} \quad & \tfrac{1}{2}\mathbf{x}^T Q\mathbf{x} + (\mathbf{c}+u)^T\mathbf{x} \\ s.t. \quad & A\mathbf{x} \geq \mathbf{b}. \end{aligned}$$

where Q is a positive semidefinite matrix, and u is the external signal.

Suppose that the parameter we seek to learn is $\mathbf{c}$, all the others are given. If for each $u \in \mathcal{U}$, the optimal solution for the above program is in the interior of the feasible region, which essentially occurs when the external signal u does not has a large range for the constrained QP. Then,

$$S(u,\mathbf{c}_1) = -Q^{-1}(\mathbf{c}_1+u); \quad S(u,\mathbf{c}_2) = -Q^{-1}(\mathbf{c}_2+u); \quad S(u,\alpha\mathbf{c}_1+\beta\mathbf{c}_2) = -Q^{-1}(\alpha\mathbf{c}_1+\beta\mathbf{c}_2+u);$$

Then, we have

$$0 = \|\alpha S(u,\mathbf{c}_1) + \beta S(u,\mathbf{c}_2) - S(u,\alpha\mathbf{c}_1+\beta\mathbf{c}_2)\|_2 \leq \alpha\beta\|S(u,\theta_1) - S(u,\theta_2)\|_2/(2(B+R)).$$

D Data for the applications

D.1 Data for learning the consumer behavior

Table 1: True $\mathbf{r}$

1.180	1.733	1.564	0.040	2.443	1.055	4.760	5.000	1.258	4.933

Table 2: True Q

2.360	0	0	0	0	0	0	0	0	0
0	3.465	0	0	0	0	0	0	0	0
0	0	3.127	0	0	0	0	0	0	0
0	0	0	0.0791	0	0	0	0	0	0
0	0	0	0	4.886	0	0	0	0	0
0	0	0	0	0	2.110	0	0	0	0
0	0	0	0	0	0	9.519	0	0	0
0	0	0	0	0	0	0	9.999	0	0
0	0	0	0	0	0	0	0	2.517	0
0	0	0	0	0	0	0	0	0	9.867

D.2 Data for learning the transportation cost

We let $\lambda_1 = 2$, $\lambda_2 = 10$, $u_e = 1.3$ for all $e \in E$, $y_1 = 3$ and $y_2 = 1.5$.

Table 3: True transportation cost for each edge

c_{13}	c_{14}	c_{23}	c_{25}	c_{34}	c_{35}
3.124	4.119	3.814	1.071	5.398	2.899

References

A. Aswani, Z.-J. Shen, and A. Siddiq. Inverse optimization with noisy data. *Operations Research*, 2018.

J. F. Bonnans and A. Shapiro. Optimization problems with perturbations: A guided tour. *SIAM Review*, 40(2):228–264, 1998.

B. Kulis and P. L. Bartlett. Implicit online learning. In *Proceedings of the 27th International Conference on Machine Learning (ICML-10)*, pages 575–582, 2010.