Appendix

A Proofs of Local Convergence

We define a sequence of constants, $\{C_J\}_{J=0,1,\cdots}$, that satisfy

$$C_0 = 1, C_1 = 3, \text{ and } C_J = C_{J-1} + (4J^2 + 2J)C_{J-2} \text{ for } J \geq 2. \tag{11}$$

By construction, we can upper bound C_J,

$$\begin{aligned} C_J &\leq C_{J-2} + (4J^2 + 2J)C_{J-2} + (4(J-1)^2 + 2J - 2)C_{J-3} \\ &\leq C_{J-2} + (4J^2 + 2J)C_{J-2} + (4(J-1)^2 + 2J - 2)C_{J-2} \\ &\leq 8J^2 C_{J-2} \\ &\leq (3J)^J. \end{aligned} \tag{12}$$

A.1 Some Lemmata

We first introduce some lemmata, whose proofs can be found in Sec. A.4.

Lemma 1 (Proposition 1.1 in [20]). *If $\boldsymbol{x} \sim \mathcal{N}(\mathbf{0}, I_d)$ and $\Sigma \in \mathbb{R}^{d\times d}$ is a fixed positive semi-definite matrix, then for all $t > 0$, w. p. $1 - e^{-t}$, we have*

$$\boldsymbol{x}^T \Sigma \boldsymbol{x} \leq \operatorname{tr}(\Sigma) + 2\sqrt{\operatorname{tr}(\Sigma^2)t} + 2\|\Sigma\|t.$$

By taking $t = P\log(d) + \log(n)$ for some $n \geq d$ and some constant $P \geq 1$, we have the following corollary.

Corollary 2. *If $\boldsymbol{x} \sim \mathcal{N}(\mathbf{0}, I_d)$ and $\Sigma \in \mathbb{R}^{d\times d}$ is a fixed positive semi-definite matrix, then for a fixed positive constant $P \geq 1$, we have, w. p. $1 - \frac{1}{n}d^{-P}$,*

$$\boldsymbol{x}^T \Sigma \boldsymbol{x} \leq \operatorname{tr}(\Sigma) + 2\sqrt{\operatorname{tr}(\Sigma^2)(P\log(d) + \log(n))} + 2\|\Sigma\|(P\log(d) + \log(n)) \leq (4P+5)\operatorname{tr}(\Sigma)\log(n).$$

Setting $\Sigma = \boldsymbol{\beta}\boldsymbol{\beta}^T$ in Corollary 2, we have the following corollary.

Corollary 3. *If $\boldsymbol{x} \sim \mathcal{N}(\mathbf{0}, I_d)$ and $P \geq 1$ is a constant, then given any fixed $\boldsymbol{\beta} \in \mathbb{R}^d$, w. p. $1 - \frac{1}{n}d^{-P}$, we have*

$$(\boldsymbol{\beta}^T \boldsymbol{x})^2 \leq (4P + 5)\|\boldsymbol{\beta}\|^2 \log n.$$

Setting $\Sigma = I$ in Corollary 2, we have the following corollary.

Corollary 4. *If $\boldsymbol{x} \sim \mathcal{N}(\mathbf{0}, I_d)$ and $P \geq 1$ is a constant, then w. p. $1 - \frac{1}{n}d^{-P}$, we have*

$$\|\boldsymbol{x}\|^2 \leq (4P + 5)d\log n.$$

Lemma 2 (Stein-type Lemma). *Let $\boldsymbol{x} \sim \mathcal{N}(\mathbf{0}, I_d)$ and $f(\boldsymbol{x})$ be a function of $\boldsymbol{x}$ whose second derivative exists. Then*

$$\mathbb{E}\left[\!\left[f(\boldsymbol{x})\boldsymbol{x}\boldsymbol{x}^T \right]\!\right] = \mathbb{E}[\![f(\boldsymbol{x})]\!]I + \mathbb{E}\left[\!\left[\nabla^2 f(\boldsymbol{x})\right]\!\right]$$

Lemma 3. *Let $\boldsymbol{x} \sim \mathcal{N}(\mathbf{0}, I_d)$ and $A_k \succeq 0$ for all $k = 1, 2, \cdots, K$, then*

$$\Pi_{k=1}^K \operatorname{tr}(A_k) I \preceq \mathbb{E}\left[\!\left[\Pi_{k=1}^K (\boldsymbol{x}^T A_k \boldsymbol{x})\boldsymbol{x}\boldsymbol{x}^T\right]\!\right] \preceq C_K \Pi_{k=1}^K \operatorname{tr}(A_k) I, \tag{13}$$

where C_K is a constant depending only on K, which is defined in Eq. (11).

Lemma 4. *Let $\boldsymbol{x}_i \sim \mathcal{N}(\mathbf{0}, I_d)$ i.i.d., for all $i \in [n]$ and $A_k \succeq 0$ for all $k = 1, 2, \cdots, K$. Let $B := \mathbb{E}\left[\!\left[\Pi_{k=1}^K (\boldsymbol{x}^T A_k \boldsymbol{x})\boldsymbol{x}\boldsymbol{x}^T\right]\!\right]$, $B_i := \Pi_{k=1}^K (\boldsymbol{x}_i^T A_k \boldsymbol{x}_i)\boldsymbol{x}_i\boldsymbol{x}_i^T$ and $\hat{B} = \frac{1}{n}\sum_{i=1}^n B_i$.*

If $n \geq O(\frac{1}{\delta^2}\log^K(\frac{1}{\delta})(PK)^K d\log^{K+1} d)$ and $\delta > \frac{\sqrt{4KC_{2K+1}}}{\sqrt{nd^P}}$ for some $0 < \delta \leq 1$ and $P \geq 1$, then w.p. $1 - O(Kd^{-P})$, we have

$$\|\hat{B} - B\| \leq \delta\|B\|. \tag{14}$$

Lemma 5. *Let $\boldsymbol{x} \sim \mathcal{N}(\mathbf{0}, I_d)$. Then given $\boldsymbol{\beta}, \boldsymbol{\gamma} \in \mathbb{R}^d$ and $A_k \succeq 0$ for all $k = 1, 2, \cdots, K$, we have*

$$\|\boldsymbol{\beta}\|\|\boldsymbol{\gamma}\|\Pi_{k=1}^K \operatorname{tr}(A_k) \leq \|\mathbb{E}[\![(\boldsymbol{\beta}^T\boldsymbol{x})(\boldsymbol{\gamma}^T\boldsymbol{x})\Pi_{k=1}^K(\boldsymbol{x}^T A_k \boldsymbol{x})\boldsymbol{x}\boldsymbol{x}^T]\!]\| \leq \sqrt{3C_{2K+1}}\|\boldsymbol{\beta}\|\|\boldsymbol{\gamma}\|\Pi_{k=1}^K \operatorname{tr}(A_k) I. \tag{15}$$

Lemma 6. *Let $\boldsymbol{x}_i \sim \mathcal{N}(\mathbf{0}, I_d)$ i.i.d., for all $i \in [n]$, $\boldsymbol{\beta}, \boldsymbol{\gamma} \in \mathbb{R}^d$ and $A_k \succeq 0$ for all $k = 1, 2, \cdots, K$. Let $B := \mathbb{E}[\![(\boldsymbol{\beta}^T\boldsymbol{x})(\boldsymbol{\gamma}^T\boldsymbol{x})\Pi_{k=1}^K(\boldsymbol{x}^T A_k \boldsymbol{x})\boldsymbol{x}\boldsymbol{x}^T]\!]$, $B_i := (\boldsymbol{\beta}^T\boldsymbol{x}_i)(\boldsymbol{\gamma}^T\boldsymbol{x}_i)\Pi_{k=1}^K(\boldsymbol{x}_i^T A_k \boldsymbol{x}_i)\boldsymbol{x}_i\boldsymbol{x}_i^T$ and $\hat{B} = \frac{1}{n}\sum_{i=1}^n B_i$.*

If $n \geq O(\frac{1}{\delta^2}\log^{K+1}(1/\delta)(PK)^K d\log^{K+2}(d))$, $\delta > \frac{\sqrt{8KC_{2K+3}}}{\sqrt{nd^P}}$ for some $0 < \delta \leq 1$ and $P \geq 1$, then w.p. $1 - O(Kd^{-P})$, we have

$$\|\hat{B} - B\| \leq \delta\|B\|. \tag{16}$$

Lemma 7. *If $n \geq c\log^{K+1}(c)K^{4K}d\log^{K+2}(d)$, where c is a constant, then $n \geq cd\log d\log^{K+1}(n)$.*

A.2 Proof of Theorem 1

Proof. Denote the Hessian of Eq. (1), $H \in \mathbb{R}^{Kd\times Kd}$. Let $H = \sum_i H_i$, where

$$H_i := \begin{bmatrix} H_i^{11} & H_i^{12} & \cdots & H_i^{1K} \\ H_i^{21} & H_i^{22} & \cdots & H_i^{2K} \\ & & \ddots & \\ H_i^{K1} & H_i^{K2} & \cdots & H_i^{KK} \end{bmatrix} \tag{17}$$

For diagonal blocks,

$$H_i^{jj} := 2\big(\Pi_{k\neq j}(y_i - (\boldsymbol{w}_k + \delta\boldsymbol{w}_k)^T\boldsymbol{x}_i)^2\big)\boldsymbol{x}_i\boldsymbol{x}_i^T \tag{18}$$

For off-diagonal blocks,

$$H_i^{jl} := 4(y_i - (\boldsymbol{w}_j + \delta\boldsymbol{w}_j)^T\boldsymbol{x}_i)(y_i - (\boldsymbol{w}_l + \delta\boldsymbol{w}_l)^T\boldsymbol{x}_i)\big(\Pi_{k\neq j,k\neq l}(y_i - (\boldsymbol{w}_k + \delta\boldsymbol{w}_k)^T\boldsymbol{x}_i)^2\big)\boldsymbol{x}_i\boldsymbol{x}_i^T \tag{19}$$

In the following we will show that when $\boldsymbol{w}_k$ is close to the optimal solution $\boldsymbol{w}_k^*$ and $\delta\boldsymbol{w}_k$ is small enough for all k, then H will be positive definite w.h.p..

The main idea is to upper bound the off-diagonal blocks and lower bound the diagonal blocks because,

$$\begin{aligned}
\sigma_{min}(H) &= \min_{\sum_{j=1}^K \|\boldsymbol{a}_j\|^2=1} \sum_{j=1}^K \boldsymbol{a}_j^T H^{jj}\boldsymbol{a}_j + \sum_{j\neq l} 2\boldsymbol{a}_j^T H^{jl}\boldsymbol{a}_l \\
&\geq \min_{\sum_{j=1}^K \|\boldsymbol{a}_j\|^2=1} \sum_{j=1}^K \sigma_{min}(H^{jj})\|\boldsymbol{a}_j\|^2 - \sum_{j\neq l}\|H^{jl}\|\|\boldsymbol{a}_j\|\|\boldsymbol{a}_l\| \\
&\geq \min_j\{\sigma_{min}(H^{jj})\} - \max_{j\neq l}\{\|H^{jl}\|\}(K-1)(\sum_j \|\boldsymbol{a}_j\|) \\
&\geq \min_j\{\sigma_{min}(H^{jj})\} - (K-1)\max_{j\neq l}\{\|H^{jl}\|\}.
\end{aligned} \tag{20}$$

First consider the diagonal blocks. The idea is to decompose the diagonal blocks into two parts. The first one only contains $\boldsymbol{w}$ and doesn't contain $\delta\boldsymbol{w}$, so for this fixed $\boldsymbol{w}$ we apply Lemma 4 to bound this term. The second one depends on $\delta\boldsymbol{w}$. We find an upper bound for this term which only depends on the magnitude of $\delta\boldsymbol{w}$. Therefore, the bound will hold for any qualified $\delta\boldsymbol{w}$. Let's first define

$\{k_1, k_2, \cdots, k_{K-1}\} = [K]\backslash\{j\}$.

$$
\begin{aligned}
H^{jj} &\succeq \sum_{i\in S_j} H_i^{jj} \\
&= \sum_{i\in S_j} 2\big(\Pi_{s=1}^{K-1}(y_i - \boldsymbol{w}_{k_s}^T\boldsymbol{x}_i - \delta\boldsymbol{w}_{k_s}^T\boldsymbol{x}_i)^2\big)\boldsymbol{x}_i\boldsymbol{x}_i^T \\
&\succeq \sum_{i\in S_j} 2\big((y_i - \boldsymbol{w}_{k_1}^T\boldsymbol{x}_i)^2 - 2|y_i - \boldsymbol{w}_{k_1}^T\boldsymbol{x}_i|\|\delta\boldsymbol{w}_{k_1}\|\|\boldsymbol{x}_i\|\big)\big(\Pi_{s=2}^{K-1}(y_i - \boldsymbol{w}_{k_s}^T\boldsymbol{x}_i - \delta\boldsymbol{w}_{k_s}^T\boldsymbol{x}_i)^2\big)\boldsymbol{x}_i\boldsymbol{x}_i^T \\
&\succeq \underbrace{\sum_{i\in S_j} 2(y_i - \boldsymbol{w}_{k_1}^T\boldsymbol{x}_i)^2\big(\Pi_{s=2}^{K-1}(y_i - \boldsymbol{w}_{k_s}^T\boldsymbol{x}_i - \delta\boldsymbol{w}_{k_s}^T\boldsymbol{x}_i)^2\big)\boldsymbol{x}_i\boldsymbol{x}_i^T}_{F_1} \\
&\quad - \underbrace{\sum_{i\in S_j} 4\|\Delta\boldsymbol{w}_{jk_1}^* - \Delta\boldsymbol{w}_{k_1}\|\|\delta\boldsymbol{w}_{k_1}\|\|\boldsymbol{x}_i\|^2\big(\Pi_{s=2}^{K-1}(y_i - \boldsymbol{w}_{k_s}^T\boldsymbol{x}_i - \delta\boldsymbol{w}_{k_s}^T\boldsymbol{x}_i)^2\big)\boldsymbol{x}_i\boldsymbol{x}_i^T}_{E_1}
\end{aligned}
\tag{21}
$$

$$
\begin{aligned}
F_1 &\succeq \sum_{i\in S_j} 2(y_i - \boldsymbol{w}_{k_1}^T\boldsymbol{x}_i)^2(y_i - \boldsymbol{w}_{k_2}^T\boldsymbol{x}_i)^2\big(\Pi_{s=3}^{K-1}(y_i - \boldsymbol{w}_{k_s}^T\boldsymbol{x}_i - \delta\boldsymbol{w}_{k_s}^T\boldsymbol{x}_i)^2\big)\boldsymbol{x}_i\boldsymbol{x}_i^T \\
&\quad - \sum_{i\in S_j} 4(y_i - \boldsymbol{w}_{k_1}^T\boldsymbol{x}_i)^2\|\Delta\boldsymbol{w}_{jk_2}^* - \Delta\boldsymbol{w}_{k_2}\|\|\delta\boldsymbol{w}_{k_2}\|\|\boldsymbol{x}_i\|^2\big(\Pi_{s=3}^{K-1}(y_i - \boldsymbol{w}_{k_s}^T\boldsymbol{x}_i - \delta\boldsymbol{w}_{k_s}^T\boldsymbol{x}_i)^2\big)\boldsymbol{x}_i\boldsymbol{x}_i^T \\
&\succeq \underbrace{\sum_{i\in S_j} 2(y_i - \boldsymbol{w}_{k_1}^T\boldsymbol{x}_i)^2(y_i - \boldsymbol{w}_{k_2}^T\boldsymbol{x}_i)^2\big(\Pi_{s=3}^{K-1}(y_i - \boldsymbol{w}_{k_s}^T\boldsymbol{x}_i - \delta\boldsymbol{w}_{k_s}^T\boldsymbol{x}_i)^2\big)\boldsymbol{x}_i\boldsymbol{x}_i^T}_{F_2} \\
&\quad - \underbrace{\sum_{i\in S_j} 4\|\Delta\boldsymbol{w}_{jk_1}^* - \Delta\boldsymbol{w}_{k_1}\|^2\|\Delta\boldsymbol{w}_{jk_2}^* - \Delta\boldsymbol{w}_{k_2}\|\|\delta\boldsymbol{w}_{k_2}\|\|\boldsymbol{x}_i\|^4\big(\Pi_{s=3}^{K-1}(y_i - \boldsymbol{w}_{k_s}^T\boldsymbol{x}_i - \delta\boldsymbol{w}_{k_s}^T\boldsymbol{x}_i)^2\big)\boldsymbol{x}_i\boldsymbol{x}_i^T}_{E_2}
\end{aligned}
\tag{22}
$$

Similarly, we decompose $F_n = F_{n+1} - E_{n+1}$, for $n = 1, 2, \cdots, K-1$. Then, recursively, we have

$$
H^{jj} \succeq F_1 - E_1 \succeq F_2 - E_2 - E_1 \succeq \cdots \succeq F_{K-1} - E_{K-1} - E_{K-2} - \cdots - E_1 \tag{23}
$$

So H^{jj} is decomposed into F_{K-1}, which contains only $\boldsymbol{w}$, and $E_1, E_2, \cdots, E_{K-1}$, each of which contains a separate term of $\|\delta\boldsymbol{w}\|$.

By Lemma 3 and Lemma 4,

$$
\begin{aligned}
E_1 \preceq& 4\sum_{i\in S_j} \|\Delta\boldsymbol{w}_{jk_1}^* - \Delta\boldsymbol{w}_{k_1}\|\|\delta\boldsymbol{w}_{k_1}\|(\Pi_{s=2}^{K-1}\|\Delta\boldsymbol{w}_{jk_s}^* - \Delta\boldsymbol{w}_{k_s} - \delta\boldsymbol{w}_{k_s}\|^2)\|\boldsymbol{x}_i\|^{2(K-1)}\boldsymbol{x}_i\boldsymbol{x}_i^T \\
\preceq& 4c_f(1 + c_m + c_f)^{2K-3}\Pi_{k:k\neq j}\|\Delta\boldsymbol{w}_{jk}^*\|^2\sum_{i\in S_j}\|\boldsymbol{x}_i\|^{2(K-1)}\boldsymbol{x}_i\boldsymbol{x}_i^T \\
\preceq& 6c_f(1 + c_m + c_f)^{2K-3}\Pi_{k:k\neq j}\|\Delta\boldsymbol{w}_{jk}^*\|^2 p_j N C_{K-1} d^{K-1} I
\end{aligned}
\tag{24}
$$

and similarly, for all $r = 1, 2, \cdots, K-1$,

$$
E_r \preceq 6c_f(1 + c_m + c_f)^{2K-3}\Pi_{k:k\neq j}\|\Delta\boldsymbol{w}_{jk}^*\|^2 p_j N C_{K-1} d^{K-1} I. \tag{25}
$$

For F_{K-1}, we have

$$
\begin{aligned}
F_{K-1} =& \sum_{i\in S_j} 2\big(\Pi_{k\neq j}(y_i - \boldsymbol{w}_k^T\boldsymbol{x}_i)^2\big)\boldsymbol{x}_i\boldsymbol{x}_i^T \\
\overset{\xi_1}{\succeq}& p_j N \Pi_{k\neq j}\|\Delta\boldsymbol{w}_{jk}^* - \Delta\boldsymbol{w}_k\|^2 I \\
\succeq& p_j N \Pi_{k\neq j}(\|\Delta\boldsymbol{w}_{jk}^*\| - \|\Delta\boldsymbol{w}_k\|)^2 I \\
\succeq& p_j N (1 - c_m)^{2(K-1)}\Pi_{k\neq j}\|\Delta\boldsymbol{w}_{jk}^*\|^2 I
\end{aligned}
\tag{26}
$$

where ξ_1 is because of Lemma 3 and Lemma 4 by setting $A_k = (\Delta \boldsymbol{w}_{jk}^* - \Delta \boldsymbol{w}_k)(\Delta \boldsymbol{w}_{jk}^* - \Delta \boldsymbol{w}_k)^T$ and $\delta = 1/(2C_{K-1})$.

Now combining Eq. (26), Eq. (23) and Eq. (25), we can lower bound the eigenvalues of H^{jj},

$$H^{jj} \succeq \Big((1-c_m)^{2(K-1)} - 6c_f(K-1)(1+c_m+c_f)^{2K-3}C_{K-1}d^{K-1}\Big)p_jN\Pi_{k\neq j}\|\Delta \boldsymbol{w}_{jk}^*\|^2 I \tag{27}$$

Next consider the off-diagonal blocks for $j \neq l$,

$$\begin{aligned}
&\sum_{i\in S_q} H_i^{jl} \\
=&\sum_{i\in S_q} 4(y_i - (\boldsymbol{w}_j+\delta\boldsymbol{w}_j)^T\boldsymbol{x}_i)(y_i - (\boldsymbol{w}_l+\delta\boldsymbol{w}_l)^T\boldsymbol{x}_i)\big(\Pi_{k\neq j,k\neq l}(y_i-(\boldsymbol{w}_k+\delta\boldsymbol{w}_k)^T\boldsymbol{x}_i)^2\big)\boldsymbol{x}_i\boldsymbol{x}_i^T \\
\preceq&\sum_{i\in S_q} 4(y_i - \boldsymbol{w}_j^T\boldsymbol{x}_i)(y_i - (\boldsymbol{w}_l+\delta\boldsymbol{w}_l)^T\boldsymbol{x}_i)\big(\Pi_{k\neq j,k\neq l}(y_i-(\boldsymbol{w}_k+\delta\boldsymbol{w}_k)^T\boldsymbol{x}_i)^2\big)\boldsymbol{x}_i\boldsymbol{x}_i^T \\
&+\sum_{i\in S_q} 4\|\delta\boldsymbol{w}_j^T\boldsymbol{x}_i\||y_i - (\boldsymbol{w}_l+\delta\boldsymbol{w}_l)^T\boldsymbol{x}_i|\big(\Pi_{k\neq j,k\neq l}(y_i-(\boldsymbol{w}_k+\delta\boldsymbol{w}_k)^T\boldsymbol{x}_i)^2\big)\boldsymbol{x}_i\boldsymbol{x}_i^T \\
\preceq&\sum_{i\in S_q} 4(y_i - \boldsymbol{w}_j^T\boldsymbol{x}_i)(y_i - \boldsymbol{w}_l^T\boldsymbol{x}_i)\big(\Pi_{k\neq j,k\neq l}(y_i-(\boldsymbol{w}_k+\delta\boldsymbol{w}_k)^T\boldsymbol{x}_i)^2\big)\boldsymbol{x}_i\boldsymbol{x}_i^T \\
&+\sum_{i\in S_q} 4|y_i - \boldsymbol{w}_j^T\boldsymbol{x}_i|\|\delta\boldsymbol{w}_l^T\boldsymbol{x}_i\|\big(\Pi_{k\neq j,k\neq l}(y_i-(\boldsymbol{w}_k+\delta\boldsymbol{w}_k)^T\boldsymbol{x}_i)^2\big)\boldsymbol{x}_i\boldsymbol{x}_i^T \\
&+\sum_{i\in S_q} 4\|\delta\boldsymbol{w}_j\|\|\boldsymbol{w}_q^* - \boldsymbol{w}_l - \delta\boldsymbol{w}_l)\|\big(\Pi_{k\neq j,k\neq l}\|\boldsymbol{w}_q^*-\boldsymbol{w}_k+\delta\boldsymbol{w}_k\|^2\big)\|\boldsymbol{x}_i\|^{2(K-1)}\boldsymbol{x}_i\boldsymbol{x}_i^T \\
\preceq& \\
&\vdots \\
\preceq&\sum_{i\in S_q} 4(y_i - \boldsymbol{w}_j^T\boldsymbol{x}_i)(y_i - \boldsymbol{w}_l^T\boldsymbol{x}_i)\big(\Pi_{k\neq j,k\neq l}(y_i-\boldsymbol{w}_k^T\boldsymbol{x}_i)^2\big)\boldsymbol{x}_i\boldsymbol{x}_i^T \\
&+8(K-1)c_f(1+c_m+c_f)^{2K-3}\Delta_{max}^{2K-2}\sum_{i\in S_q}\|\boldsymbol{x}_i\|^{2(K-1)}\boldsymbol{x}_i\boldsymbol{x}_i^T \\
\preceq&\sum_{i\in S_q} 4(y_i - \boldsymbol{w}_j^T\boldsymbol{x}_i)(y_i - \boldsymbol{w}_l^T\boldsymbol{x}_i)\big(\Pi_{k\neq j,k\neq l}(y_i-\boldsymbol{w}_k^T\boldsymbol{x}_i)^2\big)\boldsymbol{x}_i\boldsymbol{x}_i^T \\
&+12(K-1)c_f(1+c_m+c_f)^{2K-3}\Delta_{max}^{2K-2}p_qNC_{K-1}d^{K-1}I
\end{aligned} \tag{28}$$

For the first term above,

$$\begin{aligned}
&\|\sum_{i\in S_q} 4(\boldsymbol{w}_q^*-\boldsymbol{w}_j)^T\boldsymbol{x}_i(\boldsymbol{w}_q^*-\boldsymbol{w}_l)^T\boldsymbol{x}_i\big(\Pi_{k\neq j,k\neq l}((\boldsymbol{w}_q^*-\boldsymbol{w}_k)^T\boldsymbol{x}_i)^2\big)\boldsymbol{x}_i\boldsymbol{x}_i^T\| \\
\overset{\xi_1}{\leq}&6p_qN\|\mathbb{E}\big[\!\big[(\boldsymbol{w}_q^*-\boldsymbol{w}_j)^T\boldsymbol{x}_i(\boldsymbol{w}_q^*-\boldsymbol{w}_l)^T\boldsymbol{x}_i\big(\Pi_{k\neq j,k\neq l}((\boldsymbol{w}_q^*-\boldsymbol{w}_k)^T\boldsymbol{x}_i)^2\big)\boldsymbol{x}_i\boldsymbol{x}_i^T\big]\!\big]\| \\
\overset{\xi_2}{\leq}&6p_qN\sqrt{3C_{2K-3}}\|\boldsymbol{w}_q^*-\boldsymbol{w}_j\|\|\boldsymbol{w}_q^*-\boldsymbol{w}_l\|\big(\Pi_{k\neq j,k\neq l}\|\boldsymbol{w}_q^*-\boldsymbol{w}_k\|^2\big) \\
\leq&6p_qN\sqrt{3C_{2K-3}}\|\Delta\boldsymbol{w}_{qj}^*-\Delta\boldsymbol{w}_j\|\|\Delta\boldsymbol{w}_{ql}^*-\Delta\boldsymbol{w}_l\|\big(\Pi_{k\neq j,k\neq l}\|\Delta\boldsymbol{w}_{qk}^*-\Delta\boldsymbol{w}_k\|^2\big),
\end{aligned} \tag{29}$$

where ξ_1 is because of Lemma 6 and ξ_2 is because of Lemma 5.

We consider three cases: $\{q\neq j, q\neq l\}$, $q=j$ and $q=l$. When $q\neq j$ and $q\neq l$,

$$\begin{aligned}
&\|\Delta\boldsymbol{w}_{qj}^*-\Delta\boldsymbol{w}_j\|\|\Delta\boldsymbol{w}_{ql}^*-\Delta\boldsymbol{w}_l\|\big(\Pi_{k\neq j,k\neq l}\|\Delta\boldsymbol{w}_{qk}^*-\Delta\boldsymbol{w}_k\|^2\big) \\
\leq&(1+c_m)^{2K-2}c_m^2\|\Delta\boldsymbol{w}_{qj}^*\|\|\Delta\boldsymbol{w}_{ql}^*\|\big(\Pi_{k\neq j,k\neq l}\|\Delta\boldsymbol{w}_{qk}^*\|^2\big)
\end{aligned} \tag{30}$$

When $q = j$,

$$\begin{aligned}
&\|\Delta \boldsymbol{w}_{qj}^* - \Delta \boldsymbol{w}_j\| \|\Delta \boldsymbol{w}_{ql}^* - \Delta \boldsymbol{w}_l\| \big(\Pi_{k\neq j, k\neq l} \|\Delta \boldsymbol{w}_{qk}^* - \Delta \boldsymbol{w}_k\|^2\big) \\
\leq &(1+c_m)^{2K-1} c_m \|\Delta \boldsymbol{w}_{qj}^*\| \|\Delta \boldsymbol{w}_{ql}^*\| \big(\Pi_{k\neq j, k\neq l} \|\Delta \boldsymbol{w}_{qk}^*\|^2\big)
\end{aligned} \tag{31}$$

For $q = l$, we have similar results. Therefore,

$$\begin{aligned}
\|H^{jl}\| \leq &\sum_{q=1}^{K} \|\sum_{i\in S_q} H_i^{jl}\| \\
\leq &\sum_q (1+c_m)^{2K-1} c_m 6 p_q N \sqrt{3C_{2K-3}} \Delta_{max}^{2K-2} \\
&+ \sum_q 12(K-1) c_f (1+c_m+c_f)^{2K-3} p_q N C_{K-1} d^{K-1} \Delta_{max}^{2K-2} \\
\leq &(1+c_m)^{2K-1} c_m 6 N \sqrt{3C_{2K-3}} \Delta_{max}^{2K-2} \\
&+ 12(K-1) c_f (1+c_m+c_f)^{2K-3} N C_{K-1} d^{K-1} \Delta_{max}^{2K-2}
\end{aligned} \tag{32}$$

Now we obtain the lower bound for the minimal eigenvalue of the Hessian. When $c_m \leq \frac{p_{min}\Delta_{min}^{2K-2}}{500K\sqrt{C_{2K-3}}\Delta_{max}^{2K-2}}$ and $c_f \leq \frac{p_{min}\Delta_{min}^{2K-2}}{1000(K-1)^2 C_{K-1} d^{K-1}\Delta_{max}^{2K-2}}$, we have $(1-c_m)^{2K-2} \geq (1-\frac{1}{2K})^{2K-2} \geq \frac{1}{4}$, $(1+c_m+c_f)^{2K-2} \leq 3$. Hence,

$$\|H^{jl}\| \leq \frac{1}{16(K-1)} p_{min} N \Delta_{min}^{2K-2}, \tag{33}$$

Combining Eq.(20), Eq.(27) and Eq.(33), we have

$$\sigma_{min}(H) \geq \frac{1}{8} p_{min} N \Delta_{min}^{2K-2}, \tag{34}$$

which is a positive constant.

In the following we upper bound the maximal eigenvalue of the Hessian.

$$\begin{aligned}
\sigma_{max}(H) = &\max_{\sum_{j=1}^K \|\boldsymbol{a}_j\|^2=1} \sum_{j=1}^{K} \boldsymbol{a}_j^T H^{jj} \boldsymbol{a}_j + \sum_{j\neq l} 2\boldsymbol{a}_j^T H^{jl} \boldsymbol{a}_l \\
\leq &\max_{\sum_{j=1}^K \|\boldsymbol{a}_j\|^2=1} \sum_{j=1}^{K} \|(H^{jj})\| \|\boldsymbol{a}_j\|^2 + \sum_{j\neq l} \|H^{jl}\| \|\boldsymbol{a}_j\| \|\boldsymbol{a}_l\| \\
\leq &\max_j \{\|H^{jj}\|\} + \max_{j\neq l}\{\|H^{jl}\|\}(K-1)(\sum_j \|\boldsymbol{a}_j\|) \\
\leq &\max_j \{\|H^{jj}\|\} + (K-1)\max_{j\neq l}\{\|H^{jl}\|\}.
\end{aligned} \tag{35}$$

Consider the diagonal blocks and define $\{k_1, k_2, \cdots, k_{K-1}\} = [K]\backslash\{j\}$.

$$\begin{aligned}
H_i^{jj} = &2\big(\Pi_{s=1}^{K-1}(y_i - \boldsymbol{w}_{k_s}^T \boldsymbol{x}_i - \delta \boldsymbol{w}_{k_s}^T \boldsymbol{x}_i)^2\big) \boldsymbol{x}_i \boldsymbol{x}_i^T \\
\preceq &2\big((y_i - \boldsymbol{w}_{k_1}^T \boldsymbol{x}_i)^2 + 2|y_i - \boldsymbol{w}_{k_1}^T \boldsymbol{x}_i||\delta \boldsymbol{w}_{k_1}^T \boldsymbol{x}_i| + (\delta \boldsymbol{w}_{k_1}^T \boldsymbol{x}_i)^2\big)\big(\Pi_{s=2}^{K-1}(y_i - \boldsymbol{w}_{k_s}^T \boldsymbol{x}_i - \delta \boldsymbol{w}_{k_s}^T \boldsymbol{x}_i)^2\big) \boldsymbol{x}_i \boldsymbol{x}_i^T \\
\preceq &\underbrace{2(y_i - \boldsymbol{w}_{k_1}^T \boldsymbol{x}_i)^2 \big(\Pi_{s=2}^{K-1}(y_i - \boldsymbol{w}_{k_s}^T \boldsymbol{x}_i - \delta \boldsymbol{w}_{k_s}^T \boldsymbol{x}_i)^2\big) \boldsymbol{x}_i \boldsymbol{x}_i^T}_{\tilde{F}_1} \\
&+ \underbrace{2(2\|\Delta \boldsymbol{w}_{jk_1}^* - \Delta \boldsymbol{w}_{k_1}\| + \|\delta \boldsymbol{w}_{k_1}\|)\|\delta \boldsymbol{w}_{k_1}\| \|\boldsymbol{x}_i\|^2 \big(\Pi_{s=2}^{K-1}(y_i - \boldsymbol{w}_{k_s}^T \boldsymbol{x}_i - \delta \boldsymbol{w}_{k_s}^T \boldsymbol{x}_i)^2\big) \boldsymbol{x}_i \boldsymbol{x}_i^T}_{\tilde{E}_1}
\end{aligned} \tag{36}$$

For $\tilde{E}_1$,

$$\tilde{E}_1 \preceq 4 c_f (1+c_m+c_f)^{2K-3} \Delta_{max}^{2K-2} \|\boldsymbol{x}_i\|^{2K-2} \boldsymbol{x}_i \boldsymbol{x}_i^T \tag{37}$$

For $\tilde{F}_1$,

$$\begin{aligned}
&\tilde{F}_1\\
&\preceq \underbrace{2(y_i - \boldsymbol{w}_{k_1}^T\boldsymbol{x}_i)^2(y_i - \boldsymbol{w}_{k_2}^T\boldsymbol{x}_i)^2\big(\Pi_{s=3}^{K-1}(y_i - \boldsymbol{w}_{k_s}^T\boldsymbol{x}_i - \delta\boldsymbol{w}_{k_s}^T\boldsymbol{x}_i)^2\big)\boldsymbol{x}_i\boldsymbol{x}_i^T}_{\tilde{F}_2}\\
&+ \underbrace{2(y_i - \boldsymbol{w}_{k_1}^T\boldsymbol{x}_i)^2|(2\Delta\boldsymbol{w}_{jk_2}^* - 2\Delta\boldsymbol{w}_{k_2} - \delta\boldsymbol{w}_{k_2})^T\boldsymbol{x}_i||\delta\boldsymbol{w}_{k_2}^T\boldsymbol{x}_i|\big(\Pi_{s=3}^{K-1}(y_i - \boldsymbol{w}_{k_s}^T\boldsymbol{x}_i - \delta\boldsymbol{w}_{k_s}^T\boldsymbol{x}_i)^2\big)\boldsymbol{x}_i\boldsymbol{x}_i^T}_{\tilde{E}_2}
\end{aligned} \tag{38}$$

We also have for $\tilde{E}_2$

$$\tilde{E}_2 \preceq 4c_f(1 + c_m + c_f)^{2K-3}\Delta_{max}^{2K-2}\|\boldsymbol{x}_i\|^{2K-2}\boldsymbol{x}_i\boldsymbol{x}_i^T \tag{39}$$

Therefore, recursively, we have

$$\begin{aligned}
H_i^{jj} \preceq &\underbrace{2\Pi_{s=1}^{K-1}(y_i - \boldsymbol{w}_{k_s}^T\boldsymbol{x}_i)^2\boldsymbol{x}_i\boldsymbol{x}_i^T}_{\tilde{F}_{K-1}}\\
&+ 4Kc_f(1 + c_m + c_f)^{2K-3}\Delta_{max}^{2K-2}\|\boldsymbol{x}_i\|^{2K-2}\boldsymbol{x}_i\boldsymbol{x}_i^T
\end{aligned} \tag{40}$$

Now applying Lemma 3 and Lemma 4,

$$\begin{aligned}
H^{jj} =& \sum_q \sum_{i\in S_q} H_i^{jj}\\
\preceq& 6c_fK(1 + c_m + c_f)^{2K-3}NC_{K-1}d^{K-1}\Delta_{max}^{2K-2}I + \sum_q\sum_{i\in S_q} 2\big(\Pi_{k\neq q}((\Delta\boldsymbol{w}_{jk}^* - \Delta\boldsymbol{w}_k)^T\boldsymbol{x}_i)^2\big)\boldsymbol{x}_i\boldsymbol{x}_i^T\\
\preceq& 6c_fK(1 + c_m + c_f)^{2K-3}NC_{K-1}d^{K-1}\Delta_{max}^{2K-2}I + 3\sum_q p_qNC_{K-1}\big(\Pi_{k\neq q}\|\Delta\boldsymbol{w}_{jk}^* - \Delta\boldsymbol{w}_k\|^2\big)\\
=& 6c_fK(1 + c_m + c_f)^{2K-3}NC_{K-1}d^{K-1}\Delta_{max}^{2K-2}I\\
&+ 3p_jNC_{K-1}(1 + c_m)^{2K-2}\big(\Pi_{k\neq j}\|\Delta\boldsymbol{w}_{jk}^*\|^2\big)\\
&+ 3\sum_{q:q\neq j} p_qNC_{K-1}c_m^2(1 + c_m)^{2K-4}\big(\Pi_{k:k\neq j}\|\Delta\boldsymbol{w}_{jk}^*\|^2\big)\\
\preceq& 9NC_{K-1}\Delta_{max}^{2K-2}I
\end{aligned} \tag{41}$$

Combining the off-diagonal blocks bound in Eq. (33), applying union bound on the probabilities of the lemmata and Eq. (12) complete the proof. □

A.3 Proof of Theorem 2

We first introduce a corollary of Theorem 1, which shows the strong convexity on a line between a current iterate and the optimum.

Corollary 5 (Positive Definiteness on the Line between $\boldsymbol{w}$ and $\boldsymbol{w}^*$). *Let $\{\boldsymbol{x}_i, y_i\}_{i=1,2,\cdots,N}$ be sampled from the MLR model* (3). *Let $\{\boldsymbol{w}_k\}_{k=1,2,\cdots,K}$ be independent of the samples and lie in the neighborhood of the optimal solution, defined in Eq.* (4). *Then, if $N \geq O(K^Kd\log^{K+2}(d))$, w.p. $1 - O(Kd^{-2})$, for all $\lambda \in [0,1]$,*

$$\frac{1}{8}p_{\min}N\Delta_{\min}^{2K-2}I \preceq \nabla^2 f(\lambda\boldsymbol{w}^* + (1-\lambda)\boldsymbol{w}) \preceq 10N(3K)^K\Delta_{\max}^{2K-2}I. \tag{42}$$

Proof. We set d^{K-1} anchor points equally along the line $\lambda\boldsymbol{w}^* + (1-\lambda)\boldsymbol{w}$ for $\lambda \in [0,1]$. Then based on these anchors, according to Theorem 1, by setting $P = K+1$, we complete the proof. □

Now we show the proof of Theorem 2.

Proof. Let $\alpha := \frac{1}{8} p_{\min} N \Delta_{\min}^{2K-2}$ and $\beta := 10N(3K)^K \Delta_{\max}^{2K-2}$.

$$\begin{aligned}\|\boldsymbol{w}^+ - \boldsymbol{w}^*\|^2 =& \|\boldsymbol{w} - \eta \nabla f(\boldsymbol{w}) - \boldsymbol{w}^*\|^2 \\ =& \|\boldsymbol{w} - \boldsymbol{w}^*\|^2 - 2\eta \nabla f(\boldsymbol{w})^T(\boldsymbol{w} - \boldsymbol{w}^*) + \eta^2 \|\nabla f(\boldsymbol{w})\|^2\end{aligned} \tag{43}$$

$$\begin{aligned}\nabla f(\boldsymbol{w}) =& \left(\int_0^1 \nabla^2 f(\boldsymbol{w}^* + \gamma(\boldsymbol{w} - \boldsymbol{w}^*)) d\gamma\right)(\boldsymbol{w} - \boldsymbol{w}^*) \\ =:& \hat{H}(\boldsymbol{w} - \boldsymbol{w}^*)\end{aligned} \tag{44}$$

According to Corollary 5,

$$\alpha I \preceq \hat{H} \preceq \beta I. \tag{45}$$

$$\|\nabla f(\boldsymbol{w})\|^2 = (\boldsymbol{w} - \boldsymbol{w}^*)^T \hat{H}^2 (\boldsymbol{w} - \boldsymbol{w}^*) \leq \beta(\boldsymbol{w} - \boldsymbol{w}^*)^T \hat{H} (\boldsymbol{w} - \boldsymbol{w}^*) \tag{46}$$

Therefore,

$$\begin{aligned}\|\boldsymbol{w}^+ - \boldsymbol{w}^*\|^2 \leq& \|\boldsymbol{w} - \boldsymbol{w}^*\|^2 - (-\eta^2\beta + 2\eta)(\boldsymbol{w} - \boldsymbol{w}^*)^T \hat{H}(\boldsymbol{w} - \boldsymbol{w}^*) \\ \leq& \|\boldsymbol{w} - \boldsymbol{w}^*\|^2 - (-\eta^2\beta + 2\eta)\alpha\|\boldsymbol{w} - \boldsymbol{w}^*\|^2 \\ =& \|\boldsymbol{w} - \boldsymbol{w}^*\|^2 - \frac{\alpha}{\beta}\|\boldsymbol{w} - \boldsymbol{w}^*\|^2 \\ \leq& (1 - \frac{\alpha}{\beta})\|\boldsymbol{w} - \boldsymbol{w}^*\|^2\end{aligned} \tag{47}$$

where the third equality holds by setting $\eta = \frac{1}{\beta}$. □

A.4 Proof of the lemmata

A.4.1 Proof of Lemma 2

Proof. Let $g(\boldsymbol{x}) = \frac{1}{(2\pi)^{d/2}} e^{-\|\boldsymbol{x}\|^2/2}$ and we have $\boldsymbol{x} g(\boldsymbol{x}) \mathrm{d}\boldsymbol{x} = -\mathrm{d}g(\boldsymbol{x})$.

$$\begin{aligned}\mathbb{E}\left[\!\left[f(\boldsymbol{x})\boldsymbol{x}\boldsymbol{x}^T\right]\!\right] &= \int f(\boldsymbol{x})\boldsymbol{x}\boldsymbol{x}^T g(\boldsymbol{x})\mathrm{d}\boldsymbol{x} \\ &= -\int f(\boldsymbol{x})(\mathrm{d}g(\boldsymbol{x}))\boldsymbol{x}^T \\ &= \int \nabla f(\boldsymbol{x})\boldsymbol{x}^T g(\boldsymbol{x}))\mathrm{d}\boldsymbol{x} + \int f(\boldsymbol{x})g(\boldsymbol{x}) I \mathrm{d}\boldsymbol{x} \\ &= -\int \nabla f(\boldsymbol{x})(\mathrm{d}g(\boldsymbol{x}))^T + \mathbb{E}[\![f(\boldsymbol{x})]\!] I \\ &= \mathbb{E}\left[\!\left[\nabla^2 f(\boldsymbol{x})\right]\!\right] + \mathbb{E}[\![f(\boldsymbol{x})]\!] I\end{aligned} \tag{48}$$

□

A.4.2 Proof of Lemma 3

Proof. Let $G_K := \mathbb{E}\left[\!\left[\Pi_{k=1}^K (\boldsymbol{x}^T A_k \boldsymbol{x})\boldsymbol{x}\boldsymbol{x}^T\right]\!\right]$. First we show the lower bound.

$$\begin{aligned}\sigma_{min}(G_K) &= \min_{\|\boldsymbol{a}\|=1} \mathbb{E}\left[\!\left[\Pi_{k=1}^K (\boldsymbol{x}^T A_k \boldsymbol{x})(\boldsymbol{x}^T \boldsymbol{a})^2\right]\!\right] \\ &\geq \Pi_{k=1}^K \mathbb{E}\left[\!\left[(\boldsymbol{x}^T A_k \boldsymbol{x})\right]\!\right] \min_{\|\boldsymbol{a}\|=1} \mathbb{E}\left[\!\left[(\boldsymbol{x}^T \boldsymbol{a})^2\right]\!\right] \\ &= \Pi_{k=1}^K \operatorname{tr}(A_k)\end{aligned} \tag{49}$$

Next, we show the upper bound. As we know, when $K = 1$, $G_1 = \operatorname{tr}(A_1)I + 2A_1$ and for any $K > 1$, G_K should have an explicit closed-form. However, it is too complicated to derive and formulate it for general K. Fortunately we only need the property of Eq. (13) in our proofs. We prove it by induction. First, it is obvious that Eq. (13) holds for $K = 1$ and $C_1 = 3$. We assume that, for any $J < K$, there exists a constant C_J depending only on J, such that

$$G_J \preceq C_J \Pi_{k=1}^{J} \operatorname{tr}(A_k) I \tag{50}$$

Then by Stein-type lemma, Lemma 2,

$$\begin{aligned}
G_K =& \mathbb{E}\left[\!\left[\Pi_{k=1}^{K}(\boldsymbol{x}^T A_k \boldsymbol{x})\boldsymbol{x}\boldsymbol{x}^T\right]\!\right] \\
=& \mathbb{E}\left[\!\left[\Pi_{k=1}^{K}(\boldsymbol{x}^T A_k \boldsymbol{x})\right]\!\right] I + 2\sum_{j=1}^{K} \mathbb{E}\left[\!\left[(\Pi_{k\neq j}^{K}(\boldsymbol{x}^T A_k \boldsymbol{x}))A_j\right]\!\right] \\
&+ 4\sum_{j,l:j\neq l} A_j \mathbb{E}\left[\!\left[(\Pi_{k:k\neq j,k\neq l}(\boldsymbol{x}^T A_k \boldsymbol{x}))\boldsymbol{x}\boldsymbol{x}^T\right]\!\right] A_l \\
\preceq& C_{K-1}\Pi_{k=1}^{K}\operatorname{tr}(A_k) I + 2\sum_{j=1}^{K} C_{K-2}(\Pi_{k\neq j}\operatorname{tr}(A_k))A_j \\
&+ 4\sum_{j,l:j\neq l} C_{K-2}\|A_j\|\|A_l\|\Pi_{k:k\neq j,k\neq l}\operatorname{tr}(A_k) I \\
\preceq& \big(C_{K-1} + (2K+4K^2)C_{K-2}\big)\Pi_{k=1}^{K}\operatorname{tr}(A_k) I
\end{aligned} \tag{51}$$

So $C_K = C_{K-1} + (4K^2 + 2K)C_{K-2}$. Note that $C_0 = 1$. □

A.4.3 Proof of Lemma 4

Proof. Proof Sketch: We use matrix Bernstein inequality to prove this lemma. However, the spectral norm of the random matrix B_i is not uniformly bounded, which is required by matrix Bernstein inequality. So we define a new random matrix,

$$M_i := \mathbf{1}(\mathcal{E}_i)\Pi_{k=1}^{K}(\boldsymbol{x}_i^T A_k \boldsymbol{x}_i)\boldsymbol{x}_i\boldsymbol{x}_i^T,$$

where $\mathcal{E}_i$ is an event when $\|B_i\|$ is bounded, which will hold with high probability and $\mathbf{1}()$ is the indicate function of value 1 and 0, i.e., $\mathbf{1}(\mathcal{E}) = 1$ if $\mathcal{E}$ holds and $\mathbf{1}(\mathcal{E}) = 0$ otherwise. Then

$$\|\hat{B} - B\| \leq \|\hat{B} - \hat{M}\| + \|\hat{M} - M\| + \|M - B\|,$$

where $M = \mathbb{E}[\![M_i]\!]$ and $\hat{M} = \frac{1}{n}\sum_{i=1}^{n} M_i$. We show that

1. $\hat{M} = \hat{B}$ w.h.p. by the union bound
2. $\|\hat{M} - M\|$ is bounded by matrix Bernstein inequality
3. $\|M - B\|$ is bounded because $\mathbb{E}[\![\mathbf{1}(\mathcal{E}^c)]\!]$ is small.

Proof Details:

Step 1. First we show that $\|B_i\|$ is bounded w.h.p.. First,

$$\|B_i\| = \Pi_{k=1}^{K}(\boldsymbol{x}_i^T A_k \boldsymbol{x}_i)\|\boldsymbol{x}_i\|^2$$

Since $\boldsymbol{x} \sim \mathcal{N}(0, I_d)$, by Corollary 4, we have $\mathbb{P}\left[\!\left[\|\boldsymbol{x}\|^2 \geq (4P+5)d\log n\right]\!\right] \leq n^{-1}d^{-P}$. By Corollary 2, $\mathbb{P}\left[\!\left[\boldsymbol{x}^T A_k \boldsymbol{x} > (4P+5)\operatorname{tr}(A_k)\log n\right]\!\right] \leq n^{-1}d^{-P}$. Therefore w.p. $1-(K+1)n^{-1}d^{-P}$,

$$\|B_i\| \leq (4P+5)^{K+1} \times (\Pi_{k=1}^{K}\operatorname{tr}(A_k)) d\log^{K+1}(n).$$

Define

$$m := (4P+5)^{K+1}(\Pi_{k=1}^{K}\operatorname{tr}(A_k)) d\log^{K+1}(n). \tag{52}$$

and the event

$$\mathcal{E}_i = \{\|B_i\| \leq m\},$$

Let $\mathcal{E}^c$ be the complementary set of $\mathcal{E}$, thus $\mathbb{P}[\![\mathcal{E}_i^c]\!] \leq (K+1)n^{-1}d^{-P}$. By union bound, w.p. $1-(K+1)d^{-P}$, $\|B_i\| \leq m$ for all $i \in [n]$ and $\hat{M} = \hat{B}$.

Step 2. Now we bound $\|\hat{M} - M\|$ by Matrix Bernstein's inequality[26].

Set $Z_i := M_i - M$. Thus $\mathbb{E}[\![Z_i]\!] = 0$ and $\|Z_i\| \le 2m$. And

$$\|\mathbb{E}[\![Z_i^2]\!]\| = \|\mathbb{E}[\![M_i^2]\!] - M^2\| \le \|\mathbb{E}[\![M_i^2]\!]\| + \|M^2\|$$

Since M is PSD, $\|\mathbb{E}[\![M_i^2]\!]\| \le m\|M\|$. Now by matrix Bernstein's inequality, for any $\delta > 0$,

$$\mathbb{P}\left[\!\!\left[\frac{1}{n}\|\sum_{i=1}^{n} Z_i\| \ge \delta\|M\|\right]\!\!\right] \le 2d\exp(-\frac{\delta^2 n^2\|M\|^2/2}{mn\|M\| + 2mn\delta\|M\|/3}) = 2d\exp(-\frac{\delta^2 n\|M\|/2}{m + 2m\delta/3}) \tag{53}$$

Setting

$$n \ge (P+1)(\frac{4}{3\delta} + \frac{2}{\delta^2})m\|M\|^{-1}\log d, \tag{54}$$

we have w.p. at least $1 - 2d^{-P}$,

$$\|\frac{1}{n}\sum M_i - M\| \le \delta\|M\| \tag{55}$$

Step 3. Now we bound $\|M - B\|$. For simplicity, we replace $\boldsymbol{x}_i$ by $\boldsymbol{x}$ and $\mathcal{E}_i$ by $\mathcal{E}$.

$$\begin{aligned}
&\|M - B\| \\
=&\|\mathbb{E}[\![B_i \mathbf{1}(\mathcal{E}_i^c)]\!]\| \\
=&\max_{\|\boldsymbol{a}\|=1} \mathbb{E}[\![(\boldsymbol{a}^T\boldsymbol{x})^2\Pi_{k=1}^K(\boldsymbol{x}^T A_k \boldsymbol{x})\mathbf{1}(\mathcal{E}^c)]\!] \\
\overset{\zeta_1}{\le}&\max_{\|\boldsymbol{a}\|=1} \mathbb{E}[\![(\boldsymbol{a}^T\boldsymbol{x})^4\Pi_{k=1}^K(\boldsymbol{x}^T A_k \boldsymbol{x})^2]\!]^{1/2}\mathbb{E}[\![\mathbf{1}(\mathcal{E}^c)]\!]^{1/2} \\
=&\max_{\|\boldsymbol{a}\|=1} \langle \boldsymbol{a}\boldsymbol{a}^T, \mathbb{E}[\![(\boldsymbol{x}^T\boldsymbol{a}\boldsymbol{a}^T\boldsymbol{x})\Pi_{k=1}^K(\boldsymbol{x}^T A_k \boldsymbol{x})^2\boldsymbol{x}\boldsymbol{x}^T]\!]\rangle^{1/2}\mathbb{E}[\![\mathbf{1}(\mathcal{E}^c)]\!]^{1/2} \\
\overset{\zeta_2}{\le}&\max_{\|\boldsymbol{a}\|=1} \langle \boldsymbol{a}\boldsymbol{a}^T, C_{2K+1}\Pi_{k=1}^K \operatorname{tr}(A_k)^2 I\rangle^{1/2}\mathbb{E}[\![\mathbf{1}(\mathcal{E}^c)]\!]^{1/2} \\
\overset{\zeta_3}{\le}&\frac{\sqrt{(K+1)C_{2K+1}}}{\sqrt{nd^P}}\Pi_{k=1}^K \operatorname{tr}(A_k)
\end{aligned} \tag{56}$$

where ζ_1 is from Holder's inequality, ζ_2 is because of Lemma 3 and ζ_3 is because $\mathbb{E}[\![\mathbf{1}(\mathcal{E}^c)]\!] = \mathbb{P}[\![\mathcal{E}^c]\!]$. Assume $n \ge 4(K+1)C_{2K+1}/d^P$, we have $\|M - B\| \le \frac{1}{2}\|B\|$ and $\frac{3}{2}\|B\| \ge \|M\| \ge \frac{1}{2}\|B\|$. So combining this result with Eq. (52), Eq. (54), and Eq. (55), if

$$n \ge \max\{4(K+1)C_{2K+1}/d^P, c_1\frac{1}{\delta^2}(4P+5)^{K+2}d\log^{K+1}(n)\log d\}, \tag{57}$$

we obtain

$$\|\frac{1}{n}\sum M_i - M\| \le \frac{1}{3}\delta\|M\| \le \frac{1}{2}\delta\|B\|. \tag{58}$$

According to Lemma 7, $n \ge O(\frac{1}{\delta^2}\log^{K+1}(\frac{1}{\delta})(PK)^K d\log^{K+2} d)$ will imply Eq. (57). By further setting $\delta > \frac{\sqrt{4(K+1)C_{2K+1}}}{\sqrt{nd^P}}$, we have $\|M - B\| \le \frac{1}{2}\delta\|B\|$, completing the proof.

□

A.4.4 Proof of Lemma 5

Proof.

$$\begin{aligned}
&\|\mathbb{E}[\![(\boldsymbol{\beta}^T\boldsymbol{x})(\boldsymbol{\gamma}^T\boldsymbol{x})\Pi_{k=1}^K(\boldsymbol{x}^T A_k\boldsymbol{x})\boldsymbol{x}\boldsymbol{x}^T]\!]\| \\
&\ge \mathbb{E}[\![(\boldsymbol{\beta}^T\boldsymbol{x})^2(\boldsymbol{\gamma}^T\boldsymbol{x})^2\Pi_{k=1}^K(\boldsymbol{x}^T A_k\boldsymbol{x})]\!]/(\|\boldsymbol{\beta}\|\|\boldsymbol{\gamma}\|) \\
&\ge \|\boldsymbol{\beta}\|\|\boldsymbol{\gamma}\|\Pi_{k=1}^K \operatorname{tr}(A_k).
\end{aligned} \tag{59}$$

$$
\begin{aligned}
&\|\mathbb{E}[\![(\boldsymbol{\beta}^T\boldsymbol{x})(\boldsymbol{\gamma}^T\boldsymbol{x})\Pi_{k=1}^K(\boldsymbol{x}^TA_k\boldsymbol{x})\boldsymbol{x}\boldsymbol{x}^T]\!]\| \\
&=\max_{\boldsymbol{a},\boldsymbol{b}}\mathbb{E}[\![(\boldsymbol{\beta}^T\boldsymbol{x})(\boldsymbol{\gamma}^T\boldsymbol{x})(\boldsymbol{a}^T\boldsymbol{x})(\boldsymbol{b}^T\boldsymbol{x})\Pi_{k=1}^K(\boldsymbol{x}^TA_k\boldsymbol{x})]\!]/(\|\boldsymbol{a}\|\|\boldsymbol{b}\|) \\
&\le \mathbb{E}[\![(\boldsymbol{a}^T\boldsymbol{x})^2(\boldsymbol{b}^T\boldsymbol{x})^2]\!]^{1/2}\mathbb{E}[\![(\boldsymbol{\beta}^T\boldsymbol{x})^2(\boldsymbol{\gamma}^T\boldsymbol{x})^2\Pi_{k=1}^K(\boldsymbol{x}^TA_k\boldsymbol{x})^2]\!]^{1/2}/(\|\boldsymbol{a}\|\|\boldsymbol{b}\|) \\
&\le \sqrt{3C_{2K+1}}\|\boldsymbol{\beta}\|\|\boldsymbol{\gamma}\|\Pi_{k=1}^K\operatorname{tr}(A_k)
\end{aligned} \tag{60}
$$

□

A.4.5 Proof of Lemma 6

Proof. Note that the matrix B_i is probably not PSD. Thus we can't apply Lemma 4 directly. But the proof is similar to that for Lemma 4.

Define

$$
m := (4P+5)^{K+2}\|\boldsymbol{\beta}\|\|\boldsymbol{\gamma}\|(\Pi_{k=1}^K\operatorname{tr}(A_k))d\log^{K+1}(n), \tag{61}
$$

and the event, $\mathcal{E}_i := \{\|B_i\| \le m\}$. Then by Corollary 3,

$$
\mathbb{P}[\![\mathcal{E}_i]\!] \ge 1 - 2Kn^{-1}d^{-P}.
$$

Define a new random matrix $M_i := \mathbf{1}(\mathcal{E}_i)B_i$, its expectation $M := \mathbb{E}[\![M_i]\!]$ and its empirical average $\hat{M} = \frac{1}{n}\sum_{i=1}^n M_i$.

Step 1. By union bound, we have w.p. $1-2Kd^{-P}$, $M_i = B_i$ for all i, i.e., $\hat{M} = \hat{B}$.

Step 2. We now bound $\|M-B\|$, For simplicity, we replace $\boldsymbol{x}_i$ by $\boldsymbol{x}$ and $\mathcal{E}_i$ by $\mathcal{E}$.

$$
\begin{aligned}
&\|M-B\| \\
=&\|\mathbb{E}[\![B_i\mathbf{1}(\mathcal{E}_i^c)]\!]\| \\
=&\max_{\|\boldsymbol{a}\|=\|\boldsymbol{b}\|=1}\mathbb{E}[\![(\boldsymbol{a}^T\boldsymbol{x})(\boldsymbol{b}^T\boldsymbol{x})(\boldsymbol{\beta}^T\boldsymbol{x})(\boldsymbol{\gamma}^T\boldsymbol{x})\Pi_{k=1}^K(\boldsymbol{x}^TA_k\boldsymbol{x})\mathbf{1}(\mathcal{E}^c)]\!] \\
\overset{\zeta_1}{\le}&\max_{\|\boldsymbol{a}\|=\|\boldsymbol{b}\|=1}\mathbb{E}[\![(\boldsymbol{a}^T\boldsymbol{x})^2(\boldsymbol{b}^T\boldsymbol{x})^2(\boldsymbol{\beta}^T\boldsymbol{x})^2(\boldsymbol{\gamma}^T\boldsymbol{x})^2\Pi_{k=1}^K(\boldsymbol{x}^TA_k\boldsymbol{x})^2]\!]^{1/2}\mathbb{E}[\![\mathbf{1}(\mathcal{E}^c)]\!]^{1/2} \\
=&\max_{\|\boldsymbol{a}\|=\|\boldsymbol{b}\|=1}\langle\boldsymbol{a}\boldsymbol{a}^T,\mathbb{E}[\![(\boldsymbol{b}^T\boldsymbol{x})^2(\boldsymbol{\beta}^T\boldsymbol{x})^2(\boldsymbol{\gamma}^T\boldsymbol{x})^2\Pi_{k=1}^K(\boldsymbol{x}^TA_k\boldsymbol{x})^2\boldsymbol{x}\boldsymbol{x}^T]\!]\rangle^{1/2}\mathbb{E}[\![\mathbf{1}(\mathcal{E}^c)]\!]^{1/2} \\
\overset{\zeta_2}{\le}&\max_{\|\boldsymbol{a}\|=\|\boldsymbol{b}\|=1}\langle\boldsymbol{a}\boldsymbol{a}^T,C_{2K+3}\|\boldsymbol{\beta}\|^2\|\boldsymbol{\gamma}\|^2\Pi_{k=1}^K\operatorname{tr}(A_k)^2I\rangle^{1/2}\mathbb{E}[\![\mathbf{1}(\mathcal{E}^c)]\!]^{1/2} \\
\overset{\zeta_3}{\le}&\frac{\sqrt{2KC_{2K+3}}}{\sqrt{nd^P}}\|\boldsymbol{\beta}\|\|\boldsymbol{\gamma}\|\Pi_{k=1}^K\operatorname{tr}(A_k)
\end{aligned} \tag{62}
$$

where ζ_1 is from Holder's inequality, ζ_2 is because of Lemma 3 and ζ_3 is because $\mathbb{E}[\![\mathbf{1}(\mathcal{E}^c)]\!] = \mathbb{P}[\![\mathcal{E}^c]\!]$.

According to Eq. (62) and Lemma 5, if $\frac{\sqrt{2KC_{2K+3}}}{\sqrt{nd^P}} \le \delta/2$, then

$$
\|M-B\| \le \frac{1}{2}\delta\|\boldsymbol{\beta}\|\|\boldsymbol{\gamma}\|\Pi_{k=1}^K\operatorname{tr}(A_k) \le \frac{1}{2}\delta\|B\| \tag{63}
$$

Since $\delta \le 1$, we also have $\|M-B\| \le \frac{1}{2}\|B\|$, so by Lemma 5,

$$
\frac{3}{2}\|B\| \ge \|M\| \ge \frac{1}{2}\|B\| \ge \frac{1}{2}\|\boldsymbol{\beta}\|\|\boldsymbol{\gamma}\|\Pi_{k=1}^K\operatorname{tr}(A_k) \tag{64}
$$

Step 3. Now we bound $\|M - \hat{M}\|$. $\|M\| \le m$ automatically holds. Since M is probably not PSD, we don't have $\|\mathbb{E}[\![M_i^2]\!]\| \le m\|M\|$. However, we can still show that $\mathbb{E}[\![M_i^2]\!] \le O(m)\|M\|$.

$$\begin{aligned}
&\|\mathbb{E}[\![M_i^2]\!]\| \\
&\le \|\mathbb{E}[\![B_i^2]\!]\| \\
&= \|\mathbb{E}[\![(\boldsymbol{\beta}^T\boldsymbol{x})^2(\boldsymbol{\gamma}^T\boldsymbol{x})^2\Pi_{k=1}^K(\boldsymbol{x}^T A_k\boldsymbol{x})^2\|\boldsymbol{x}\|^2\boldsymbol{x}\boldsymbol{x}^T]\!]\| \\
&\le C_{2K+3} d \times (\|\boldsymbol{\beta}\|\|\boldsymbol{\gamma}\|\Pi_{k=1}^K \operatorname{tr}(A_k))^2 \\
&\le \frac{2C_{2K+3}}{(4P+5)^{K+2}} m\|M\|
\end{aligned} \tag{65}$$

We can use matrix Bernstein inequality now. Let $Z_i := M_i - M$. $\|Z_i\| \le 2m$. $\|\mathbb{E}[\![Z_i^2]\!]\| \le (\frac{2C_{2K+3}}{(4P+5)^{K+2}} + 1)m\|M\|$. Define $\hat{C}_K := \frac{2C_{2K+3}}{(4P+5)^{K+2}} + 1$, then

$$\mathbb{P}\left[\!\!\left[\frac{1}{n}\|\sum_{i=1}^n Z_i\| \ge \delta\|M\|\right]\!\!\right] \le 2d\exp(-\frac{\delta^2 n^2\|M\|^2/2}{\hat{C}_K mn\|M\| + 2mn\delta\|M\|/3}) \le 2d\exp(-\frac{\delta^2 n\|M\|/2}{\hat{C}_K m + 2m\delta/3}) \tag{66}$$

Thus, when $n \ge (P+1)(\frac{\hat{C}_K}{\delta^2} + \frac{2}{3\delta})m/\|M\|\log d$, we have w.p., $1 - c_2 d^{-P}$,

$$\|\hat{M} - M\| \le \frac{1}{3}\delta\|M\| \le \frac{1}{2}\delta\|B\|.$$

By Eq. (61) and Eq. (64),

$$\begin{aligned}
(P+1)(\frac{\hat{C}_K}{\delta^2} + \frac{2}{3\delta})m/\|M\|\log d &\le c_1\frac{\hat{C}_K}{\delta^2} \times (4P+5)^{K+3} d\log^{K+1}(n)\log(d) \\
&\le \frac{c_1}{\delta^2}(2C_{2K+3}(P+1) + (4P+5)^{K+2})d\log^{K+1}(n)\log(d)
\end{aligned}$$

Applying the fact, $\|\hat{B} - B\| \le \|\hat{B} - \hat{M}\| + \|\hat{M} - M\| + \|M - B\|$, and Lemma 7 completes the proof. $\square$

A.5 Proof of Lemma 7

Proof. Assume we require $n \geq cd\log(d)\log^{K+1}(n)$ and we have $n \geq bcd\log(d)\log^{A}(d)$, where b, A depends only on K.

$$
\begin{aligned}
& n \geq cd\log d\log^{K+1}(n) \\
& \Uparrow \\
& \frac{n}{\log^{K+1}(n)} \geq cd\log d \\
& \Uparrow \\
& \frac{bcd\log(d)\log^{A}(d)}{\log^{K+1}(bcd\log(d)\log^{A}(d))} \geq cd\log d \\
& \Uparrow \\
& b\log^{A}(d) \geq \log^{K+1}(bcd\log(d)\log^{A}(d)) \\
& \Uparrow \\
& \log b + A\log\log(d) \geq (K+1)\log(\log(b)+\log(c)+\log(d)+(A+1)\log\log(d)) \\
& \Uparrow \\
& \log b + A\log\log(d) \geq (K+1)\log(4\max\{\log(b),\log(c),\log(d),(A+1)\log\log(d)\}) \\
& \Uparrow \\
& \begin{cases} \log b \geq (K+1)\log(4\log(b)) \\ \log b \geq (K+1)\log(4\log(c)) \\ \log b + A\log\log(d) \geq (K+1)\log(4\log(d)) \\ \log b + A\log\log(d) \geq (K+1)\log(4(A+1)\log\log(d)) \end{cases} \\
& \Uparrow \\
& \begin{cases} b \geq K^{4K} \\ b \geq 4^{K+1}\log^{K+1}(c) \\ A \geq K+1 \\ b \geq (4(A+1))^{K+1} \end{cases} \\
& \Uparrow \\
& \begin{cases} b = K^{4K}\log^{K+1}(c) \\ A = K+1 \end{cases}
\end{aligned}
\tag{67}
$$

□

B Proofs of Tensor Method for Initialization

B.1 Some Lemmata

We will use the following lemma to guarantee the robust tensor power method. The proofs of these lemmata will be found in Sec. B.4.

Lemma 8 (Some properties of thrid-order tensor). *If $T \in \mathbb{R}^{d\times d\times d}$ is a supersymmetric tensor, i.e.,T_{ijk} is equivalent for any permutation of the index, then the operator norm defined as*

$$\|T\|_{op} := \sup_{\|\boldsymbol{a}\|=1} |T(\boldsymbol{a},\boldsymbol{a},\boldsymbol{a})|$$

Property 1. $\|T\|_{op} = \sup_{\|\boldsymbol{a}\|=\|\boldsymbol{b}\|=\|\boldsymbol{c}\|=1} |T(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c})|$
Property 2. $\|T\|_{op} \leq \|T_{(1)}\| \leq \sqrt{K}\|T\|_{op}$
Property 3. *If T is a rank-one tensor, then $\|T_{(1)}\| = \|T\|_{op}$*
Property 4. *For any matrix $W \in \mathbb{R}^{d\times d'}$, $\|T(W,W,W)\|_{op} \leq \|T\|_{op}\|W\|^3$*

Lemma 9 (Approximation error for the second moment)**.** *Let $\{\boldsymbol{x}_i, y_i\}_{i\in[n]}$ be generated from the mixed linear regression model* (3)*. Define $M_2 := \sum_{k=[K]} 2p_k \boldsymbol{w}_k^* \otimes \boldsymbol{w}_k^*$ and $\hat{M}_2 := \frac{1}{n}\sum_{i\in[n]} y_i^2(\boldsymbol{x}_i \otimes$*

$\boldsymbol{x}_i - I$). Then with $n \geq c_1 \frac{1}{p_{min}\delta_2^2} d \log^2(d)$, we have w.p. $1 - c_2 K d^{-2}$,

$$\|\hat{M}_2 - M_2\| \leq \delta_2 \sum_k p_k \|\boldsymbol{w}_k^*\|^2 \tag{68}$$

where c_1, c_2 are universal constants.

And for any fixed orthogonal matrix $Y \in \mathbb{R}^{d \times K}$, with the same condition, we have

$$\|Y^T(\hat{M}_2 - M_2)Y\| \leq \delta_2 \sum_k p_k \|\boldsymbol{w}_k^*\|^2 \tag{69}$$

Lemma 10 (Subspace Estimation)**.** *Let M_2, M_3 be*

$$M_2 = \sum_{k=[K]} 2p_k \boldsymbol{w}_k^* \otimes \boldsymbol{w}_k^*, \ \text{ and } \ M_3 = \sum_{k=[K]} 6p_k \boldsymbol{w}_k^* \otimes \boldsymbol{w}_k^* \otimes \boldsymbol{w}_k^*, \tag{70}$$

and $\hat{M}_2$ be an estimate of M_2. Assume $\|\hat{M}_2 - M_2\| \leq \delta \sigma_K(M_2)$ and $\delta \leq \frac{1}{6}$. Let Y be the returned matrix of the power method after $O(\log(1/\delta))$ steps. Define $R_2 = Y^T M_2 Y$ and $R_3 = M_3(Y, Y, Y)$. Then $\|R_2\| \leq \|M_2\|$ and $\|R_3\|_{op} \leq \|M_3\|_{op}$. We also have

$$\|YY^T \boldsymbol{w}_k^* - \boldsymbol{w}_k^*\| \leq 3\delta \|\boldsymbol{w}_k^*\|, \forall k \tag{71}$$

and

$$\sigma_K(R_2) \geq \frac{3}{4} \sigma_K(M_2)$$

Lemma 11 (Approximation error for the third moment)**.** *Let $\{\boldsymbol{x}_i, y_i\}_{i \in [n]}$ be drawn from the mixed linear regression model* (3)*. Let $Y \in \mathbb{R}^{d \times K}$ be any fixed orthogonal matrix that satisfies, $\|YY^T \boldsymbol{w}_k^* - \boldsymbol{w}_k^*\| \leq \frac{1}{2}\|\boldsymbol{w}_k^*\|, \forall k$, and $\boldsymbol{r}_i = Y^T \boldsymbol{x}_i$, for all $i \in [n]$. Let*

$$\hat{R}_3 = \frac{1}{n} \sum_{i \in [n]} y_i^3 (\boldsymbol{r}_i \otimes \boldsymbol{r}_i \otimes \boldsymbol{r}_i - \sum_{j \in [K]} \boldsymbol{e}_j \otimes \boldsymbol{r}_i \otimes \boldsymbol{e}_j - \sum_{j \in [K]} \boldsymbol{e}_j \otimes \boldsymbol{e}_j \otimes \boldsymbol{r}_i - \sum_{j \in [K]} \boldsymbol{r}_i \otimes \boldsymbol{e}_j \otimes \boldsymbol{e}_j)$$

and

$$R_3 = \sum_{k=[K]} 6p_k (Y^T \boldsymbol{w}_k^*) \otimes (Y^T \boldsymbol{w}_k^*) \otimes (Y^T \boldsymbol{w}_k^*)$$

Then if $n \geq c_3 \frac{1}{p_{min}\delta_3^2} K^3 \log^4(d)$ and $3\sqrt{C_5} n^{-1/2} d^{-1} \leq \frac{\delta_3}{4}$, we have w.p. $1 - c_4 K d^{-2}$

$$\|\hat{R}_3 - R_3\|_{op} \leq \delta_3 \sum_{k \in [K]} p_k \|\boldsymbol{w}_k^*\|^3,$$

where c_3 and c_4 are universal constant.

Lemma 12 (Robust Tensor Power Method. Similar to Lemma 4 in [7])**.** *Let $R_2 = \sum_{k=1}^K p_k \boldsymbol{u}_k \otimes \boldsymbol{u}_k$ and $R_3 = \sum_{k=1}^K p_k \boldsymbol{u}_k \otimes \boldsymbol{u}_k \otimes \boldsymbol{u}_k$, where $\boldsymbol{u}_k \in \mathbb{R}^K$ can be any fixed vector. Define $\sigma_K := \sigma_K(R_2)$. Assume the estimations of R_2 and R_3, $\hat{R}_2$ and $\hat{R}_3$ respectively, satisfy $\|R_2 - \hat{R}_2\|_{op} \leq \epsilon_2$ and $\|R_3 - \hat{R}_3\|_{op} \leq \epsilon_3$ with*

$$\epsilon_2 \leq \sigma_K/3, \ 8\|R_3\|_{op} \sigma_K^{-5/2} \epsilon_2 + 2\sqrt{2} \sigma_K^{-3/2} \epsilon_3 \leq c_T \frac{1}{K\sqrt{p_{max}}}, \tag{72}$$

for some constant c_T. Let the whitening matrix $\hat{W} = \hat{U}_2 \hat{\Lambda}_2^{-1/2} \hat{U}_2^T$, where $\hat{R}_2 = \hat{U}_2 \hat{\Lambda}_2 \hat{U}_2^T$ is the eigendecomposition of $\hat{R}_2$. Then w.p. $1 - \eta$, the eigenvalues $\{\hat{a}_k\}_{k=1}^K$ and the eigenvectors $\{\hat{\boldsymbol{v}}_k\}_{k=1}^K$ computed from the whitened tensor $\hat{R}_3(\hat{W}, \hat{W}, \hat{W}) \in \mathbb{R}^{K \times K \times K}$ by using the robust tensor power method [2] will satisfy

$$\|(\hat{W}^T)^\dagger (\hat{a}_k \hat{\boldsymbol{v}}_k) - \boldsymbol{u}_k\| \leq \kappa_2 \epsilon_2 + \kappa_3 \epsilon_3$$

where $\kappa_2 = 3\|R_2\|^{1/2} \sigma_K^{-1} + 200\|R_2\|^{1/2} \|R_3\|_{op} \sigma_K^{-5/2}$, $\kappa_3 = 75\|R_2\|^{1/2} \sigma_K^{-3/2}$ and η is related to the computational time by $O(\log(1/\eta))$.

Remark: This lemma differs from Lemma 4 of [7] in the requirement on ϵ_2, ϵ_3. Lemma 4 in [7] treats ϵ_2, ϵ_3 in the same order (that are bounded by the same value), however, they should have different order because one is for second-order moments and the other is for third-order moments.

B.2 Proof of Theorem 3

Proof Details. We state the proof outline here,

1. $\|\hat{M}_2 - M_2\| \le \epsilon_{M_2}$ by Matrix Bernstein's inequality.
2. $\|YY^T\boldsymbol{w}_k^* - \boldsymbol{w}_k^*\| \le \epsilon_Y \|\boldsymbol{w}_k^*\|$ for all $k \in [K]$ by Davis-Kahan's theorem [10].
3. $\|\hat{R}_2 - R_2\| \le \epsilon_2$ by Matrix Bernstein's inequality.
4. $\|\hat{R}_3 - R_3\|_{op} \le \epsilon_3$ by Matrix Bernstein's inequality after matricizing tensor.
5. Let $\hat{\boldsymbol{u}}_k = (\hat{W}^T)^\dagger(\hat{a}_k\hat{\boldsymbol{v}}_k)$. Then $\|\hat{\boldsymbol{u}}_k - Y^T\boldsymbol{w}_k^*\| \le \epsilon_{\boldsymbol{u}}$ by the robust tensor power method.
6. Finally, $\|\boldsymbol{w}_k^{(0)} - \boldsymbol{w}_k^*\| \le c_6\Delta_{min}$ by combining the results of Step 2 and Step 5.

The lemmata in Appendix B.1 provide the bound for the above steps: Lemma 9 for Step 1, Lemma 10 for Step 2 and Step 3, Lemma 11 for Step 4, and Lemma 12 for Step 5. Now we show the details. Define

$$\bar{\kappa}_2 := 4\|M_2\|^{1/2}\sigma_K^{-1}(M_2) + 412\|M_2\|^{1/2}\|M_3\|_{op}\sigma_K^{-5/2}(M_2)$$

and

$$\bar{\kappa}_3 := 116\|M_2\|^{1/2}\sigma_K^{-3/2}(M_2).$$

By Lemma 10, we have $\bar{\kappa}_3 \ge \kappa_3$ and $\bar{\kappa}_2 \ge \kappa_2$ for any orthogonal matrix Y.

$$\begin{aligned}
\|\boldsymbol{w}_k^{(0)} - \boldsymbol{w}_k^*\| &\overset{\xi_1}{\le} \|Y\hat{\boldsymbol{u}}_k - YY^T\boldsymbol{w}_k^*\| + \|YY^T\boldsymbol{w}_k^* - \boldsymbol{w}_k^*\| \\
&\overset{\xi_2}{\le} \bar{\kappa}_2\|\hat{R}_2 - R_2\| + \bar{\kappa}_3\|\hat{R}_3 - R_3\|_{op} + \frac{2}{3}\delta_{M_2}\|\boldsymbol{w}_k^*\|\sigma_K^{-1}(M_2)\sum_k p_k\|\boldsymbol{w}_k^*\|^2 \\
&\overset{\xi_3}{\le} \bar{\kappa}_2\delta_2\sum_k p_k\|\boldsymbol{w}_k^*\|^2 + \bar{\kappa}_3\delta_3\sum_k p_k\|\boldsymbol{w}_k^*\|^3 + \frac{2}{3}\delta_{M_2}\sigma_K^{-1}(M_2)(\max_k\|\boldsymbol{w}_k^*\|)\sum_k p_k\|\boldsymbol{w}_k^*\|^2
\end{aligned} \tag{73}$$

where ξ_1 is due to triangle inequality, ξ_2 is due to Lemma 12, Lemma 10 and Lemma 9, and ξ_3 is due to Lemma 9 and Lemma 11. Therefore, we can set

$$\delta_2 \le \frac{c_6\Delta_{min}}{3\bar{\kappa}_2\sum_k p_k\|\boldsymbol{w}_k^*\|^2},$$

$$\delta_3 \le \frac{c_6\Delta_{min}}{3\bar{\kappa}_3\sum_{k\in[K]} p_k\|\boldsymbol{w}_k^*\|^3}$$

and

$$\delta_{M_2} \le \frac{c_6\Delta_{min}}{2\sigma_K^{-1}(M_2)(\max_k\|\boldsymbol{w}_k^*\|)\sum_k p_k\|\boldsymbol{w}_k^*\|^2},$$

such that $\|\boldsymbol{w}_k^{(0)} - \boldsymbol{w}_k^*\| \le c_6\Delta_{min}$. Note that Lemma 12 also requires Eq. (72), which can be satisfied if

$$\|\hat{R}_2 - R_2\| \le \min\{\frac{\sigma_K(M_2)}{4}, \frac{c_T\sigma_K(M_2)^{5/2}}{34\|M_3\|_{op}K\sqrt{p_{max}}}\}$$

and

$$\|\hat{R}_3 - R_3\|_{op} \le \frac{c_T\sigma_K(M_2)^{3/2}}{6K\sqrt{p_{max}}}.$$

Therefore, we require

$$\delta_2 \le \delta_2^* := \frac{1}{\sum_k p_k\|\boldsymbol{w}_k^*\|^2}\min\{\frac{\sigma_K(M_2)}{4}, \frac{c_T\sigma_K(M_2)^{5/2}}{34\|M_3\|_{op}K\sqrt{p_{max}}}, \frac{c_6\Delta_{min}}{3\bar{\kappa}_2}\}$$

$$\delta_3 \le \delta_3^* := \frac{1}{\sum_{k\in[K]} p_k\|\boldsymbol{w}_k^*\|^3}\min\{\frac{c_6\Delta_{min}}{3\bar{\kappa}_3}, \frac{c_T\sigma_K(M_2)^{3/2}}{6K\sqrt{p_{max}}}\}$$

$$\delta_{M_2} \le \delta_{M_2}^* := \frac{c_6\Delta_{min}}{2\sigma_K^{-1}(M_2)(\max_k\|\boldsymbol{w}_k^*\|)\sum_k p_k\|\boldsymbol{w}_k^*\|^2},$$

Now we analyze the sample complexity. $\delta^*_{M_2}, \delta^*_2, \delta^*_3$ correspond to the sample sets, Ω_{M_2}, Ω_2 and Ω_3 respectively. By Lemma 9, Lemma 11, we require

$$|\Omega_{M_2}| \geq c_{M_2} \frac{1}{p_{min}\delta^{*2}_{M_2}} d \log^2(d)$$

$$|\Omega_2| \geq c_2 \frac{1}{p_{min}\delta^{*2}_2} d \log^2(d)$$

$$|\Omega_3| \geq c_3 \frac{1}{p_{min}\delta^{*2}_3} K^3 \log^{11/2}(d),$$

and $3\sqrt{C_5} n^{-1/2} d^{-1} \leq \frac{\delta_3}{4}$. For the probability, we can set $\eta = d^{-2}$ in Lemma 12 by scarifying a little more computational time, which is in the order of $O(\log(d))$. Therefore, the final probability is at least $1 - O(Kd^{-2})$.

B.3 Proof of Theorem 4

According to Theorem 2, after $T_0 = O(\log d)$ iterations, we arrive the local convexity region in Corollary 1. Then we just need one more set of samples, but still need $O(\log(1/\epsilon))$ iterations to achieve $1/\epsilon$ precision. By Theorem 1, Corollary 1, Theorem 2 and Theorem 3, we can partition the dataset into $|\Omega^{(t)}| = O(d(K \log(d))^{2K+2})$ for all $t = 0, 1, 2, \cdots, T_0 + 1$ to satisfy their sample complexity requirement. This complete the proof.

B.4 Proofs of Some Lemmata

B.4.1 Proof of Lemma 8

Proof. **Property 1.** See the proof in Lemma 21 of [19].

Property 2.

$$\|T_{(1)}\| = \max_{\|\boldsymbol{a}\|=1} \|T(\boldsymbol{a}, I, I)\|_F \leq \max_{\|\boldsymbol{a}\|=1} \sqrt{K}\|T(\boldsymbol{a}, I, I)\| = \max_{\|\boldsymbol{a}\|=\|\boldsymbol{b}\|=1} \sqrt{K}|T(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{b})| = \|T\|_{op}.$$

Obviously, $\max_{\|\boldsymbol{a}\|=1} \|T(\boldsymbol{a}, I, I)\|_F \geq \|T\|_{op}$.
Property 3. Let $T = \boldsymbol{v} \otimes \boldsymbol{v} \otimes \boldsymbol{v}$.

$$\|T_{(1)}\| = \max_{\|\boldsymbol{a}\|=1} \|T(\boldsymbol{a}, I, I)\|_F = \max_{\|\boldsymbol{a}\|=1} \|\boldsymbol{v}\|^2 (\boldsymbol{v}^T \boldsymbol{a})^2 = \|\boldsymbol{v}\|^3 = \max_{\|\boldsymbol{a}\|=1} |(\boldsymbol{v}^T \boldsymbol{a})^3| = \|T\|_{op}.$$

Property 4. There exists a $\boldsymbol{u} \in \mathbb{R}^{d'}$ with $\|\boldsymbol{u}\| = 1$ such that

$$\|T(W, W, W)\|_{op} = |T(W\boldsymbol{u}, W\boldsymbol{u}, W\boldsymbol{u})| \leq \|T\|_{op} \|W\boldsymbol{u}\|^3 \leq \|T\|_{op} \|W\|^3$$

□

B.4.2 Proof of Lemma 9

Proof. Define $M_2^{(k)} := 2\boldsymbol{w}^*_k \boldsymbol{w}^{*T}_k$ and $\hat{M}_2^{(k)} = \frac{1}{|S_k|} \sum_{i \in S_k} y_i^2 (\boldsymbol{x}_i \otimes \boldsymbol{x}_i - I)$, where $S_k \subset [n]$ is the index set for samples from the k-th model. Since we assume $|S_k| = p_k n$, $\hat{M}_2 = \sum_{k \in [K]} p_k M_2^{(k)}$. We first bound $\|\hat{M}_2^{(k)} - M_2^{(k)}\|$. By Lemma 4 with $K = 1$, $A_1 = \boldsymbol{w}^*_k \boldsymbol{w}^{*T}_k$, then if $|S_k| \geq c_1 \frac{1}{\delta^2} d \log^2(d)$, we have w.p., $1 - c_2 d^{-2}$,

$$\|\frac{1}{|S_k|} \sum_{i \in S_k} y_i^2 \boldsymbol{x}_i \boldsymbol{x}_i^T - \|\boldsymbol{w}^*_k\|^2 I - 2\boldsymbol{w}^*_k \boldsymbol{w}^*_k\| \leq \delta \|\boldsymbol{w}^*_k\|^2.$$

By Lemma 4 with $K = 0$, we have w.p. at least $1 - d^{-2}$,

$$\|\frac{1}{|S_k|} \sum_{i \in S_k} \boldsymbol{x}_i \boldsymbol{x}_i^T - I\| \leq \delta$$

Then

$$\|\frac{1}{|S_k|}\sum_{i\in S_k}(\boldsymbol{x}_i^T\boldsymbol{w}_k^*)^2-\|\boldsymbol{w}_k^*\|^2\|\le\|\frac{1}{|S_k|}\sum_{i\in S_k}\boldsymbol{x}_i\boldsymbol{x}_i^T-I\|\|\boldsymbol{w}_k^*\|^2\le\delta\|\boldsymbol{w}_k^*\|^2.$$

Thus

$$\|\frac{1}{|S_k|}\sum_{i\in S_k}y_i^2(\boldsymbol{x}_i\boldsymbol{x}_i^T-I)-2\boldsymbol{w}_k^*\boldsymbol{w}_k^*\|\le 2\delta\|\boldsymbol{w}_k^*\|^2.$$

And w.p. $1-O(Kd^{-2})$,

$$\|\hat{M}_2-M_2\|\le 2\delta\sum_k p_k\|\boldsymbol{w}_k^*\|^2.$$

□

B.4.3 Proof of Lemma 10

Proof. $\|R_2\|\le\|Y\|^2\|M_2\|=\|M_2\|$. By Property 4 in Lemma 8, $\|R_3\|_{op}\le\|Y\|^3\|M_3\|_{op}=\|M_3\|_{op}$. Let U be the top-K eigenvectors of M_2. Then $U=\text{span}(\boldsymbol{w}_1^*,\boldsymbol{w}_2^*,\cdots,\boldsymbol{w}_K^*)$. Let $\bar{Y}\in\mathbb{R}^{d\times K}$ be the top-K eigenvectors of $\hat{M}_2$. By Lemma 9 in [19] (Davis-Kahan's theorem [10] can also prove it),

$$\|(I-\bar{Y}\bar{Y}^T)UU^T\|\le\frac{3}{2}\delta.$$

According to Theorem 7.2 in [3], after t steps of the power method, we have

$$\|\bar{Y}\bar{Y}^T-Y^{(t)}Y^{(t)T}\|\le(\frac{\sigma_{K+1}(\hat{M}_2)}{\sigma_K(\hat{M}_2)})^t\|\bar{Y}\bar{Y}^T-Y^{(0)}Y^{(0)T}\|.$$

When $\delta\le 1/3$, by Weyl's inequality, we have $\sigma_{K+1}(\hat{M}_2)\le\frac{1}{3}\sigma_K(M_2)$ and $\sigma_K(\hat{M}_2)\ge\frac{2}{3}\sigma_K(M_2)$. Therefore, after $t=\log(2/(3\delta))$ steps of the power method, we have

$$\|\bar{Y}\bar{Y}^T-Y^{(t)}Y^{(t)T}\|\le\frac{3}{2}\delta$$

Let $Y=Y^{(t)}$. We have

$$\|YY^T-UU^T\|\le\|YY^T-\bar{Y}\bar{Y}^T\|+\|UU^T-\bar{Y}\bar{Y}^T\|\le 3\delta$$

and

$$\|YY^T\boldsymbol{w}_k^*-\boldsymbol{w}_k^*\|\le\|YY^T-UU^T\|\|\boldsymbol{w}_k^*\|\le 3\delta\|\boldsymbol{w}_k^*\|$$

Now we consider $\sigma_K(R_2)$. The proof is similar to that for Property 3 in Lemma 9 in [19].

$$\sigma_K(R_2)\ge\sigma_K(M_2)\sigma_K^2(Y^TU)$$

Note that $\|Y_\perp^TU\|=\|YY^T-UU^T\|$, where $Y_\perp$ is the subspace orthogonal to Y. For any normalized vector $\boldsymbol{v}$,

$$\|Y^TU\boldsymbol{v}\|^2=\|U\boldsymbol{v}\|^2-\|Y_\perp^TU\boldsymbol{v}\|^2\ge 1-(3\delta)^2\ge\frac{3}{4}$$

Therefore, we have $\sigma_K(R_2)\ge\frac{3}{4}\sigma_K(M_2)$. □

B.4.4 Proof of Lemma 11

Proof. We prove it by matricizing the tensor. Define

$$G_i=y_i^3(\boldsymbol{r}_i\otimes\boldsymbol{r}_i\otimes\boldsymbol{r}_i-\sum_{j\in[K]}\boldsymbol{e}_j\otimes\boldsymbol{r}_i\otimes\boldsymbol{e}_j-\sum_{j\in[K]}\boldsymbol{e}_j\otimes\boldsymbol{e}_j\otimes\boldsymbol{r}_i-\sum_{j\in[K]}\boldsymbol{r}_i\otimes\boldsymbol{e}_j\otimes\boldsymbol{e}_j).$$

Like in Lemma 9, we first bound $\|\hat{R}_3^{(k)}-R_3^{(k)}\|_{op}$, where $\hat{R}_3^{(k)}=\frac{1}{|S_k|}\sum_{i\in S_k}G_i$, and $R_3^{(k)}=6(Y^T\boldsymbol{w}_k^*)\otimes(Y^T\boldsymbol{w}_k^*)\otimes(Y^T\boldsymbol{w}_k^*)$.

$$\|R_3^{(k)}\|_{op}=6\|Y^T\boldsymbol{w}_k^*\|^3.$$

By Lemma 10, $\frac{1}{2}\|\boldsymbol{w}_k^*\| \le \|Y^T\boldsymbol{w}_k^*\| \le \frac{3}{2}\|\boldsymbol{w}_k^*\|$. Thus

$$\frac{3}{4}\|\boldsymbol{w}_k^*\|^3 \le \|R_3^{(k)}\|_{op} \le \frac{81}{4}\|\boldsymbol{w}_k^*\|^3. \tag{74}$$

Then

$$\|G_i\|_{op} \le 4|\boldsymbol{x}_i^T\boldsymbol{w}_k^*|^3\|\boldsymbol{r}_i\|^3, \tag{75}$$

By Corollary 4, we have w.p., $1-n^{-1}d^{-2}$, $\|\boldsymbol{r}_i\|^2 \le 4K\log n$. Thus, w.p. $1-4n^{-1}d^{-2}$,

$$\|G_i\|_{op} \le 4\times 12^{3/2}\|\boldsymbol{w}_k^*\|^3\log^3(n)(4K)^{3/2}$$

Define $m := c_6\|\boldsymbol{w}_k^*\|^3K^{3/2}\log^3(n)$ for constant $c_6 = 4\times(48)^{3/2}$, and the event

$$\mathcal{E}_i := \{\|G_i\|_{op} \le m\}$$

Then $\mathbb{P}[\![\mathcal{E}_i^c]\!] \le 4n^{-1}d^{-2}$. Define a new tensor $B_i = \mathbf{1}(\mathcal{E}_i)G_i$, its expectation $B = \mathbb{E}[\![B_i]\!]$ (the expectation is over all samples from the k-th components) and its empirical average $\hat{B} = \frac{1}{|S_k|}\sum_{i\in[S_k]}B_i$.

Step 1. So we have $B_i = G_i$ for all $i \in S_k$ w.p. $1-4d^{-2}$, i.e.,

$$\hat{R}_3^{(k)} = \hat{B} \tag{76}$$

Step 2. We bound $\|B - R_3^{(k)}\|_{op}$

$$\begin{aligned}
\|B - R_3^{(k)}\|_{op} &= \|\mathbb{E}[\![\mathbf{1}(\mathcal{E}_i^c)G_i]\!]\|_{op}\\
&= \max_{\|\boldsymbol{a}\|=1}|\mathbb{E}[\![\mathbf{1}(\mathcal{E}_i^c)G_i(\boldsymbol{a},\boldsymbol{a},\boldsymbol{a})]\!]|\\
&\le \mathbb{E}[\![\mathbf{1}(\mathcal{E}_i^c)]\!]^{1/2}\max_{\|\boldsymbol{a}\|=1}|\mathbb{E}\left[\!\left[G_i(\boldsymbol{a},\boldsymbol{a},\boldsymbol{a})^2\right]\!\right]|^{1/2}\\
&\le 2n^{-1/2}d^{-1}\max_{\|\boldsymbol{a}\|=1}|\mathbb{E}\left[\!\left[(y_i^3((\boldsymbol{r}_i^T\boldsymbol{a})^3 - 3\boldsymbol{r}_i^T\boldsymbol{a}))^2\right]\!\right]|^{1/2}\\
&\le 2n^{-1/2}d^{-1}\max_{\|\boldsymbol{a}\|=1}|\mathbb{E}\left[\!\left[(\boldsymbol{w}_k^{*T}\boldsymbol{x}_i)^6((\boldsymbol{x}_i^TY\boldsymbol{a})^6 + 9(\boldsymbol{x}_i^TY\boldsymbol{a})^2)\right]\!\right]|^{1/2}\\
&\le 2n^{-1/2}d^{-1}\sqrt{2C_5}\|\boldsymbol{w}_k^*\|^3\\
&\overset{\xi}{\le} 3\sqrt{C_5}n^{-1/2}d^{-1}\|R_3^{(k)}\|_{op},
\end{aligned} \tag{77}$$

where ξ is due to Eq. (74). Therefore, if $3\sqrt{C_5}n^{-1/2}d^{-1} \le \frac{\delta_3}{4}$, we have

$$\|B - R_3^{(k)}\|_{op} \le \frac{3\delta_3}{8}\|\boldsymbol{w}_k^*\|^3 \le \frac{\delta_3}{2}\|R_3^{(k)}\|_{op} \tag{78}$$

And further if $\delta_3 \le 1$, combining Eq. (74),

$$\frac{3}{8}\|\boldsymbol{w}_k^*\|^3 \le \frac{1}{2}\|R_3^{(k)}\|_{op} \le \|B\|_{op} \le \frac{3}{2}\|R_3^{(k)}\|_{op} \le 32\|\boldsymbol{w}_k^*\|^3$$

Step 3. We bound $\|\hat{B} - B\|_{op}$. Let $Z_i = (B_i - B)_{(1)}$.

$$\begin{aligned}
\|B_{(1)}\| &\le \max_{\|\boldsymbol{a}\|=1}\|B_{(1)}\boldsymbol{a}\|\\
&= \max_{\|\boldsymbol{a}\|=1}\|B(\boldsymbol{a},I,I)\|_F\\
&\le \max_{\|\boldsymbol{a}\|=1}K\|B(\boldsymbol{a},I,I)\|\\
&\le \max_{\|\boldsymbol{a}\|=1}\max_{\|\boldsymbol{b}\|=1}\sqrt{K}|B(\boldsymbol{a},\boldsymbol{b},\boldsymbol{b})|\\
&\overset{\xi}{=} \sqrt{K}\|B\|_{op}\\
&\le 32\sqrt{K}\|\boldsymbol{w}_k^*\|^3
\end{aligned} \tag{79}$$

where ξ is due to Lemma 8.

$$\|Z_i\| \leq \|B_{i(1)}\| + \|B_{(1)}\| \leq \sqrt{K}(\|B_i\|_{op} + \|B\|_{op}) \leq 2\sqrt{K}m$$

Now consider $\|\mathbb{E}[\![Z_iZ_i^T]\!]\|$ and $\|\mathbb{E}[\![Z_i^TZ_i]\!]\|$.

$$\mathbb{E}[\![Z_iZ_i^T]\!] = \mathbb{E}[\![(B_{i(1)} - B_{(1)})(B_{i(1)} - B_{(1)})^T]\!] = \mathbb{E}\left[\!\!\left[B_{i(1)}B_{i(1)}^T\right]\!\!\right] - B_{(1)}B_{(1)}^T$$

$$\begin{aligned}
\|\mathbb{E}\left[\!\!\left[B_{i(1)}B_{i(1)}^T\right]\!\!\right]\| &\leq \|\mathbb{E}\left[\!\!\left[G_{i(1)}G_{i(1)}^T\right]\!\!\right]\| \\
&\leq \|\mathbb{E}[\![(\boldsymbol{w}_k^{*T}\boldsymbol{x})^6(\|\boldsymbol{r}\|^4\boldsymbol{r}\boldsymbol{r}^T + 2\|\boldsymbol{r}\|^2 I + (K+6)\boldsymbol{r}\boldsymbol{r}^T - 6\|\boldsymbol{r}\|^2\boldsymbol{r}\boldsymbol{r}^T)]\!]\| \\
&\leq \|\mathbb{E}[\![(\boldsymbol{w}_k^{*T}\boldsymbol{x})^6(\|Y^T\boldsymbol{x}\|^4 Y^T\boldsymbol{x}\boldsymbol{x}^T Y + 2\|Y^T\boldsymbol{x}\|^2 I + (K+6)Y^T\boldsymbol{x}\boldsymbol{x}^T Y)]\!]\| \\
&\leq 2C_5K^2\|\boldsymbol{w}_k^*\|^6,
\end{aligned} \tag{80}$$

where the last inequality is due to Lemma 3. Thus

$$\|\mathbb{E}[\![Z_iZ_i^T]\!]\| \leq 3C_5K^2\|\boldsymbol{w}_k^*\|^6$$

Similarly $\mathbb{E}[\![Z_i^TZ_i]\!] = \mathbb{E}\left[\!\!\left[B_{i(1)}^TB_{i(1)}\right]\!\!\right] - B_{(1)}^TB_{(1)}$ and $\|B_{(1)}^TB_{(1)}\| \leq \|B_{(1)}\|^2$.

$$\begin{aligned}
&\|\mathbb{E}\left[\!\!\left[B_{i(1)}^TB_{i(1)}\right]\!\!\right]\| \\
&\leq \|\mathbb{E}\left[\!\!\left[G_{i(1)}^TG_{i(1)}\right]\!\!\right]\| \\
&\leq \max_{\|A\|_F=1, A \text{ sym.}} \mathbb{E}[\![y_i^6\|\boldsymbol{r}^TA\boldsymbol{r}\boldsymbol{r} - (2A\boldsymbol{r} + \operatorname{tr}(A)\boldsymbol{r})\|^2]\!] \\
&\leq \max_{\|A\|_F=1, A \text{ sym.}} \mathbb{E}[\![(\boldsymbol{w}_k^{*T}\boldsymbol{x})^6((\boldsymbol{r}^TA\boldsymbol{r})^2\|\boldsymbol{r}\|^2 + 4\boldsymbol{r}^TA^2\boldsymbol{r} + \operatorname{tr}^2(A)\|\boldsymbol{r}\|^2 + |\operatorname{tr}(A)\boldsymbol{r}^TA\boldsymbol{r}|(4+2\|\boldsymbol{r}\|^2))]\!] \\
&\leq \max_{\|A\|_F=1, A \text{ sym.}} \mathbb{E}[\![(\boldsymbol{w}_k^{*T}\boldsymbol{x})^6(\|\boldsymbol{r}\|^6\|A\|_F^2 + 4\|\boldsymbol{r}\|^2\operatorname{tr}(A^2) + \operatorname{tr}^2(A)\|\boldsymbol{r}\|^2 + (4+2\|\boldsymbol{r}\|^2)\|\boldsymbol{r}\|^2|\operatorname{tr}(A)|\|A\|_F)]\!] \\
&\leq \mathbb{E}\left[\!\!\left[(\boldsymbol{w}_k^{*T}\boldsymbol{x})^6(\|\boldsymbol{r}\|^6 + 4\|\boldsymbol{r}\|^2 + K\|\boldsymbol{r}\|^2 + \sqrt{K}(4+2\|\boldsymbol{r}\|^2)\|\boldsymbol{r}\|^2)\right]\!\!\right] \\
&= \mathbb{E}\left[\!\!\left[(\boldsymbol{w}_k^{*T}\boldsymbol{x})^6(\|Y^T\boldsymbol{x}\|^6 + 2\sqrt{K}\|Y^T\boldsymbol{x}\|^4 + 4(\sqrt{K}+1)\|Y^T\boldsymbol{x}\|^2 + K\|Y^T\boldsymbol{x}\|^2)\right]\!\!\right] \\
&\leq 2C_5K^3\|\boldsymbol{w}_k^*\|^6
\end{aligned} \tag{81}$$

Therefore,

$$\|\mathbb{E}[\![Z_i^TZ_i]\!]\| \leq 3C_5K^3\|\boldsymbol{w}_k^*\|^6,$$

and

$$\max\{\|\mathbb{E}[\![Z_i^TZ_i]\!]\|, \|\mathbb{E}[\![Z_iZ_i^T]\!]\|\} \leq 3C_5K^3\|\boldsymbol{w}_k^*\|^6 \leq c_{m2}K^{3/2}m\|\boldsymbol{w}_k^*\|^3$$

Now we are ready to apply matrix Bernstein's inequality.

$$\mathbb{P}\left[\!\!\left[\frac{1}{|S_k|}\|\sum_{i\in S_k} Z_i\| \geq t\right]\!\!\right] \leq 2K^2\exp\left(-\frac{-|S_k|t^2/2}{c_{m2}K^{3/2}m\|\boldsymbol{w}_k^*\|^3 + 2\sqrt{K}mt/3}\right) \tag{82}$$

Setting $t = \delta_3\|\boldsymbol{w}_k^*\|^3$, we have when

$$|S_k| \geq \hat{c}_3\frac{1}{\delta_3^2}K^3\log^3(n)\log(d) \tag{83}$$

w.p. $1 - d^{-2}$,

$$\|\hat{B} - B\|_{op} \leq \|\frac{1}{|S_k|}\sum_{i\in S_k} Z_i\| \leq \delta_3\|\boldsymbol{w}_k^*\|^3, \tag{84}$$

for some universal constant $\hat{c}_3$. And there exists some constant c_3, such that $|S_k| \geq \hat{c}_3 \frac{1}{\delta_3^2} K^3 \log^4(d)$ will imply (83). **Step 4.** Combing all the K components. With above three steps for k-th component, i.e., Eq. (76), Eq. (78) and Eq. (84), w.h.p., we have

$$\|\hat{R}_3^{(k)} - R_3^{(k)}\|_{op} \leq \delta_3 \|\boldsymbol{w}_k^*\|^3$$

Now we can complete the proof by combing all the K components, w.p. $1 - O(Kd^{-2})$

$$\|\hat{R}_3 - R_3\|_{op} \leq \sum_{k \in [K]} p_k \|\hat{R}_3^{(k)} - R_3^{(k)}\|_{op} \leq \delta_3 \sum_{k \in [K]} p_k \|\boldsymbol{w}_k^*\|^3 \tag{85}$$

□

B.4.5 Proof of Lemma 12

Proof. Most part of the proof follows the proof of Lemma 4 in [7]. Let $\hat{W}^T R_2 \hat{W} = U \Lambda U^T$. Define $W := \hat{W} U \Lambda^{-1/2} U^T$, then W is the whitening matrix of R_2, i.e., $W^T R_2 W = I$. Define the whitened tensor $T = R_3(W, W, W)$, i.e.,

$$\begin{aligned} T &:= \sum_{k=1}^{K} p_k W^T \boldsymbol{u}_k \otimes W^T \boldsymbol{u}_k \otimes W^T \boldsymbol{u}_k \\ &= \sum_{k=1}^{K} p_k^{-1/2} (p_k^{1/2} W^T \boldsymbol{u}_k) \otimes (p_k^{1/2} W^T \boldsymbol{u}_k) \otimes (p_k^{1/2} W^T \boldsymbol{u}_k) \\ &= \sum_{k=1}^{K} p_k^{-1/2} \boldsymbol{v}_k \otimes \boldsymbol{v}_k \otimes \boldsymbol{v}_k, \end{aligned} \tag{86}$$

where $\{\boldsymbol{v}_k := p_k^{1/2} W^T \boldsymbol{u}_k\}_{k=1}^K$ are orthogonal basis because $\sum_{k=1}^K \boldsymbol{v}_k \boldsymbol{v}_k^T = W^T R_2 W = I_K$. In practice, we have $\hat{T} := \hat{M}_3(\hat{W}, \hat{W}, \hat{W})$, an estimation of T. Define $\epsilon_T := \|\hat{T} - T\|_{op}$. Similar to the proof of Lemma 4 in [7], we have

$$\begin{aligned} \epsilon_T =& \|R_3(W, W, W) - \hat{R}_3(\hat{W}, \hat{W}, \hat{W})\|_{op} \\ \leq& \|R_3(W, W, W) - R_3(W, W, \hat{W})\|_{op} + \|R_3(W, W, \hat{W}) - R_3(W, \hat{W}, \hat{W})\|_{op} \\ &+ \|R_3(W, \hat{W}, \hat{W}) - R_3(\hat{W}, \hat{W}, \hat{W})\|_{op} + \|R_3(\hat{W}, \hat{W}, \hat{W}) - \hat{R}_3(\hat{W}, \hat{W}, \hat{W})\|_{op} \\ =& \|R_3(W, W, W - \hat{W})\|_{op} + \|R_3(W, W - \hat{W}, \hat{W})\|_{op} \\ &+ \|R_3(W - \hat{W}, \hat{W}, \hat{W})\|_{op} + \|R_3(\hat{W}, \hat{W}, \hat{W}) - \hat{R}_3(\hat{W}, \hat{W}, \hat{W})\|_{op} \\ \leq& \|R_3\|_{op} (\|W\|^2 + \|W\| \|\hat{W}\| + \|\hat{W}\|^2) \epsilon_W + \|\hat{W}\|^3 \epsilon_3 \end{aligned} \tag{87}$$

where $\epsilon_W = \|\hat{W} - W\|$.

If $\epsilon_2 \leq \sigma_K/3$, we have $|\sigma_K(\hat{R}_2) - \sigma_K| \leq \epsilon_2 \leq \sigma_K/3$. Then $\frac{2}{3}\sigma_K \leq \sigma_K(\hat{R}_2) \leq \frac{4}{3}\sigma_K$ and $\|\hat{W}\| \leq \sqrt{2}\sigma_K^{-1/2}$.

$$\epsilon_W = \|\hat{W} - W\| = \|\hat{W}(I - U\Lambda^{-1/2}U^T)\| \leq \|\hat{W}\| \|I - \Lambda^{-1/2}\| \tag{88}$$

Since we have $\|I - \Lambda\| = \|\hat{W}^T R_2 \hat{W} - \hat{W}^T \hat{R}_2 \hat{W}\| \leq \|\hat{W}\|^2 \epsilon_2 = 2\sigma_K^{-1} \epsilon_2$. Thus

$$\|I - \Lambda^{-1/2}\| \leq \max\{|1 - (1 + 2\epsilon_2/\sigma_K)^{-1/2}|, |1 - (1 - 2\epsilon_2/\sigma_K)^{-1/2}|\} \leq \epsilon_2/\sigma_K$$

Therefore,

$$\epsilon_W \leq \sqrt{2}\epsilon_2 \sigma_K^{-3/2} \tag{89}$$

Now we have

$$\epsilon_T \leq 8\|R_3\|_{op} \sigma_K^{-5/2} \epsilon_2 + 2\sqrt{2}\sigma_K^{-3/2} \epsilon_3 \tag{90}$$

Thus we can apply Theorem 5.1 [2] to show the guarantees of the robust tensor power method to recover $\{\boldsymbol{v}_k\}_{k=1}^K$ and $\{p_k\}_{k=1}^K$. It can be stated as below, for some universal constant c_T and a small value η (the computational complexity is related to η by $O(\log(1/\eta))$), if $\epsilon_T \leq c_T \frac{1}{K\sqrt{p_{max}}}$, w.p. $1-\eta$ the returned eigenvectors $\{\hat{\boldsymbol{v}}_k\}_{k=1}^K$ and eigenvalues $\{\hat{a}_k\}_{k=1}^K$ satisfy

$$\|\hat{\boldsymbol{v}}_k - \boldsymbol{v}_k\| \leq 8\epsilon_T\sqrt{p_k} \leq 8\epsilon_T\sqrt{p_{max}},\ |\hat{a}_k - \frac{1}{\sqrt{p_k}}| \leq 5\epsilon_T \tag{91}$$

Let $a_k = \frac{1}{\sqrt{p_k}}$. Now we show

$$\begin{aligned}
\|(\hat{W}^T)^\dagger(\hat{a}_k\hat{\boldsymbol{v}}_k) - \boldsymbol{u}_k\| &= \|(\hat{W}^T)^\dagger(\hat{a}_k\hat{\boldsymbol{v}}_k) - W^\dagger a_k\boldsymbol{v}_k\| \\
&\leq \|(\hat{W}^T)^\dagger(\hat{a}_k\hat{\boldsymbol{v}}_k) - (\hat{W}^T)^\dagger(a_k\boldsymbol{v}_k)\| + \|(\hat{W}^T)^\dagger(a_k\boldsymbol{v}_k) - (W^T)^\dagger a_k\boldsymbol{v}_k\| \\
&\leq \|(\hat{W}^T)^\dagger\|(\|\hat{a}_k\hat{\boldsymbol{v}}_k - \hat{a}_k\boldsymbol{v}_k\| + \|\hat{a}_k\boldsymbol{v}_k - a_k\boldsymbol{v}_k\|) + \|(\hat{W}^T)^\dagger - (W^T)^\dagger\|\|a_k\boldsymbol{v}_k\| \\
&\leq \|(\hat{W}^T)^\dagger\|(\hat{a}_k 8\epsilon_T/a_k + 5\epsilon_T) + \|(\hat{W}^T)^\dagger - (W^T)^\dagger\|a_k
\end{aligned} \tag{92}$$

If $\epsilon_T \leq \frac{1}{10\sqrt{p_{max}}}$, we have $\hat{a}_k/a_k \leq 3/2$. If $\epsilon_2 \leq \sigma_K/3$,

$$\|(\hat{W}^T)^\dagger\| = \|\hat{\Lambda}_2\|^{1/2} \leq \sqrt{2}\|R_2\|^{1/2} \tag{93}$$

and

$$\begin{aligned}
\|(\hat{W}^T)^\dagger - (W^T)^\dagger\| &= \|(\hat{W}^T)^\dagger(I - U\Lambda^{1/2}U^T)\| \\
&= \|(\hat{W}^T)^\dagger\|\|I - \Lambda^{1/2}\| \\
&\leq 2\sqrt{2}\|R_2\|^{1/2}\epsilon_2/\sigma_K
\end{aligned} \tag{94}$$

$$\begin{aligned}
\|(\hat{W}^T)^\dagger(\hat{a}_k\hat{\boldsymbol{v}}_k) - \boldsymbol{u}_k\| &\leq \|R_2\|^{1/2}(25\epsilon_T + 3\epsilon_2/\sigma_K) \\
&\leq (3\|R_2\|^{1/2}\sigma_K^{-1} + 200\|R_2\|^{1/2}\|R_3\|_{op}\sigma_K^{-5/2})\epsilon_2 + (75\|R_2\|^{1/2}\sigma_K^{-3/2})\epsilon_3
\end{aligned} \tag{95}$$

□

C Proofs of Subspace Clustering

C.1 Some Properties of the Distance between Subspaces

According to [16], $D(U,V) = \sqrt{r - \|U^TV\|_F^2} = \sqrt{\mathrm{tr}(I_r - U^TVV^TU)} = \|U_\perp^TV\|_F = \|V_\perp^TU\|_F$. We briefly give the proof.

$$\|UU^T - VV^T\|_F^2 = \|(I - VV^T)UU^T - VV^T(I - UU^T)\|_F^2$$

Since $(I-VV^T)UU^T(VV^T(I-UU^T))^T = 0$ and $(VV^T(I-UU^T))^T(I-VV^T)UU^T = 0$, we have

$$\begin{aligned}
&\|(I - VV^T)UU^T - VV^T(I - UU^T)\|_F^2 \\
=&\|(I - VV^T)UU^T\|_F^2 + \|VV^T(I - UU^T)\|_F^2 \\
=&2\,\mathrm{tr}(VV^T - V^TUU^TV) \\
=&2(r - \|V^TU\|_F^2)
\end{aligned} \tag{96}$$

By the property of Frobineous norm we see $D(\cdot,\cdot)$ is a metric, so we can use triangular inequality. We will also use the following inequality, which is due to the dual property of matrices, ref. Lemma 3.2 in [5]. Let A, B be two matrices.

$$\begin{aligned}
&\|AB\|_F \\
&= \langle AB, AB\rangle^{1/2} \\
&= \left\langle BB^T, A^TA\right\rangle^{1/2} \\
&\leq \|BB^T\|^{1/2}\|A^TA\|_*^{1/2} \\
&= \|B\|\|A\|_F
\end{aligned} \tag{97}$$

Similarly, we also have $\|AB\|_F \leq \|A\|\|B\|_F$.

C.2 Proof of Theorem 5

Proof. For simplicity, we use U_j to denote U_j^t for $j \in [K]$. Consider fixing $\{U_k\}_{k\neq j}$ and updating U_j.

$$
\begin{aligned}
\bar{U}_j &= \sum_{i=1}^{N} \big(\Pi_{k\neq j}\langle I_d - U_k U_k^T, \boldsymbol{z}_i \boldsymbol{z}_i^T\rangle\big)\boldsymbol{z}_i \boldsymbol{z}_i^T U_j \\
&= \sum_{q=1}^{K} \sum_{i\in\Omega_q^{(t)}} \big(\Pi_{k\neq j}\langle I_d - U_k U_k^T, U_q^* \boldsymbol{s}_i \boldsymbol{s}_i^T U_q^{*T}\rangle\big) U_q^* \boldsymbol{s}_i \boldsymbol{s}_i^T U_q^{*T} U_j \\
&= \sum_{q=1}^{K} U_q^* \sum_{i\in\Omega_q^{(t)}} \big(\Pi_{k\neq j}\langle I_r - U_q^{*T} U_k U_k^T U_q^*, \boldsymbol{s}_i \boldsymbol{s}_i^T\rangle\big)\boldsymbol{s}_i \boldsymbol{s}_i^T U_q^{*T} U_j
\end{aligned} \tag{98}
$$

where $\Omega_q^{(t)}$ is the set of data points belongs to q-th subspace in t-th iteration.
Define

$$
B_{jq} := \mathbb{E}\big[\!\big[\Pi_{k\neq j}(\boldsymbol{s}^T(I_r - U_q^{*T} U_k U_k^T U_q^*)\boldsymbol{s})\boldsymbol{s}\boldsymbol{s}^T\big]\!\big] \tag{99}
$$

$$
\hat{B}_{jq} := \frac{1}{|\Omega_q^{(t)}|} \sum_{i=\Omega_q^{(t)}} \Pi_{k\neq j}(\boldsymbol{s}_i^T(I_r - U_q^{*T} U_k U_k^T U_q^*)\boldsymbol{s}_i)\boldsymbol{s}_i \boldsymbol{s}_i^T \tag{100}
$$

According to Lemma 3, we have

$$
\Pi_{k\neq j}\,\mathrm{tr}\,(I_r - U_q^{*T} U_k U_k^T U_q^*) I \preceq B_{jq} \preceq C_{K-1}\Pi_{k\neq j}\,\mathrm{tr}\,(I_r - U_q^{*T} U_k U_k^T U_q^*) I \tag{101}
$$

Note that $\mathrm{tr}(I_r - U_q^{*T} U_k U_k^T U_q^*) = D(U_q^*, U_k)^2$. We have

$$
\Pi_{k\neq j} D(U_q^*, U_k)^2 I \preceq B_{jq} \preceq C_{K-1}\Pi_{k\neq j} D(U_q^*, U_k)^2 I \tag{102}
$$

If the conditions about n and r in Theorem 5 are satisfied, because of Lemma 4 with $\delta = \frac{1}{2C_{K-1}}$, we have, w.p. $1 - O(Kr^{-2})$,

$$
\|B_{jq} - \hat{B}_{jq}\| \le \frac{1}{2C_{K-1}}\|B_{jq}\| \le \frac{1}{2}\Pi_{k\neq j}\,\mathrm{tr}\,(I_r - U_q^{*T} U_k U_k^T U_q^*) \le \frac{1}{2}\sigma_{min}(B_{jq})
$$

Therefore,

$$
\frac{1}{2}\Pi_{k\neq j}\,\mathrm{tr}\,(I_r - U_q^{*T} U_k U_k^T U_q^*) I \preceq \hat{B}_{jq} \preceq (C_{K-1}+1)\Pi_{k\neq j}\,\mathrm{tr}\,(I_r - U_q^{*T} U_k U_k^T U_q^*) I \tag{103}
$$

Given the condition, $D(U_k^*, U_k) \le c_s \min_{q\neq j}\{D(U_q^*, U_j^*)\}$, we have for $q \neq k$

$$
D(U_q^*, U_k)^2 \le (D(U_q^*, U_k^*) + D(U_k^*, U_k))^2 \le (1+c_s)^2 D(U_q^*, U_k^*)^2
$$

Similarly,

$$
D(U_q^*, U_k)^2 \ge (1-c_s)^2 D(U_q^*, U_k^*)^2
$$

Therefore, for $j \neq q$

$$
\|\hat{B}_{jq}\| \le (C_{K-1}+1)\big(\Pi_{k:k\neq j,k\neq q}(1+c_s)^2 D(U_q^*, U_k^*)^2\big) D(U_q^*, U_q)^2 \tag{104}
$$

For $j = q$

$$
\sigma_{min}(\hat{B}_{jj}) \ge \frac{1}{2}\Pi_{k:k\neq j}(1-c_s)^2 D(U_j^*, U_k^*)^2 \tag{105}
$$

Shown in Eq. (98), $\bar{U}_j = \sum_{q=1}^K U_q^* p_q \hat{B}_{jq} U_q^{*T} U_j$. Let $[U_j^+, \bar{R}_j] := \text{QR}(\bar{U}_j)$

$$
\begin{aligned}
&D(U_j^+, U_j^*) \\
&= \|U_{j\perp}^{*T} \bar{U}_j \bar{R}_j^{-1}\|_F \\
&\le \|U_{j\perp}^{*T} \bar{U}_j\|_F \|\bar{R}_j^{-1}\| \\
&= \|U_{j\perp}^{*T} \sum_{q=1}^{K} U_q^* p_q \hat{B}_{jq} U_q^{*T} U_j\|_F \|\bar{R}_j^{-1}\| \\
&\le (\sum_{q\neq j} p_q \|U_{j\perp}^{*T} U_q^*\|_F \|\hat{B}_{jq}\| \|U_q^{*T} U_j\|) \|\bar{R}_j^{-1}\| \\
&\le (\sum_{q\neq j} p_q \|U_{j\perp}^{*T} U_q^*\|_F \|\hat{B}_{jq}\|) \|\bar{R}_j^{-1}\| \\
&\le (C_{K-1}+1)(\sum_{q\neq j} p_q D(U_j^*, U_q^*) \Big((1+c_s)^{2(K-2)} \Pi_{k:k\neq j, k\neq q} D(U_q^*, U_k^*)^2\Big) D(U_q^*, U_q)^2) \|\bar{R}_j^{-1}\| \\
&\le (C_{K-1}+1)(1+c_s)^{2(K-2)} \left(\sum_{q\neq j} p_q D(U_j^*, U_q^*) \big(\Pi_{k:k\neq j, k\neq q} D(U_q^*, U_k^*)^2\big) D(U_q^*, U_q)^2 \right) \|\bar{R}_j^{-1}\| \\
&\le (C_{K-1}+1)(1+c_s)^{2(K-2)} p_{max} (K-1) D_{max}^{2K-3} D(U_q^*, U_q)^2 \|\bar{R}_j^{-1}\|
\end{aligned} \tag{106}
$$

Now we show,

$$
\begin{aligned}
\|\bar{R}_j^{-1}\| &\le \sigma_{min}^{-1}(\bar{R}_j) = \sigma_{min}^{-1}(\bar{U}_j) \le (p_j \sigma_{min}(\hat{B}_{jj}))^{-1} \\
&\le \big((p_j/2)\Pi_{k:k\neq j}(1-c_s)^2 D(U_j^*, U_k^*)^2\big)^{-1} \\
&\le \frac{2}{p_{min}(1-c_s)^{2K-2} D_{min}^{2K-2}}
\end{aligned} \tag{107}
$$

Combing Eq.(106), Eq. (107) and the condition on c_s,

$$
\begin{aligned}
D(U_j^+, U_j^*) &\le 2(C_{K-1}+1)(K-1)(1+c_s)^{2(K-2)}(1-c_s)^{-2(K-1)} \frac{p_{max} D_{max}^{2K-3}}{p_{min} D_{min}^{2K-2}} D(U_q^*, U_q)^2 \\
&\le \frac{1}{2c_s D_{min}} \Delta_t^2
\end{aligned} \tag{108}
$$

Using the initialization condition, we can easily obtain $\frac{\Delta_t}{2c_s D_{min}} \le \frac{1}{2}$ by induction. Also, the condition, $D(U_j^+, U_j^*) \le c_s D_{min}$, still holds after each update. So we have super-linear convergence rate. $\square$

D More Experimental Results

In Fig. 2, we show that, to achieve an initial error $\epsilon^{(0)} = c$ for some constant $c < 1$, our tensor method only requires N to be proportional to d. Note that the naive initialization methods, random initialization (using normal distribution) or all-zero initialization, will lead to $\epsilon^{(0)} \approx 1.4$ and $\epsilon^{(0)} = 1$ respectively.

In Fig. 3 we compare our methods with EM in terms of iterations. In Fig. 4 we compare EM and our methods for larger K, $K = 6$. Note that the per-iteration cost of MLR will be K times more than the per-iteration cost of EM. So when K is larger, MLR will be slower than EM.

In Fig. 5, we show the sample complexities for different methods. Our methods (MLR) have a better sample complexity than EM. And the tensor initialization outperforms random initialization significantly.

Fig. 6 shows whatever the ambient dimension d is, the clusters will be exactly recovered when N is proportional to r by a constant factor.

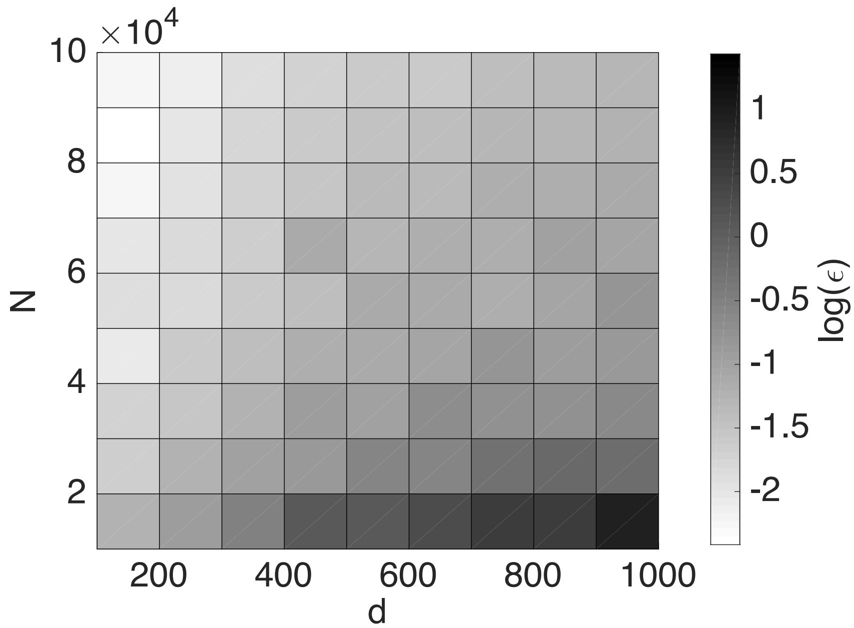
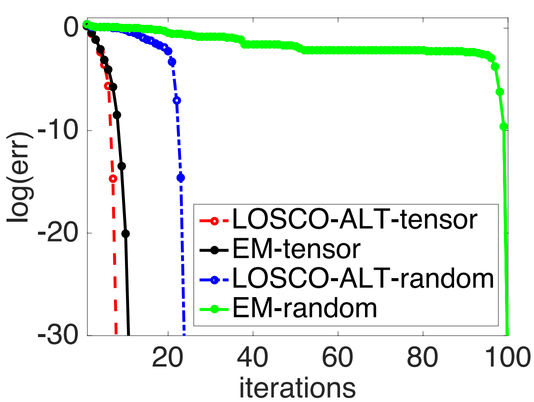
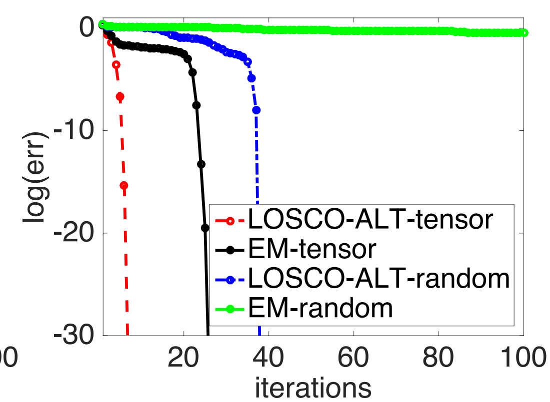


Figure 2: Initialization error for tensor method.



(a) $d = 100$, $N = 6k$, $K = 3$

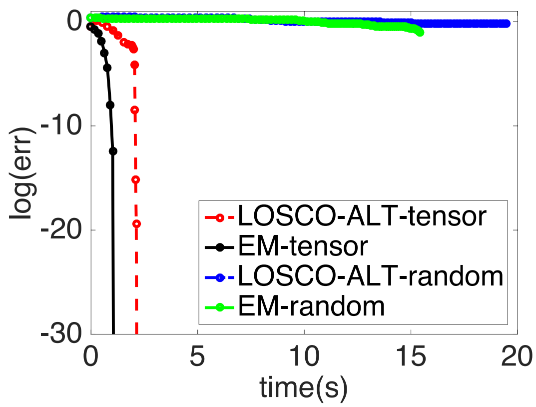


(b) $d = 1k$, $N = 60k$, $K = 3$

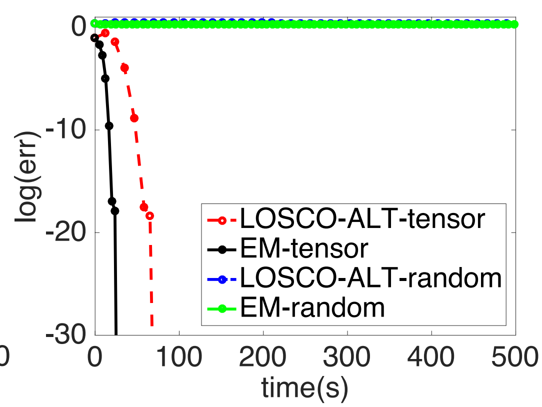
Figure 3: Comparison with EM in terms of iterations.

Table 2: Corresponding CE's for the results in Table 1

N/K	SSC	SSC-OMP	LRR	TSC	NSN+spectral	NSN+GSR	**PSC**
200	0	0.0190	0.0010	0.0650	0	0	0
400	0	0.0090	0.0015	0.0190	0	0	0
600	0	0.0027	0	0.0120	0	0	0
800	0	0.0027	0	0.0030	0	0	0
1000	0	0.0014	0	0.0022	0	0	0

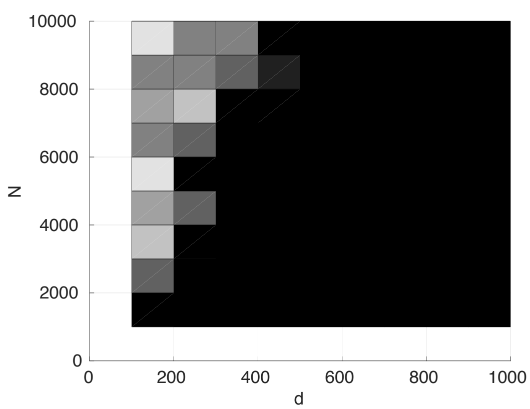


(c) $d = 60$, $N = 36k$, $K = 6$

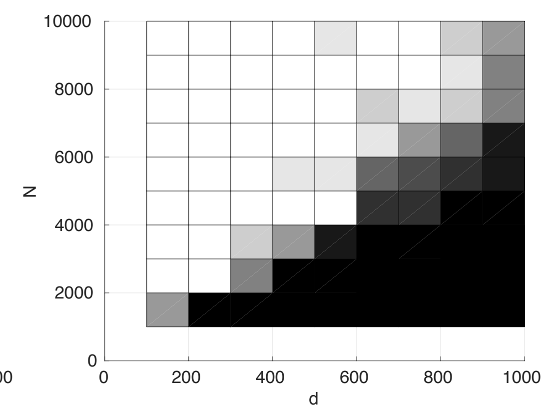


(d) $d = 300$, $N = 180k$, $K = 6$

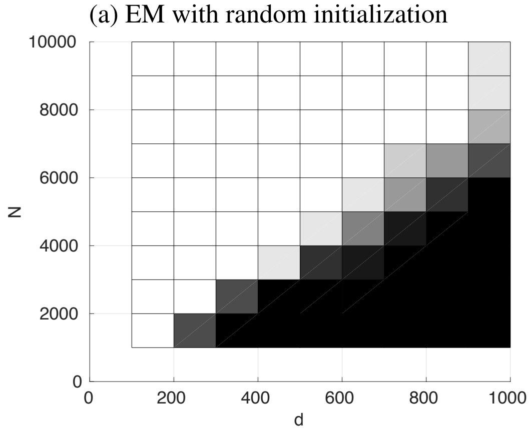
Figure 4: Comparison with EM for larger K, $K = 6$



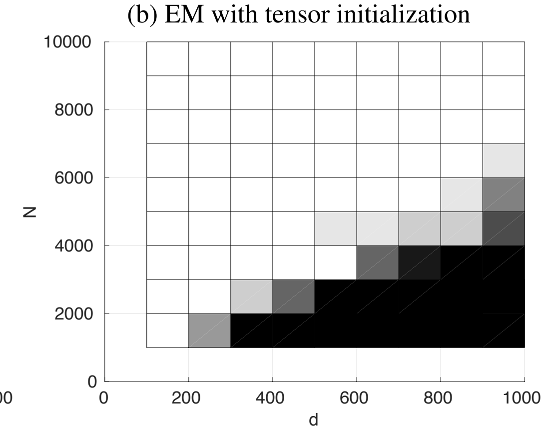
(a) EM with random initialization



(b) EM with tensor initialization



(a) MLR with random initialization



(b) MLR with tensor initialization

Figure 5: Sample complexities for different methods

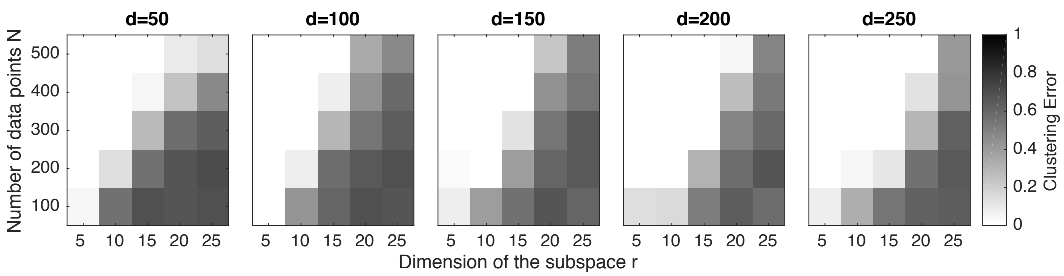


Figure 6: Subspace Clustering error for different N, d and r.