Sparse Support Recovery with Non-smooth Loss Functions: Supplementary Material

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1 Proof of Theorem 1 for $\alpha = 1$

As the proof presented in the paper for $\alpha = \infty$, our proof consists in construction a vector supported on J, obeying the implicit relationship (6) which becomes in this case

$$x_{\tau,J} = x_{0,J} + \widetilde{\Phi}^{-1}\Theta w - \tau v_{\infty,J}, \tag{12}$$

and which is indeed a solution to $(\mathcal{P}^\tau_\infty(\Phi x_0 + w))$ for an appropriate regime of the parameters $(\tau, \|w\|_\alpha)$. Note that we assume that (INJ_1) holds and in particular, $\tilde{\Phi}$ is invertible. If we define $Z_p \overset{\text{def.}}{=} \text{sat}(p)$, note that

$$\partial\|p\|_\infty = \left\{ u \;\middle|\; u_{Z_p^c} = 0, \langle u_{Z_p}, \text{sign}(p_{Z_p})\rangle = 1, \text{sign}(u_{Z_p}) = \text{sign}(p_{Z_p}) \right\},$$

so that for an optimal primal-dual pair (x, p), the condition (8) reads,

$$y_{Z_p^c} = \Phi_{Z_p^c,\cdot}x\ , \quad \langle \Phi^*_{Z_p,\cdot}\,\text{sign}(p_{Z_p}),\, x\rangle = \langle \text{sign}(p_{Z_p}),\, y_{Z_p}\rangle - \tau, \tag{13}$$

and

$$\text{sign}(y_{Z_p} - \Phi_{Z_p}x) = \text{sign}(p_{Z_p}). \tag{14}$$

To simplify the notations, we use

$$\Theta_p \overset{\text{def.}}{=} \begin{bmatrix} \text{Id}_{Z_p^c,\cdot} \\ \text{sign}(p^*_{Z_p})\text{Id}_{Z_p,\cdot} \end{bmatrix} \quad \text{and} \quad \widetilde{\Phi}_p \overset{\text{def.}}{=} \Theta_p \Phi_{\cdot,J}.$$

One should then look for a candidate primal-dual pair $(\hat{x}, \hat{p})$ such that $\text{supp}(\hat{x}) = J$ and satisfying

$$\widetilde{\Phi}_{\hat{p}}\hat{x}_J = \tilde{y}_{\hat{p}} = \Theta_{\hat{p}}y - \tau\delta_{|J|} = \widetilde{\Phi}_{\hat{p}}x_{0,J} + \Theta_{\hat{p}}w - \tau\delta_{|J|}. \tag{15}$$

We now need to show that the first order conditions (7) and (8) hold for some $p = \hat{p}$ solution of the "perturbed" dual problem $(\mathcal{D}^\tau_\infty(\Phi x_0 + w))$ with $x = \hat{x}$. Actually, we will show that under the conditions of the theorem, this holds for $\hat{p} = p_\infty$, *i.e.*, p_∞ is solution of $(\mathcal{D}^\tau_\infty(\Phi x_0 + w))$ so that

$$\hat{x}_J = x_{0,J} + \widetilde{\Phi}^{-1}\Theta w - \tau\widetilde{\Phi}^{-1}\delta_{|J|} = x_{0,J} + \widetilde{\Phi}^{-1}\Theta w - \tau v_{\infty,J}.$$

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We remind that as defined in Section 1.4, $\Theta = \Theta_{p_\infty}$ and $\widetilde{\Phi} = \widetilde{\Phi}_{p_\infty}$ and that $v_{\infty,J} = \widetilde{\Phi}^{-1}\delta_{|J|}$. Let us start by proving the equality part of (7), $\Phi^*_{\cdot,J} p_\infty = \mathrm{sign}(\hat{x}_J)$. Noting $\mathrm{Id}_{I,J}$ the restriction from J to I, we have

$$\mathrm{sign}\left(x_{0,I} + \mathrm{Id}_{I,J}\widetilde{\Phi}^{-1}\Theta w - \tau v_{\infty,I}\right) = \mathrm{sign}\left(x_{0,I}\right)$$

as soon as

$$\left|\left(\widetilde{\Phi}^{-1}\Theta w\right)_i - \tau v_{\infty,i}\right| < |x_{0,I}| \quad \forall i \in I.$$

It is sufficient to require

$$\|\mathrm{Id}_{I,J}\widetilde{\Phi}^{-1}\Theta w - \tau v_{\infty,I}\|_\infty < \underline{x}$$
$$\|\widetilde{\Phi}^{-1}\Theta\|_{\infty,\infty}\|w\|_\infty + \tau\|v_{\infty,I}\|_\infty < \underline{x},$$

with $\underline{x} = \min_{i\in I}|x_{0,I}|$. Injecting the fact that $\|w\|_\infty < c_1\tau$ (the value of c_1 will be derived later), we get the condition

$$\tau\left(bc_1 + \nu\right) \leqslant \underline{x},$$

with $b = \|\widetilde{\Phi}^{-1}\Theta\|_{\infty,\infty}$ and $\nu = \|v_\infty\|_\infty \leqslant b$. Rearranging the terms, we obtain

$$\tau \leqslant \frac{\underline{x}}{bc_1 + \nu} = c_2\underline{x},$$

which guarantees $\mathrm{sign}(\hat{x}_I) = \mathrm{sign}(x_{0,I})$. Outside I, defining $\mathrm{Id}_{\tilde{J},J}$ as the restriction from J to $\tilde{J}$, we must have

$$\Phi^*_{\cdot,\tilde{J}} p_\infty = \mathrm{sign}\left(\mathrm{Id}_{\tilde{J},J}\widetilde{\Phi}^{-1}\Theta w - \tau v_{\infty,\tilde{J}}\right).$$

From Lemma 1, we know that $-\mathrm{sign}(v_{\infty,\tilde{J}}) = \Phi^*_{\cdot,\tilde{J}} p_\infty$, so that the condition is satisfied as soon as

$$\left|\left(\widetilde{\Phi}^{-1}\Theta w\right)_j\right| < \tau|v_{\infty,j}| \quad \forall j \in \tilde{J}.$$

Noting $\underline{v} = \min_{j\in\tilde{J}}|v_{\infty,j}|$, we get the sufficient condition for (7),

$$\|\widetilde{\Phi}^{-1}\Theta w\|_\infty < \tau\underline{v},$$
$$\|w\|_\infty < \tau\frac{\underline{v}}{b}. \tag{c_1a}$$

We can now verify (13) and (14). From (15) we see that (13) is satisfied *i.e.*,

$$y_{Z^c} = \Phi_{Z^c,J}\hat{x}_J \quad \text{and} \quad \mathrm{sign}(p_{\infty,Z})^*\Phi_{Z,J}\hat{x}_J = \mathrm{sign}(p_{\infty,Z})^* y_Z - \tau.$$

On Z, we have

$$y_Z - \Phi_{Z,\cdot}\hat{x} = w_Z - \Phi_{Z,J}\widetilde{\Phi}^{-1}\Theta w + \tau\Phi_{Z,J}v_{\infty,J}.$$

We know from Lemma 1 that

$$\mathrm{sign}(p_{\infty,Z}) = \mathrm{sign}\left(\Phi_{Z,J}v_{\infty,J}\right)$$

so that (14) holds, *i.e.*, $\mathrm{sign}(y - \Phi\hat{x})_Z = \mathrm{sign}(p_{\infty,Z})$ as soon as

$$\|w_Z - \Phi_{Z,J}\widetilde{\Phi}^{-1}\Theta w\|_\infty \leqslant \tau\min_{i\in Z}|\Phi_{i,J}v_{\infty,J}|$$
$$\|\mathrm{Id}_{Z,\cdot} - \Phi_{Z,J}\widetilde{\Phi}^{-1}\Theta\|_{1,\infty}\|w\|_1 \leqslant \underline{z}\tau$$

with $\underline{z} \overset{\text{def.}}{=} \min_{i\in Z}|\Phi_{i,J}v_{\infty,J}|$ and finally, noting $a \overset{\text{def.}}{=} \|\mathrm{Id}_{Z,\cdot} - \Phi_{Z,J}\widetilde{\Phi}^{-1}\Theta\|_{1,\infty}$,

$$\|w\|_1 \leqslant \frac{\underline{z}}{a}\tau. \tag{c_1b}$$

(c_1a) and (c_1b) together give the value of c_1. This ensures that $\hat{x}$ is solution to $(\mathcal{P}_1^\tau(\Phi x_0 + w))$ and p_∞ solution to $(\mathcal{D}_\infty^\tau(\Phi x_0 + w))$, which concludes the proof. □