A Appendix

For notational convenience, let

$$Y_t^k(\mathbf{x}) = \sum_{\mathbf{z}: A_t^k \mathbf{z} = b_t^k \mod 2} p_\theta(\mathbf{x}, \mathbf{z})$$

denote a random variable representing the projected marginal likelihood, where the randomness is over the choice of the matrices A_t^k, b_t^k, and

$$\delta_t^{k,\mathcal{Q}}(\mathbf{x}) = \min_{q \in \mathcal{Q}} D_{KL}\big(q_\phi(\mathbf{z}|\mathbf{x}) || R^k_{A_t^k, b_t^k}[p_\theta(\mathbf{z}|\mathbf{x})]\big).$$

denote the minimum KL-divergence within an approximating family of distributions $\mathcal{Q}$ and the true posterior projected using A_t^k, b_t^k.

Before proving Theorem 3.1 and Theorem 3.2, we first extend an important result from earlier work to our setting.

A.1 Extension of Theorem 2 from [Hsu et al., 2016]

Lemma A.1. *For any $\Delta > 0$, let $T \geq \frac{1}{\alpha}(\log(2n/\Delta))$. Let $A_t^k \in \{0,1\}^{k \times n} \overset{iid}{\sim}$ Bernoulli($\frac{1}{2}$) and $b_t^k \in \{0,1\}^k \overset{iid}{\sim}$ Bernoulli($\frac{1}{2}$) for $k \in \{0, 1, \ldots, n\}$ and $t \in \{1, \ldots, T\}$. Let $\mathcal{D}$ denote the set of degenerate (deterministic) probability distributions. Then there exists a positive constant α such that with probability at least $(1 - \Delta)$*

$$p_\theta(\mathbf{x})/32 \leq \sum_{k=0}^{n} \exp\bigg(\mathop{Median}_{t \in [T]}\Big(-\delta_t^{k,\mathcal{D}}(\mathbf{x}) + \log Y_t^k(\mathbf{x})\Big)\bigg) 2^{k-1} \leq 32 p_\theta(\mathbf{x}) \tag{8}$$

i.e., it is a 32-approximation to $p_\theta(\mathbf{x})$.

Proof. By definition,

$$\delta_t^{k,\mathcal{D}}(\mathbf{x}) = \min_{q \in \mathcal{D}} \sum_{\mathbf{z}: A_t^k \mathbf{z} = b_t^k \mod 2} q_\phi(\mathbf{z}|\mathbf{x})\big[\log q_\phi(\mathbf{z}|\mathbf{x}) - \log p_\theta(\mathbf{x}, \mathbf{z})\big] + \log Y_t^k(\mathbf{x}).$$

For a degenerate distribution, $q \in \mathcal{D}$, the entropy is zero and all its mass is at a single point. Hence,

$$\begin{aligned}
\delta_t^{k,\mathcal{D}}(\mathbf{x}) &= -\max_{q \in \mathcal{D}} \sum_{\mathbf{z}: A_t^k \mathbf{z} = b_t^k \mod 2} q_\phi(\mathbf{z}|\mathbf{x}) \cdot \log p_\theta(\mathbf{x}, \mathbf{z}) + \log Y_t^k(\mathbf{x}) \\
&= -1 \cdot \max_{\mathbf{z}: A_t^k \mathbf{z} = b_t^k \mod 2} \log p_\theta(\mathbf{x}, \mathbf{z}) + \log Y_t^k(\mathbf{x}).
\end{aligned}$$

Rearranging terms,

$$-\delta_t^{k,\mathcal{D}}(\mathbf{x}) + \log Y_t^k(\mathbf{x}) = \max_{\mathbf{z}: A_t^k \mathbf{z} = b_t^k \mod 2} \log p_\theta(\mathbf{x}, \mathbf{z}).$$

Substituting the above expression into Eq. (8), we get

$$\begin{aligned}
&\sum_{k=0}^{n} \exp\left(Median\left(\max_{\mathbf{z}: A_1^k \mathbf{z} = b_1^k \mod 2} \log p_\theta(\mathbf{x}, \mathbf{z}), \cdots, \max_{\mathbf{z}: A_T^k \mathbf{z} = b_T^k \mod 2} \log p_\theta(\mathbf{x}, \mathbf{z})\right)\right) 2^{k-1} \\
&\quad = \sum_{k=0}^{n} Median\left(\exp\left(\max_{\mathbf{z}: A_1^k \mathbf{z} = b_1^k \mod 2} \log p_\theta(\mathbf{x}, \mathbf{z})\right), \cdots, \exp\left(\max_{\mathbf{z}:\mathbf{z}: A_T^k \mathbf{z} = b_T^k \mod 2} \log p_\theta(\mathbf{x}, \mathbf{z})\right)\right) 2^{k-1} \\
&\quad = \sum_{k=0}^{n} Median\left(\max_{\mathbf{z}: A_1^k \mathbf{z} = b_1^k \mod 2} p_\theta(\mathbf{x}, \mathbf{z}), \cdots, \max_{\mathbf{z}: A_T^k \mathbf{z} = b_T^k \mod 2} p_\theta(\mathbf{x}, \mathbf{z})\right) 2^{k-1}.
\end{aligned}$$

The result then follows directly from Theorem 1 from [Ermon et al., 2013b]. $\square$

A.2 Proof of Theorem 3.1: Upper bound based on mean aggregation

From the non-negativity of KL divergence we have that for any $q \in Q$,

$$\log Y_t^k(\mathbf{x}) \geq \sum_{\mathbf{z}:A_t^k \mathbf{z}=b_t^k \mod 2} q_\phi(\mathbf{z}|\mathbf{x})\big[\log p_\theta(\mathbf{x},\mathbf{z}) - \log q_\phi(\mathbf{z})\big]$$
$$\geq \max_{q\in Q}\left(\sum_{\mathbf{z}:A_t^k \mathbf{z}=b_t^k \mod 2} q_\phi(\mathbf{z}|\mathbf{x})\big[\log p_\theta(\mathbf{x},\mathbf{z}) - \log q_\phi(\mathbf{z})\big]\right)$$

Exponentiating both sides,

$$Y_t^k(\mathbf{x}) \geq \exp\left(\max_{q\in Q}\left(\sum_{\mathbf{z}:A_t^k \mathbf{z}=b_t^k \mod 2} q_\phi(\mathbf{z}|\mathbf{x})\big[\log p_\theta(\mathbf{x},\mathbf{z}) - \log q_\phi(\mathbf{z})\big]\right)\right) \stackrel{\text{def}}{=} \gamma_t^k(\mathbf{x}). \tag{9}$$

Taking an expectation on both sides w.r.t A_t^k, b_t^k,

$$\mathbb{E}_{A_t^k,b_t^k}[Y_t^k(\mathbf{x})] \geq \mathbb{E}_{A_t^k,b_t^k}[\gamma_t^k(\mathbf{x})]$$

Using Lemma 3.1, we get:

$$\mathbb{E}_{A_t^k,b_t^k}[\gamma_t^k(\mathbf{x})] \leq 2^{-k} p_\theta(\mathbf{x})$$

A.3 Proof of Theorem 3.2: Upper bound based on median aggregation

From Markov's inequality, since $Y_t^k(\mathbf{x})$ is non-negative,

$$\mathbb{P}\left[Y_t^k(\mathbf{x}) \geq c\mathbb{E}[Y_t^k(\mathbf{x})]\right] \leq \frac{1}{c}.$$

Using Lemma 3.1,

$$\mathbb{P}\left[Y_t^k(\mathbf{x})2^k \geq c p_\theta(\mathbf{x})\right] \leq \frac{1}{c}.$$

Since $Y_t^k(\mathbf{x}) \geq \gamma_t^k(\mathbf{x})$ from Eq. (9), setting $c = 4$ and $k = k^\star$ we get

$$\mathbb{P}\left[\gamma_t^{k^\star}(\mathbf{x})2^{k^\star} \geq 4 p_\theta(\mathbf{x})\right] \leq \frac{1}{4}. \tag{10}$$

From Chernoff's inequality, if for any non-negative $\epsilon \leq 0.5$,

$$\mathbb{P}\left[\gamma_t^{k^\star}(\mathbf{x})2^{k^\star} \geq 4 p_\theta(\mathbf{x})\right] \leq \left(\frac{1}{2} - \epsilon\right) \tag{11}$$

then,

$$\mathbb{P}\left[4p_\theta(\mathbf{x}) \leq Median\left(\gamma_1^{k^\star}(\mathbf{x}), \cdots, \gamma_T^{k^\star}(\mathbf{x})\right)2^{k^\star}\right] \leq \exp(-2\epsilon^2 T) \tag{12}$$

From Eq. (10) and Eq. (11), $\epsilon \leq 0.25$. Hence, taking the complement of Eq. (12) and given a positive constant $\alpha \leq 0.125$ such that for any $\Delta > 0$, if $T \geq \frac{1}{\alpha}\log(2n/\Delta) \geq \frac{1}{\alpha}\log(1/\Delta)$, then

$$\mathbb{P}\left[4p_\theta(\mathbf{x}) \geq Median\left(\gamma_1^{k^\star}(\mathbf{x}), \cdots, \gamma_T^{k^\star}(\mathbf{x})\right)2^{k^\star}\right] \geq 1 - \Delta.$$

A.4 Proof of Theorem 3.2: Lower bound based on median aggregation

Since the conditions of Lemma A.1 are satisfied, we know that Eq. (8) holds with probability at least $1 - \delta$. Also, since the terms in the sum are non-negative we have that the maximum element is at least $1/(n+1)$ of the sum. Hence,

$$\max_k \exp\left(Median\left(-\delta_1^{k,\mathcal{D}}(\mathbf{x}) + \log Y_1^k(\mathbf{x}), \cdots, -\delta_T^{k,\mathcal{D}}(\mathbf{x}) + \log Y_T^k(\mathbf{x})\right)\right)2^{k-1} \geq \frac{1}{32} p_\theta(\mathbf{x})\frac{1}{n+1}. \tag{13}$$

Therefore, there exists $k^\star$ (corresponding to the arg max in Eq. (13)) such that

$$Median\left(-\delta_1^{k^\star,\mathcal{D}}(\mathbf{x}) + \log Y_1^{k^\star}(\mathbf{x}), \cdots, -\delta_T^{k^\star,\mathcal{D}}(\mathbf{x}) + \log Y_T^{k^\star}(\mathbf{x})\right) + (k^\star - 1)\log 2 \geq -\log 32 + \log p_\theta(\mathbf{x}) - \log(n+1).$$

Since $\mathcal{D} \subseteq \mathcal{Q}$, we also have

$$\delta_t^{k^\star,\mathcal{Q}}(\mathbf{x}) \leq \delta_t^{k^\star,\mathcal{D}}(\mathbf{x})).$$

Thus,

$$Median\left(-\delta_1^{k^\star,\mathcal{Q}}(\mathbf{x})+\log Y_1^{k^\star}(\mathbf{x}),\cdots,-\delta_T^{k^\star,\mathcal{Q}}(\mathbf{x})+\log Y_T^{k^\star}(\mathbf{x})\right)+(k^\star-1)\log 2\geq-\log 32+\log p_\theta(\mathbf{x})-\log(n+1).$$

From Eq. (5), note that

$$\begin{aligned}\log\gamma_t^k(\mathbf{x})&=\max_{q\in\mathcal{Q}}\sum_{\mathbf{z}:A_t^k\mathbf{z}=b_t^k \bmod 2}q_\phi(\mathbf{z}|\mathbf{x})\big(\log p_\theta(\mathbf{x},\mathbf{z})-\log q_\phi(\mathbf{z}|\mathbf{x})\big)\\&=\max_{q\in\mathcal{Q}}\sum_{\mathbf{z}:A_t^k\mathbf{z}=b_t^k \bmod 2}q_\phi(\mathbf{z}|\mathbf{x})\big(\log p_\theta(\mathbf{z}|\mathbf{x})-\log q_\phi(\mathbf{z}|\mathbf{x})\big)+\log p_\theta(\mathbf{x})\\&=-\delta_t^{k,\mathcal{Q}}(\mathbf{x})+\log Y_t^k(\mathbf{x}).\end{aligned}$$

Plugging in we get,

$$Median\left(\log\gamma_1^{k^\star}(\mathbf{x}),\cdots,\log\gamma_T^{k^\star}(\mathbf{x})\right)+(k^\star-1)\log 2\geq-\log 32-\log(n+1)+\log p_\theta(\mathbf{x})$$

and also

$$Median\left(\log\gamma_1^{k^\star}(\mathbf{x}),\cdots,\log\gamma_T^{k^\star}(\mathbf{x})\right)+k^\star\log 2\geq-\log 32-\log(n+1)+\log p_\theta(\mathbf{x}).$$

with probability at least $1-\Delta$.

$$Median\left(\gamma_1^{k^\star}(\mathbf{x}),\cdots,\gamma_T^{k^\star}(\mathbf{x})\right)2^{k^\star}\geq\frac{p_\theta(\mathbf{x})}{32(n+1)}.$$

Combining the lower and upper bounds, we get

$$4p_\theta(\mathbf{x})\geq\mathcal{L}_{Md}^{k^\star,T}(\mathbf{x})\geq\frac{p_\theta(\mathbf{x})}{32(n+1)}$$

with probability at least $1-2\Delta$ by union bound by choosing a small enough value for α.

A.5 Proof of Theorem 3.1: Lower bound based on mean aggregation

We first prove a useful inequality relating to the mean and median of non-negative reals.

Lemma A.2. *For a set of non-negative reals $F=\{f_i\}_{i=1}^{\ell}$,*

$$\frac{1}{\ell}\sum_{i=1}^{\ell}f_i\geq\frac{1}{2}Median(F).$$

Proof. Without loss of generality, we assume for notational convenience that elements in F are sorted by their indices, i.e., $f_1\leq f_2\cdots\leq f_\ell$. By definition of median, we have for all $i\in\{\lfloor\ell/2\rfloor,\lfloor\ell/2\rfloor+1,\ldots,\ell\}$

$$f_i\geq Median(F).$$

Adding all the above inequalities, we get

$$\sum_{i=\lfloor\ell/2\rfloor}^{\ell}f_i\geq\left\lceil\frac{\ell}{2}\right\rceil Median(F).$$

Since all f_i are non-negative,

$$\sum_{i=1}^{\ell}f_i\geq\left\lceil\frac{\ell}{2}\right\rceil Median(F).$$

The median of non-negative reals is also non-negative, and hence,

$$\sum_{i=1}^{\ell}f_i\geq\frac{\ell}{2}Median(F)$$

finishing the proof. ∎

Substituting for F in the above lemma with $\{\gamma_t^{k^\star}(\mathbf{x})\}_{t=1}^T$, we get

$$\frac{1}{T}\sum_{t=1}^{T}\gamma_t^{k^\star}(\mathbf{x})\geq\frac{1}{2}Median\left(\gamma_1^{k^\star}(\mathbf{x}),\cdots,\gamma_T^{k^\star}(\mathbf{x})\right).$$

Now using the lower bound in Theorem 3.2, with probability at least $1-2\Delta$,

$$\frac{1}{T}\sum_{t=1}^{T}\gamma_t^{k^\star}(\mathbf{x})\cdot 2^{k^\star}\geq\frac{p_\theta(\mathbf{x})}{64(n+1)}.$$