Supplementary material: a hybrid sampler for Poisson-Kingman mixture models

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1 Pseudocode

Algorithm 1 HybridSamplerσ-PK$\Big(K, V, \mathbf{c}, \{X_i\}_{i\in[n]}, \{Y_c^*\}_{c\in\Pi_n}, \{\tilde{J}_c\}_{c\in\Pi_n}, H_0, M\Big)$

for $t = 2 \to iter$ **do**
  Update $v^{(t)}$: Slice sample $\tilde{\mathbb{P}}\left(V \in \mathrm{d}v \mid \text{rest}\right)$
  Update $s_i^{(t)}$ for $i = 1, \ldots, k$: Slice sample $\tilde{\mathbb{P}}\left(\tilde{J}_i \in \mathrm{d}s_i \mid \text{rest}\right)$
  Update $\pi^{(t)}, \left\{y_c^{*(t)}\right\}_{c\in\pi}, \left\{s_c^{(t)}\right\}_{c\in\pi}$: **AddTable&ReUse**$\Big(V, \Pi_n, M, \{X_i\}_{i\in[n]}, \{Y_c^*\}_{c\in\Pi_n}, \{\tilde{J}_c\}_{c\in\Pi_n}, H_0 \mid \text{rest}\Big)$
end for

Algorithm 2 HybridSampler-MH-σPK$\Big(K, V, \mathbf{c}, \{X_i\}_{i\in[n]}, \{Y_c^*\}_{c\in\Pi_n}, \{\tilde{J}_c\}_{c\in\Pi_n}, H_0, M\Big)$

for $t = 2 \to iter$ **do**
  Update $s_i^{(t)}$ for $i = 1, \ldots, k$: Slice sample $\tilde{\mathbb{P}}\left(\tilde{J}_i \in \mathrm{d}s_i \mid \text{rest}\right)$
  Update $v^{(t)}$: MH step for $\tilde{\mathbb{P}}\left(V \in \mathrm{d}v \mid \text{rest}\right)$ with independent proposal `Stablernd`(σ) or `ExpTiltStablernd`(λ, σ).
  Update $\pi^{(t)}, \left\{y_c^{*(t)}\right\}_{c\in\pi}, \left\{s_c^{(t)}\right\}_{c\in\pi}$: **AddTable&ReUse**$\Big(V, \Pi_n, M, \{X_i\}_{i\in[n]}, \{Y_c^*\}_{c\in\Pi_n}, \{\tilde{J}_c\}_{c\in\Pi_n}, H_0 \mid \text{rest}\Big)$
end for

Algorithm 3 AddTable&ReUse$\Big(V, \Pi_n, M, \{X_i\}_{i\in[n]}, \{Y_c^*\}_{c\in\Pi_n}, \{\tilde{J}_c\}_{c\in\Pi_n}, H_0 \mid \text{rest}\Big)$

Let $c \in \Pi_n$ be such that $i \in c$
$c \leftarrow c\backslash\{i\}$
if $c = \varnothing$ **then**
  $k \sim$ UniformDiscrete$(\frac{1}{M})$
  $Y_k^e \leftarrow Y_c^*$
  $\Pi_n \leftarrow \Pi_n\backslash\{c\}$
  $V \leftarrow V + \tilde{J}_c$ ▷ Add back the discarded table size to the surplus
end if
Set c'according to$\mathbb{P}(c_i = c \mid \mathbf{c}_{-i}, \text{Rest})\propto \begin{cases} \tilde{J}_c F(x_i \mid \{X_i\}_{i\in c} Y_c^*) & \text{if existing} \\ \frac{V}{M} F(x_i \mid Y_c^*) & \text{if new} \end{cases}$
if $c' \in [M]$ **then**
  $\tilde{J}_{\text{new}} \leftarrow$**ExactSampleNewTableSize**$(V, \sigma \mid \text{rest})$
  $V \leftarrow V - \tilde{J}_{\text{new}}$ ▷ Remove it from the old surplus
  $\Pi_n \leftarrow \Pi_n \cup \{\{i\}\}$
  $Y_{\{i\}}^* \leftarrow Y_{c'}^e$
  $Y_{c'}^e \sim H_0$
else
  $c' \leftarrow c' \cup \{i\}$
end if
Draw $\{Y_j^e\}_{j=1}^M \overset{\text{i.i.d.}}{\sim} H_0$

Algorithm 4 ExactSampleNewTableSize$(V, \sigma \mid \text{rest})$

if $\sigma = 0.5$ **then**
  $G \sim$ Gamma $\left(\frac{3}{4}, 1\right)$
  $IG \sim$ Inverse Gamma $\left(\frac{1}{4}, \frac{1}{4^3} V^{-2}\right)$
  $V_{stick} = \frac{\sqrt{G}}{\sqrt{G}+\sqrt{IG}}$
  $\tilde{J}_{new} = V_{stick} V$
else
  if $\sigma < 0.5$ && $\sigma = \frac{u_\sigma}{v_\sigma}, u_\sigma, v_\sigma \in \mathbb{N}$ **then**
    $\lambda = u_\sigma^2 / v_\sigma^{\frac{v_\sigma}{u_\sigma}}$
    $IG \sim$ Inverse Gamma $\left(1 - \frac{u_\sigma}{v_\sigma}, \lambda\right)$
    $\frac{1}{G} \sim \mathcal{E}_\mathcal{T}\left(\lambda, L_{\frac{u_\sigma}{v_\sigma}}^{-1/u}\right)$ ▷ Samples an exponentially tilted random variable. See Favaro *et al.* [1] for details.
    $V_{stick} = \frac{G}{G+IG}$
    $\tilde{J}_{new} = V_{stick} V$
  end if
end if

2 Relationship to the Pitman-Yor's Stick-breaking construction

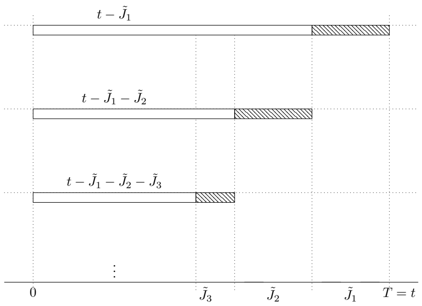

Figure 1: Generative process of Section 2.1

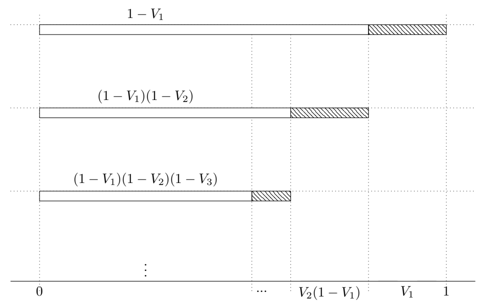

Figure 2: Pitman-Yor's stick breaking construction

$$
\begin{aligned}
T &\sim \gamma_{\mathrm{PY}} \\
\tilde{J}_1 \mid T &\sim \mathrm{SBS}(T) \\
\tilde{J}_2 \mid T, \tilde{J}_1 &\sim \mathrm{SBS}\left(T - \tilde{J}_1\right) \\
&\vdots \\
\tilde{J}_\ell \mid T, \tilde{J}_1, \ldots, \tilde{J}_{\ell-1} &\sim \mathrm{SBS}\left(T - \sum_{i<\ell} \tilde{J}_i\right) \\
&\vdots \\
P_\ell &\stackrel{d}{=} \frac{\tilde{J}_\ell}{T - \sum_{j<\ell} \tilde{J}_j}
\end{aligned}
$$

$$
\begin{aligned}
V_1 &\sim \mathrm{Beta}(v_1 \mid 1-\sigma, \theta+\sigma) \\
V_2 &\sim \mathrm{Beta}(v_2 \mid 1-\sigma, \theta+2\sigma) \\
&\vdots \\
V_\ell &\sim \mathrm{Beta}(v_\ell \mid 1-\sigma, \theta+\ell\sigma) \\
&\vdots
\end{aligned}
$$

the corresponding weights are:

$$P_\ell \stackrel{\mathrm{d}}{=} V_\ell \prod_{j<\ell} (1 - V_j).$$

The Pitman Yor's stick breaking construction from Ishwaran & James [2] can be recovered from the size-biased sampling (SBS) generative process of Section 2.1 after integrating out the total mass T, the change of variables given in Section 2.2 and if we choose a specific distribution for the total mass

$$\gamma_{\mathrm{PY}}(t) = \frac{\Gamma(\theta+1)}{\Gamma(\frac{\theta}{\sigma}+1)} t^{-\theta} f_\sigma(t) \mathbb{I}_{(0,\infty)}(t), \qquad \theta > -\sigma$$

which corresponds to the Pitman-Yor prior and f_σ is the density of a σ-Stable random variable.

References

[1] Favaro, S., Lomeli, M., Nipoti, B., & Teh, Y. W. 2014. On the Stick-Breaking representation of σ-Stable Poisson-Kingman models. *Electronic Journal of Statistics*, **8**, 1063–1085.

[2] Ishwaran, H., & James, L. F. 2001. Gibbs Sampling Methods for Stick-Breaking Priors. *Journal of the American Statistical Association*, **96**(453), 161–173.

[3] Pitman, J. 1996. Random discrete distributions invariant under size-biased permutation. *Advances in Applied Probability*, **28**, 525–539.