Online Decision-Making in General Combinatorial Spaces

A Supplement to Section 2 (Preliminaries and Background)

A.1 Online Mirror Descent (OMD) for Online Linear Optimization

Algorithm Online Mirror Descent (OMD) for Online Linear Optimization

Inputs:
Convex set $\Omega \subseteq \mathbb{R}^n$

Parameters:
$\eta > 0$
Closed convex set $\mathcal{K} \supseteq \Omega$, Legendre function $F : \mathcal{K} \to \mathbb{R}$

Initialize:
$x^1 \in \operatorname{argmin}_{x \in \Omega} F(x)$ (or $x^1 =$ any other point in Ω)

For $t = 1 \ldots T$:
– Receive loss vector $\ell^t \in \mathbb{R}^n$
– Incur loss $x^t \cdot \ell^t$
– Update:
$\widetilde{x}^{t+1} \leftarrow \nabla F^*(\nabla F(x^t) - \eta \ell^t)$
$x^{t+1} \leftarrow \operatorname{argmin}_{x \in \Omega} B_F(x, \widetilde{x}^{t+1})$

The following bound on the regret of OMD (in the linear setting) is well known (e.g. see [7]):

Theorem 4 (Regret bound for OMD)**.** *Let $B_F(x, x^1) \le D^2 \ \forall x \in \Omega$. Let $\|\cdot\|$ be any norm in $\mathbb{R}^n$ such that $\|\ell^t\| \le G \ \forall t \in [T]$, and such that the restriction of F to Ω is α-strongly convex w.r.t. $\|\cdot\|_*$, the dual norm of $\|\cdot\|$. Then setting $\eta^* = \frac{D}{G}\sqrt{\frac{2\alpha}{T}}$ gives*

$$R_T\left[\,\mathrm{OMD}(\eta^*)\right] \ \Big(= \ \sum_{t=1}^T x^t \cdot \ell^t - \inf_{x \in \Omega} \sum_{t=1}^T x \cdot \ell^t \Big) \ \le \ DG\sqrt{\frac{2T}{\alpha}}\,.$$

A.2 Hedge/Naïve OMD for Online Combinatorial Decision-Making

Algorithm Hedge/Naïve OMD for Online Combinatorial Decision-Making [10]

Inputs:
Finite set of combinatorial structures $\mathcal{C}$
Mapping $\phi : \mathcal{C} \to \mathbb{R}^d$

Parameters:
$\eta > 0$

Initialize:
$p^1 = \left(\frac{1}{|\mathcal{C}|}, \ldots, \frac{1}{|\mathcal{C}|}\right) \in \Delta_{\mathcal{C}}$

For $t = 1 \ldots T$:
– Randomly draw $c^t \sim p^t$
– Receive loss vector $\ell^t \in [0,1]^d$
– Incur loss $\phi(c^t) \cdot \ell^t$
– Update:

$$\forall c \in \mathcal{C} : \ \ p_c^{t+1} \leftarrow \frac{p_c^t \, \exp(-\eta\, \phi(c) \cdot \ell^t)}{Z^t}\,,$$

where $Z^t = \sum_{c' \in \mathcal{C}} p_{c'}^t \, \exp(-\eta\, \phi(c') \cdot \ell^t)$

A.3 Follow the Perturbed Leader (FPL) for Online Combinatorial Decision-Making

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Algorithm Follow the Perturbed Leader (FPL) for Online Combinatorial Decision-Making [13]

Inputs:
Finite set of combinatorial structures $\mathcal{C}$
Mapping $\phi : \mathcal{C} \to \mathbb{R}^d$

Parameters:
$\eta > 0$

For $t = 1 \ldots T$:
- Draw $z^t \in \left[0, \frac{1}{\eta}\right]^d$ uniformly at random
- Predict $c^t \in \operatorname{argmin}_{c \in \mathcal{C}} \ \phi(c) \cdot \left(\sum_{s=1}^{t-1} \ell^s + z^t \right)$
- Receive loss vector $\ell^t \in [0,1]^d$
- Incur loss $\phi(c^t) \cdot \ell^t$

B Supplement to Section 7 (Transportation Polytopes)

The decomposition step in applying LDOMD to transportation polytopes requires finding a suitable extreme point on each iteration. Here we give details of how one can find such an extreme point.

We start by giving a procedure which, given a matrix $X \in \mathcal{T}(a,b)$, efficiently finds an extreme point $Q \in \mathcal{T}(a,b)$ such that $X_{ij} = 0 \implies Q_{ij} = 0$ (note that such an extreme point always exists, since X can be written as a convex combination of extreme points, all of which must necessarily have a zero entry wherever X does). We will make use of the following characterization of extreme points of transportation polytopes in terms of spanning forests of complete bipartite graphs (e.g. see [6]):

Theorem 5 (Characterization of extreme points of transportation polytopes)**.** *Let $a \in \mathbb{Z}_+^m$, $b \in \mathbb{Z}_+^n$. A matrix $X \in \mathcal{T}(a,b)$ is an extreme point of $\mathcal{T}(a,b)$ if and only if the edges $\{(i,j) : X_{ij} > 0\}$ form a spanning forest of the complete bipartite graph $K_{m,n}$.*

The basic idea behind the procedure below is as follows: given $X \in \mathcal{T}(a,b)$, let $E = \{(i,j) : X_{ij} > 0\}$. If E forms a spanning forest of $K_{m,n}$, then by Lemma 5, X is already an extreme point. Otherwise, successively remove cycles from E and adjust corresponding entries in X so that X remains in $\mathcal{T}(a,b)$ while satisfying $X_{ij} > 0 \iff (i,j) \in E$. Eventually, E must be a spanning forest of $K_{m,n}$, and therefore by Lemma 5, X must be an extreme point of $\mathcal{T}(a,b)$.

Algorithm Procedure for finding an extreme point Q of $\mathcal{T}(a,b)$ such that
$X_{ij} = 0 \implies Q_{ij} = 0$ for a given matrix $X \in \mathcal{T}(a,b)$

Input:
$X \in \mathcal{T}(a,b)$ (where $a \in \mathbb{Z}_+^m, b \in \mathbb{Z}_+^n$)

Initialize:
$E \leftarrow \{(i,j) : X_{ij} > 0\}$

While $\big($ E does not form a spanning forest of $K_{m,n}$ $\big)$ **do**:
- Find a cycle $E' = \{(i_1, j_1), (i_2, j_1), (i_2, j_2), \ldots, (i_s, j_s), (i_{s+1} = i_1, j_s)\} \subseteq E$ for some $s \geq 2$
- Let $e_{\min} \in \operatorname{argmin}_{e \in E'} X_e$
- $\theta \leftarrow \begin{cases} +1 & \text{if } e_{\min} = (i_r, j_r) \text{ for some } r \in [s] \\ -1 & \text{if } e_{\min} = (i_{r+1}, j_r) \text{ for some } r \in [s] \end{cases}$
- **For** $r = 1 \ldots s$ **do:**
  $X_{i_r, j_r} \leftarrow X_{i_r, j_r} - \theta X_{e_{\min}}$
  $X_{i_{r+1}, j_r} \leftarrow X_{i_{r+1}, j_r} + \theta X_{e_{\min}}$
- $E \leftarrow \{(i,j) : X_{ij} > 0\}$

end while
$Q \leftarrow X$
Output: Q

Applying the above procedure to implement decomposition step. The above procedure can be used to implement the decomposition step for transportation polytopes in Section 7 by doing the following on each iteration k:

- Apply the above procedure to the matrix A^k, which can be verified to belong to $\mathcal{T}(\gamma_k a, \gamma_k b)$ for suitable $\gamma_k \in \mathbb{R}_+$ (specifically, $\gamma_k = 1 - \sum_{r=1}^{k-1} \alpha_r$), to get an extreme point $\widetilde{Q}^k \in \mathcal{T}(\gamma_k a, \gamma_k b)$ satisfying $A^k_{ij} = 0 \implies \widetilde{Q}^k_{ij} = 0$.
- Set $Q^k \leftarrow \frac{1}{\gamma_k} \widetilde{Q}^k$.

It can be verified that Q^k is then an extreme point of $\mathcal{T}(a, b)$ and satisfies $A^k_{ij} = 0 \implies Q^k_{ij} = 0$ as desired.