Spike Frequency Adaptation Implements Anticipative Tracking

in Continuous Attractor Neural Networks

Supplementary Information

1 Travelling Wave in a CANN with the asymmetrical neuronal interaction

The dynamics of a CANN including the asymmetrical neuronal interaction is given by

$$\tau\frac{dU(x,t)}{dt} = -U(x,t) + \rho\int_{x'}\tilde{J}(x,x')r(x',t)dx' + I^{ext}(x,t), \tag{1}$$

$$r(x,t) = \frac{U(x,t)^2}{1 + k\rho\int_{x'}U(x',t)^2dx'}, \tag{2}$$

where the neuronal interaction is written as

$$\tilde{J}(x,x') = \frac{J_0}{\sqrt{2\pi}a}\exp\left[-\frac{(x-x')^2}{2a^2}\right] + \gamma\tau\frac{J_0}{\sqrt{2\pi}a^3}(x-x')\exp\left[-\frac{(x-x')^2}{2a^2}\right]. \tag{3}$$

We check that the network has the following form of the traveling wave solution,

$$\overline{U}(x,t) = U_0\exp\left\{-\frac{[x-(z+vt)]^2}{4a^2}\right\}, \quad \overline{r}(x,t) = r_0\exp\left\{-\frac{[x-(z+vt)]^2}{2a^2}\right\}, \quad \forall z \tag{4}$$

where v is the speed of the travelling wave and $r_0 = \sqrt{2}U_0/(\rho J_0)$.

Substituting Eq.(4) into (1), we obtain

$$Left-side = \tau U_0 v\frac{x-(z+vt)}{2a^2}\exp\left\{-\frac{[x-(z+vt)]^2}{4a^2}\right\},$$

$$Right-side = \tau U_0\gamma\frac{x-(z+vt)}{2a^2}\exp\left\{-\frac{[x-(z+vt)]^2}{4a^2}\right\}.$$

Comparing two sides, we see that when $v = \gamma$, Eq.(4) is the solution of the network dynamics.

2 Travelling Wave in a CANN with SFA

The dynamics of a CANN with SFA is given by

$$\tau\frac{dU(x,t)}{dt} = -U(x,t) + \rho\int_{x'}J(x,x')r(x',t)dx', -V(x,t) + I^{ext}(x,t),$$

$$\tau_v\frac{dV(x,t)}{dt} = -V(x,t) + mU(x,t), \tag{5}$$

$$r(x,t) = \frac{U(x,t)^2}{1 + k\rho\int_{x'}U(x',t)^2dx'},. \tag{6}$$

where $J(x, x') = J_0/(\sqrt{2\pi}a)\exp\left[-(x - x')^2/(2a^2)\right]$.

When SFA is strong, the CANN may hold a self-sustained travelling wave solution. Assume the travelling wave has the following form,

$$\overline{U}(x,t) = A_u\exp\left\{-\frac{[x - z(t)]^2}{4a^2}\right\}, \tag{7}$$

$$\overline{r}(x,t) = A_r\exp\left\{-\frac{[x - z(t)]^2}{2a^2}\right\}, \tag{8}$$

$$\overline{V}(x,t) = A_v\exp\left\{-\frac{[x - z(t) + d]^2}{4a^2}\right\}, \tag{9}$$

where d is the peak separation between the profiles $\overline{U}(x,t)$ and $\overline{V}(x,t)$. Without loss of generality, we consider the bump moves from left to right. Thus, $d > 0$ reflects that $\overline{V}(x,t)$ is always lagging behind $\overline{U}(x,t)$ due to slow SFA.

To solve the network dynamics, we take advantage of an important property of CANNs, that is, the dynamics of a CANN is dominated by a few motion modes corresponding to distortions in different features of the bump state. The first two dominating modes we use correspond to the distortions in the height and position of the bump, which are given by

$$\phi_0(x|z) = \exp\left[-\frac{(x - z)^2}{4a^2}\right], \tag{10}$$

$$\phi_1(x|z) = (x - z)\exp\left[-\frac{(x - z)^2}{4a^2}\right]. \tag{11}$$

We can project the network dynamics on these dominating motion modes and simplify the network dynamics significantly. By projecting a function $f(x)$ onto a mode $\phi_n(x)$, we mean to compute $\int_x f(x)\phi_n(x)dx/\int_x \phi_n(x)^2 dx$.

Substituting Eqs.(7,8) into Eq.(6), we get

$$A_r = \frac{A_u^2}{1 + k\rho\sqrt{2\pi}aA_u^2}. \tag{12}$$

Substituting Eqs.(7-9) into Eq.(5), we obtain

$$Left - side = \tau A_u\frac{x - z(t)}{2a^2}\exp\left\{-\frac{[x - z(t)]^2}{4a^2}\right\}\frac{dz(t)}{dt},$$

$$Right - side = \left(-A_u + \frac{\rho J_0}{\sqrt{2}}A_r\right)\exp\left\{-\frac{[x - z(t)]^2}{4a^2}\right\} - A_v\exp\left\{-\frac{[x - z(t) + d]^2}{4a^2}\right\}.$$

Projecting both sides onto the motion mode $\phi_0(x|z)$, we obtain

$$Left - side = 0,$$
$$Right - side = \left(-A_u + \frac{\rho J_0}{\sqrt{2}}A_r\right)\sqrt{2\pi}a - A_v\exp\left(-\frac{d^2}{8a^2}\right)\sqrt{2\pi}a.$$

Equating both sides, we have

$$-A_u + \frac{\rho J_0}{\sqrt{2}}A_r - A_v\exp\left(-\frac{d^2}{8a^2}\right) = 0. \tag{13}$$

Projecting both sides onto the motion mode $\phi_1(x|z)$, we obtain

$$
\begin{aligned}
Left-side &= \frac{\sqrt{2\pi}a}{2}\tau A_u \frac{dz(t)}{dt}, \\
Right-side &= \frac{\sqrt{2\pi}a}{2}dA_v \exp\left(-\frac{d^2}{8a^2}\right).
\end{aligned}
$$

Equating both sides, we have

$$
\tau A_u \frac{dz(t)}{dt} = dA_v \exp\left(-\frac{d^2}{8a^2}\right). \tag{14}
$$

Substituting Eqs.(7-9) into Eq.(5), we obtain

$$
\begin{aligned}
Left-side &= \tau_v A_v \frac{x-z(t)+d}{2a^2}\exp\left\{-\frac{[x-z(t)+d]^2}{4a^2}\right\}\frac{dz(t)}{dt}, \\
Right-side &= -A_v \exp\left\{-\frac{[x-z(t)+d]^2}{4a^2}\right\} + mA_u \exp\left\{-\frac{[x-z(t)]^2}{4a^2}\right\}.
\end{aligned}
$$

Projecting both sides onto the motion mode $\phi_0(x|z)$, we obtain

$$
\begin{aligned}
Left-side &= \frac{d\sqrt{2\pi}a}{4a^2}\tau_v A_v \exp\left(-\frac{d^2}{8a^2}\right)\frac{dz(t)}{dt}, \\
Right-side &= -A_v \exp\left(-\frac{d^2}{8a^2}\right)\sqrt{2\pi}a + mA_u\sqrt{2\pi}a.
\end{aligned}
$$

Equating both sides, we have

$$
\frac{d}{4a^2}\tau_v A_v \exp\left(-\frac{d^2}{8a^2}\right)\frac{dz(t)}{dt} = -A_v \exp\left(-\frac{d^2}{8a^2}\right) + mA_u. \tag{15}
$$

Projecting both sides onto the motion mode $\phi_1(x|z)$, we obtain

$$
\begin{aligned}
Left-side &= \frac{\sqrt{2\pi}a}{2}\tau_v A_v \exp\left(-\frac{d^2}{8a^2}\right)\frac{dz(t)}{dt} - \frac{d^2}{8a^2}\tau_v A_v \frac{dz(t)}{dt}\exp\left(-\frac{d^2}{8a^2}\right)\sqrt{2\pi}a, \\
Right-side &= \frac{d\sqrt{2\pi}a}{2}A_v \exp\left(-\frac{d^2}{8a^2}\right).
\end{aligned}
$$

Equating both sides, we have

$$
\tau_v\left(1-\frac{d^2}{4a^2}\right)\frac{dz(t)}{dt} = d. \tag{16}
$$

Combining Eqs.(12-16), we get the bump heights, the separation distance d, and the speed of the

travelling wave, which are

$$A_u = \frac{\rho J_0 + \sqrt{\rho^2 J_0^2 - 8\sqrt{2\pi}k\rho a(1 + \sqrt{\frac{m\tau}{\tau_v}})^2}}{4\sqrt{\pi}k\rho a(1 + \sqrt{\frac{m\tau}{\tau_v}})}, \tag{17}$$

$$A_r = \frac{\rho J_0 + \sqrt{\rho^2 J_0^2 - 8\sqrt{2\pi}k\rho a(1 + \sqrt{\frac{m\tau}{\tau_v}})^2}}{2\sqrt{2\pi}k\rho^2 a J_0}, \tag{18}$$

$$A_v = \sqrt{\frac{m\tau}{\tau_v}}\frac{\rho J_0 + \sqrt{\rho^2 J_0^2 - 8\sqrt{2\pi}k\rho a(1 + \sqrt{\frac{m\tau}{\tau_v}})^2}}{4\sqrt{\pi}k\rho a(1 + \sqrt{\frac{m\tau}{\tau_v}})}\exp\left(\frac{1 - \sqrt{\frac{\tau}{m\tau_v}}}{2}\right), \tag{19}$$

$$v_{int} \equiv \frac{dz(t)}{dt} = \frac{2a}{\tau_v}\sqrt{\frac{m\tau_v}{\tau} - \sqrt{\frac{m\tau_v}{\tau}}}, \tag{20}$$

$$d = 2a\sqrt{1 - \sqrt{\frac{\tau}{m\tau_v}}}, \tag{21}$$

where the speed of the travelling wave dz/dt is defined as the intrinsic speed of the network v_{int}.

3   Tracking Dynamics of a CANN with SFA

When an external moving input is applied, the network state will track the moving input, provided that the input speed is not too large. Consider the external input has the form,

$$I^{ext}(x,t) = \alpha\exp\left\{-\frac{[x - z_0(t)]^2}{4a^2}\right\}, \tag{22}$$

where α denotes the input strength. The input moves at a constant speed $v_{ext} = dz_0(t)/dt$ from left to right.

We study the case that the network is able to track the moving input. Denote $s = z(t) - z_0(t)$ to be the separation between the network bump and the external input. We assume that the network state has the form as given by Eqs.(7-9). By applying the projection method, the network dynamics can be solved.

Substituting Eqs.(7-9) into Eq.(6), we get

$$A_r = \frac{A_u^2}{1 + k\rho\sqrt{2\pi}a A_u^2}. \tag{23}$$

Substituting Eqs.(7-9) into Eq.(5), we obtain

$$\begin{aligned}
Left - side &= \tau A_u \frac{x - z(t)}{2a^2}\exp\left\{-\frac{[x - z(t)]^2}{4a^2}\right\}\frac{dz(t)}{dt}, \\
Right - side &= \left(-A_u + \frac{\rho J_0}{\sqrt{2}}A_r\right)\exp\left\{-\frac{[x - z(t)]^2}{4a^2}\right\} - A_v\exp\left\{-\frac{[x - z(t) + d]^2}{4a^2}\right\} \\
&\quad + \alpha\exp\left\{-\frac{(x - z_0(t))^2}{4a^2}\right\}.
\end{aligned}$$

4

Projecting both sides onto the motion mode $\phi_0(x|z)$, we obtain

$$
\begin{aligned}
Left - side &= 0, \\
Right - side &= \left(-A_u + \frac{\rho J_0}{\sqrt{2}} A_r\right) \sqrt{2\pi}a - A_v \exp\left(-\frac{d^2}{8a^2}\right) \sqrt{2\pi}a + \alpha\sqrt{2\pi}a \exp\left(-\frac{s^2}{8a^2}\right).
\end{aligned}
$$

Equating both sides, we have

$$
-A_u + \frac{\rho J_0}{\sqrt{2}} A_r - A_v \exp\left(-\frac{d^2}{8a^2}\right) + \alpha \exp\left(-\frac{s^2}{8a^2}\right) = 0. \tag{24}
$$

Projecting both sides onto the motion mode $\phi_1(x|z)$, we obtain

$$
\begin{aligned}
Left - side &= \frac{\sqrt{2\pi}a}{2} \tau A_u \frac{dz(t)}{dt}, \\
Right - side &= \frac{\sqrt{2\pi}a}{2} dA_v \exp\left(-\frac{d^2}{8a^2}\right) - \frac{\sqrt{2\pi}a}{2} \alpha s \exp\left(-\frac{s^2}{8a^2}\right).
\end{aligned}
$$

Equating both sides, we have

$$
\tau A_u \frac{dz(t)}{dt} = dA_v \exp\left(-\frac{d^2}{8a^2}\right) - \alpha s \exp\left(-\frac{s^2}{8a^2}\right). \tag{25}
$$

Substituting Eqs.(7-9) into Eq.(5) and projecting them onto the modes $\phi_0(x|z)$ and $\phi_1(x|z)$, we get

$$
\frac{d}{4a^2} \tau_v A_v \exp\left(-\frac{d^2}{8a^2}\right) \frac{dz(t)}{dt} = -A_v \exp\left(-\frac{d^2}{8a^2}\right) + m A_u, \tag{26}
$$

$$
\tau_v \left(1 - \frac{d^2}{4a^2}\right) \frac{dz(t)}{dt} = d. \tag{27}
$$

When the network is able to track the moving input, we have

$$
\frac{ds}{dt} = \frac{dz(t)}{dt} - \frac{dz_0(t)}{dt} = 0. \tag{28}
$$

Combining Eqs.(23 - 28), the network dynamics is solved, which gives

$$
d = 2a \frac{-a + \sqrt{a^2 + (v_{ext}\tau_v)^2}}{v_{ext}\tau_v}, \tag{29}
$$

$$
s \exp\left(-\frac{s^2}{8a^2}\right) = \frac{A_u \tau}{\alpha v_{ext}} \left(\frac{md^2}{\tau \tau_v} - v_{ext}^2\right), \tag{30}
$$

$$
A_v = \frac{A_u m d}{\tau_v v_{ext}} \exp\left(\frac{d^2}{8a^2}\right). \tag{31}
$$

From Eq.(30), we see that the condition for $s > 0$, i.e., the network state leads the input, is

$$
v_{ext} < d\sqrt{\frac{m}{\tau \tau_v}} = 2a\sqrt{\frac{m}{\tau \tau_v}} \frac{-a + \sqrt{a^2 + (v_{ext}\tau_v)^2}}{v_{ext}\tau_v}. \tag{32}
$$

This condition is equivalent to

$$
v_{ext} < \frac{2a}{\tau_v} \sqrt{\sqrt{\frac{m\tau_v}{\tau}} - \sqrt{\frac{m\tau_v}{\tau}}} = v_{int}. \tag{33}
$$

Thus, when the intrinsic speed of the network $v_{int} > v_{ext}$, $s > 0$. Similarly, when $v_{int} < v_{ext}$, $s < 0$.