Statistical analysis of coupled time series with Kernel Cross-Spectral Density operators.
Supplementary information

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Proof of Lemma 3

Using the expression of the quadricumulant of centered variables, we show for any functions $f_1, f_3 \in \mathcal{H}_1$ and $f_2, f_4 \in \mathcal{H}_2$.

$$\begin{aligned}\left\langle f_1 \otimes f_2^*, (\boldsymbol{\mathcal{K}}_{1,2,3,4})(f_3 \otimes f_4^*)\right\rangle &= \mathrm{Cum}(f_1(X_1),f_2(X_2),f_3(X_3),f_4(X_4)) \\ &= \mathbb{E}\big(\left\langle X_1 \otimes X_2^*, f_1 \otimes f_2^*\right\rangle\left\langle f_3 \otimes f_4^*, X_3 \otimes X_4^*\right\rangle\big)\end{aligned}$$
$$-\left\langle \mathbf{C}_{12}, f_1 \otimes f_2^*\right\rangle\left\langle \mathbf{C}_{34}, f_3 \otimes f_4^*\right\rangle - \left\langle \mathbf{C}_{13}, f_1 \otimes f_3^*\right\rangle\left\langle \mathbf{C}_{24}, f_2 \otimes f_4^*\right\rangle - \left\langle \mathbf{C}_{14}, f_1 \otimes f_4^*\right\rangle\left\langle \mathbf{C}_{23}, f_2 \otimes f_3^*\right\rangle$$

Then we apply this formula to the scalar product by using two orthonormal systems $\{\boldsymbol{\alpha}_i\}$ and $\{\boldsymbol{\beta}_j\}$ of $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively

$$\begin{aligned}\mathbb{E}\left[\left\langle X_1, X_3\right\rangle\left\langle X_2, X_4\right\rangle\right] &= \mathbb{E}\left\langle X_1 \otimes X_2^*, X_3 \otimes X_4^*\right\rangle = \mathbb{E}\textstyle\sum_{i,j}\left\langle X_1 \otimes X_2^*, \alpha_i \otimes \beta_j^*\right\rangle\left\langle X_3 \otimes X_4^*, \alpha_i \otimes \beta_j^*\right\rangle \\ &= \textstyle\sum_{i,j}\left\langle \alpha_i \otimes \beta_j^*, (\boldsymbol{\mathcal{K}}_{1,2,3,4})(\alpha_i \otimes \beta_j^*)\right\rangle + \sum_{i,j}\left\langle \alpha_i \otimes \beta_j^*, \boldsymbol{C}_{1,2}\right\rangle\left\langle \alpha_i \otimes \beta_j^*, \boldsymbol{C}_{3,4}\right\rangle \\ &\quad + \textstyle\sum_{i,j}\left\langle \alpha_i \otimes \alpha_j^*, \boldsymbol{C}_{1,3}\right\rangle\left\langle \beta_i \otimes \beta_j^*, \boldsymbol{C}_{2,4}\right\rangle + \sum_{i,j}\left\langle \alpha_i \otimes \beta_j^*, \boldsymbol{C}_{1,4}\right\rangle\left\langle \beta_i \otimes \alpha_j^*, \boldsymbol{C}_{2,3}\right\rangle \\ &= \mathrm{Tr}\,\boldsymbol{\mathcal{K}}_{1,2,3,4} + \left\langle \boldsymbol{C}_{1,2}, \boldsymbol{C}_{3,4}\right\rangle + \mathrm{Tr}\,\boldsymbol{C}_{1,3}\,\mathrm{Tr}\,\boldsymbol{C}_{2,4} + \left\langle \boldsymbol{C}_{1,4}, \boldsymbol{C}_{3,2}\right\rangle\end{aligned}$$

Proof of Lemma 4

We prove this lemma for $k > 0$ (the alternative case is similar):

$\left|\sum_t w_T(t+k)w(t) - \sum_t w_T(t)^2\right| \leq \sum_t |w(t)| \sum_{p=0}^{k-1} |w(t+p+1) - w(t+p)|$.

Using successively the bounded and bounded variation properties we get:

$\left|\sum_t w_T(t+k)w(t) - \sum_t w_T(t)^2\right| \leq (\sup|w(t)|) \sum_{p=0}^{k-1} \sum_t |w(t+p+1) - w(t+p)| \leq C|k|$

Riemann-Lebesgue Lemma

The following lemma is introduced to prove Theorem 6

Lemma 10. *[Variant of the Riemann-Lebesgue Lemma] For* $\nu \neq 0 (mod\, 1)$*, and* w *Lebesgue integrable* $1/T \sum_{t \in \mathbb{Z}} w(k/T) \exp(-2i\pi\nu k) \underset{T \to \infty}{\to} 0$

Proof. Let us choose ϵ arbitrary small. According to the Riemann-Lebesgue Lemma, for $T > T_0$

$$\left|\int_{\mathbb{R}} w(t) \exp(-2i\pi\nu t T) dt\right| < \epsilon/2$$

Moreover, we can choose $T > T_1$ such that the piece-wise constant function $p_T(t) = w(k/T)$, $k/T \leq t < (k+1)/T$ verifies $\int |p_T(t) - w(t)| dt < \epsilon/2$ then

$$|\int (p_T(t) - w(t)) \exp(-2i\pi\nu tT) dt| < \int |p_T(t) - w(t)| < \epsilon/2$$

And thus, $|\int p_T(t) \exp(-2i\pi\nu tT) dt| < \epsilon$,

This proves that $\sum_{k=\infty}^{+\infty} w(k/T) \int_{k/T}^{(k+1)/T} \exp(-2i\pi\nu tT) dt \to 0$ Which is equivalent to

$$1/T \sum_{k=-\infty}^{+\infty} w(k/T)(\exp(-2i\pi\nu k) - \exp(-2i\pi\nu(k+1))) \to 0$$

by dividing by $(1 - \exp(-2i\pi\nu))$ we get the result

$$1/T \sum_{k} w(k/T) \exp(-2i\pi\nu k) \to 0$$

$\square$

Proof of Theorem 6

We assume that the population mean elements are used for centering (the case where empirical mean elements are used instead is asymptotically equivalent):

$$T^{-2}\mathbb{E}\big\|\sum_{k\in\mathbb{Z}} w_T(k)\mathbf{X}_1^c(k)z^{-k}\big\|^2\big\|\sum_{n\in\mathbb{Z}} w_T(n)\mathbf{X}_2^c(n)z^{-n}\big\|^2 = \\ \sum_{k,n,k',n'} w_T(k)w_T(n)w_T(k')w_T(n')z^{+k-n-k'+n'}\big[\big\langle \boldsymbol{C}_{12}(k-n), \boldsymbol{C}_{12}(k'-n')\big\rangle \dots \\ +\big\langle \boldsymbol{C}_{12}(k-n'), \boldsymbol{C}_{12}(k'-n)\big\rangle + \mathrm{Tr}(\boldsymbol{C}_{11}(k-k'))\,\mathrm{Tr}(\boldsymbol{C}_{22}(n-n')) + \mathrm{Tr}(\boldsymbol{\mathcal{K}}(n-k, k'-k, n'-k))\big]$$

We focus on the first term of the sum: as T grows,

$T^{-2}\sum_{k,n,k',n'} w_T(k)w_T(n)w_T(k')w_T(n')z^{+k-n-k'+n'}\big\langle \boldsymbol{C}_{12}(k-n), \boldsymbol{C}_{12}(k'-n')\big\rangle$

is arbitrary close to (using lemma 4 and $\sum_{k\in\mathbb{Z}} |k| \|\mathbf{C}_{12}(k)\|_{\mathrm{HS}} < +\infty$)

$T^{-2}\sum_{k,\delta,k',\delta'} w_T(k)^2 w_T(k')^2 z^{+\delta-\delta'}\big\langle \boldsymbol{C}_{12}(\delta), \boldsymbol{C}_{12}(\delta')\big\rangle \underset{T\to+\infty}{\to} \|w\|^4 \big\|\boldsymbol{S}_{12}(z)\big\|_{HS}^2$

For the second term of the sum: as T grows

$T^{-2}\sum_{k,n,k',n'} w_T(k)w_T(n)w_T(k')w_T(n')z^{+k-n-k'+n'}\big\langle \boldsymbol{C}_{12}(k-n'), \boldsymbol{C}_{12}(k'-n))\big\rangle$

Due to the bounded variation of w, it is arbitrary close to

$T^{-2}\sum_{k,k'} w_T(k)^2 w_T(k')^2 z^{+2k-2k'}\big\langle \sum_\delta \boldsymbol{C}_{12}(\delta)z^{-\delta}, \sum_{\delta'} \boldsymbol{C}_{12}(\delta')z^{-\delta'}\big\rangle \underset{T\to+\infty}{\to} 0$

(the limit is computed using Lemma 10).

For the third term of the sum: as T grows

$T^{-2}\sum_{k,n,k',n'} w_T(k)w_T(n)w_T(k')w_T(n')z^{+k-n-k'+n'}\,\mathrm{Tr}(\boldsymbol{C}_{11}(k-k'))\,\mathrm{Tr}(\boldsymbol{C}_{22}(n-n'))$

is arbitrary close to (using lemma 5 and $\sum_{k=-\infty}^{+\infty} |k|\,\mathrm{Tr}(\mathbf{C}_{ii}(k)) < +\infty$)

$$T^{-2}\sum_{k,\delta,k',\delta'} w_T(k')^2 w_T(n')^2 z^{+\delta-\delta'}\,\mathrm{Tr}(\boldsymbol{C}_{11}(\delta)\,\mathrm{Tr}(\boldsymbol{C}_{22}(\delta')) \underset{T\to+\infty}{\to} \|w\|^4\,\mathrm{Tr}(\boldsymbol{S}_{11}(z))\,\mathrm{Tr}(\boldsymbol{S}_{22}(z))$$

Finally the last term vanishes using the assumption $\sum_{k,i,j=-\infty}^{+\infty}\big|\mathrm{Tr}\left[\boldsymbol{\mathcal{K}}_{1212}(k,i,j)\right]\big| < +\infty$. $\square$