Solution of the threshold-linear negative feedback circuit dynamics

To detail the closed form solution for the threshold-linear negative feedback circuit we divide time into the intervals in between interneurons crossing threshold and consider them each in sequence since between threshold crossings the dynamics are linear first order and can be explicitly solved.

At time $t^0 = 0$, and define the first threshold crossing, t^1 and so on.

Let us first define some auxiliary variables:

$\mathbf{\Gamma}^k$: an indicator function for which interneurons are above threshold at time t^k. For instance, $\mathbf{\Gamma}^0 = (0, 0, \dots, 0)$.

$\mathbf{d}^k$: the distance between each interneuron and its threshold at time t^k. For instance, $\mathbf{d}^0 = (\lambda, \lambda, \dots, \lambda)$.

Let us write down the dynamics for $t^0 < t < t^1$:

$$\boldsymbol{p} = \boldsymbol{s} - \boldsymbol{W}\boldsymbol{a}$$

$$\delta \frac{d\boldsymbol{n}}{dt} = \boldsymbol{W}^{\boldsymbol{T}}\boldsymbol{p}$$

$$\boldsymbol{a} = \boldsymbol{0}, \qquad \text{Eq. S1}$$

since none of the interneurons are active the equations can be easily solved:

$$\boldsymbol{n}(t) = \boldsymbol{n}(t = 0) + \tfrac{1}{\delta}\boldsymbol{W}^{\boldsymbol{T}}\boldsymbol{s}t = \tfrac{1}{\delta}\boldsymbol{W}^{\boldsymbol{T}}\boldsymbol{s}t\,, \qquad \text{Eq. S2}$$

since we assume as usual that interneurons are initially at rest.

Let us define l^k as a vector of the time at which each interneuron would cross threshold if the dynamics stay the same. We note that this time will be different for each interneuron and that these are not the true times of threshold crossing since once a neuron crosses threshold the dynamics are no longer valid. To find the components of l^k we find when each neuron's activity has changed by the value of d^k. Thus, for l^1:

$$\boldsymbol{n}_{\boldsymbol{j}}(t = l^1) - \boldsymbol{n}_{\boldsymbol{j}}(t = 0) = d_j^0\,, \qquad \text{Eq. S3}$$

from the equation above we have:

$$l_j^1 = \frac{\delta\lambda}{(\boldsymbol{W}^{\boldsymbol{T}}\boldsymbol{s})_j}\,, \qquad \text{Eq. S4}$$

Where by $(\boldsymbol{W}^{\boldsymbol{T}}\boldsymbol{s})_j$ we denote the j^th^ component of the vector $\boldsymbol{W}^{\boldsymbol{T}}\boldsymbol{s}$. We are interested in the first threshold crossing:

$$t^1 = \min(\boldsymbol{l}^1) = \frac{\delta\lambda}{\max(\boldsymbol{W}^{\boldsymbol{T}}\boldsymbol{s})}\,, \qquad \text{Eq. S5}$$

Finding t^1 we update:

$$\mathbf{\Gamma}^1 = (0, 0, \dots, 1, \dots, 0) \text{ and } \mathbf{d}^1 = \left(\lambda - \frac{(\boldsymbol{W}^{\boldsymbol{T}}\boldsymbol{s})_{\boldsymbol{1}}}{\delta}, \lambda - \frac{(\boldsymbol{W}^{\boldsymbol{T}}\boldsymbol{s})_{\boldsymbol{2}}}{\delta}, \dots, 0, \dots, \lambda - \frac{(\boldsymbol{W}^{\boldsymbol{T}}\boldsymbol{s})_{\boldsymbol{n}}}{\delta}\right), \qquad \text{Eq. S6}$$

Let us know find the solution for $t^i < t < t^{i+1}$

To make the calculations more concise let us work with variables $\tilde{t} = t - t^i$, so that time starts at zero and $\widetilde{\boldsymbol{n}} = \boldsymbol{n} - \boldsymbol{n}(t = t^i)$ so that $\boldsymbol{n}$ starts at zero.

Lastly, we denote $\boldsymbol{W}_{>}^{\boldsymbol{i}}$ as the matrix collecting those columns of $\boldsymbol{W}$ for which the interneurons are above threshold immediately following the i^{th} threshold crossing, i.e. the indices for which $\boldsymbol{\Gamma}^{k} = 1$. Conversely, we denote $\boldsymbol{W}_{<}^{\boldsymbol{i}}$ collecting the columns of $\boldsymbol{W}$ for which the interneurons are under threshold immediately following the i^{th} threshold crossing. To avoid clutter we drop the superscript i above $\boldsymbol{W}$.

Thus, we can write the dynamics for the above threshold interneurons marked as $\widetilde{\boldsymbol{n}}_{>}$ and the subthreshold interneurons marked as $\widetilde{\boldsymbol{n}}_{<}$ as follows:

$$\boldsymbol{p} = \boldsymbol{s} - \boldsymbol{W}\boldsymbol{a} = \boldsymbol{s} - \boldsymbol{W}_{>}(\widetilde{\boldsymbol{n}}_{>} - \lambda \mathrm{sign}(\boldsymbol{n}_{>}))$$

$$\delta \frac{d\widetilde{\boldsymbol{n}}_{>}}{dt} = \boldsymbol{W}_{>}^{\boldsymbol{T}}\boldsymbol{p} = \boldsymbol{W}_{>}^{\boldsymbol{T}}\boldsymbol{s} - \boldsymbol{W}_{>}^{\boldsymbol{T}}\boldsymbol{W}_{>}(\widetilde{\boldsymbol{n}}_{>} - \lambda \mathrm{sign}(\boldsymbol{n}_{>})), \qquad \text{Eq. S7}$$

Defining $\boldsymbol{A} \equiv \boldsymbol{W}_{>}^{\boldsymbol{T}}\boldsymbol{W}_{>}$ and $\boldsymbol{B} \equiv \boldsymbol{W}_{>}^{\boldsymbol{T}}\boldsymbol{W}_{>}\lambda \mathrm{sign}(\boldsymbol{n}_{>})$ we can simplify the equation above to:

$$\delta \frac{d\widetilde{\boldsymbol{n}}_{>}}{dt} = -\boldsymbol{A}\widetilde{\boldsymbol{n}}_{>} + \left(\boldsymbol{W}_{>}^{\boldsymbol{T}}\boldsymbol{s} + \boldsymbol{B}\right), \qquad \text{Eq. S8}$$

Assuming $\boldsymbol{A}$ is invertible we can write down the solution:

$$\widetilde{\boldsymbol{n}}_{>} = \boldsymbol{A}^{-\boldsymbol{1}}\exp\left(\frac{-\boldsymbol{A}}{\delta}\tilde{t}\right)\widetilde{\boldsymbol{C}} + \boldsymbol{A}^{-\boldsymbol{1}}\left(\boldsymbol{W}_{>}^{\boldsymbol{T}}\boldsymbol{s} + \boldsymbol{B}\right), \qquad \text{Eq. S9}$$

where $\widetilde{\boldsymbol{C}}$ is an integration constant determined by the initial conditions, $\widetilde{\boldsymbol{n}}(\tilde{t} = 0) = 0$, yielding:

$$\widetilde{\boldsymbol{n}}_{>} = \boldsymbol{A}^{-\boldsymbol{1}}\left(1 - \exp\left(\frac{-\boldsymbol{A}}{\delta}\tilde{t}\right)\right)\left(\boldsymbol{W}_{>}^{\boldsymbol{T}}\boldsymbol{s} + \boldsymbol{B}\right), \qquad \text{Eq. S10}$$

For the subthreshold interneurons:

$$\delta \frac{d\widetilde{\boldsymbol{n}}_{<}}{dt} = \boldsymbol{W}_{<}^{\boldsymbol{T}}\boldsymbol{p} = \boldsymbol{W}_{<}^{\boldsymbol{T}}\boldsymbol{s} - \boldsymbol{W}_{<}^{\boldsymbol{T}}\boldsymbol{W}_{>}(\widetilde{\boldsymbol{n}}_{>} - \lambda \mathrm{sign}(\boldsymbol{n}_{>})), \qquad \text{Eq. S11}$$

Therefore:

$$\widetilde{\boldsymbol{n}}_{<} = \frac{1}{\delta}\boldsymbol{W}_{<}^{\boldsymbol{T}}\left(\boldsymbol{s} + \boldsymbol{W}_{>}\lambda \mathrm{sign}(\boldsymbol{n}_{>})\right)\tilde{t} - \frac{1}{\delta}\boldsymbol{W}_{<}^{\boldsymbol{T}}\boldsymbol{W}_{>}\int_{\boldsymbol{0}}^{\tilde{\boldsymbol{t}}}\widetilde{\boldsymbol{n}}_{>}(\tau)d\tau,$$

$$\widetilde{\boldsymbol{n}}_{<} = \frac{1}{\delta}\boldsymbol{W}_{<}^{\boldsymbol{T}}\left(\boldsymbol{s} + \boldsymbol{W}_{>}\lambda \mathrm{sign}(\boldsymbol{n}_{>})\right)\tilde{t} - \frac{1}{\delta}\boldsymbol{W}_{<}^{\boldsymbol{T}}\boldsymbol{W}_{>}\left[\boldsymbol{A}^{-\boldsymbol{1}}\left(\tilde{t} - \delta\boldsymbol{A}^{-1}\left(1 - \exp\left(\frac{-\boldsymbol{A}}{\delta}\tilde{t}\right)\right)\right)\left(\boldsymbol{W}_{>}^{\boldsymbol{T}}\boldsymbol{s} + \boldsymbol{B}\right)\right].$$

Eq. S12

Recall that these expressions hold only until the next threshold crossing:

$$t^{i+1} = \min\left[l^{i+1}\right], \qquad \text{Eq. S13}$$

Once the time and the neuron crossing threshold is found, we update the relevant variables and repeat the calculation. Thus, the solution proceeds from threshold crossing to threshold crossing.