Supplementary Material of
'Delay Compensation with Dynamical Synapses'

I. FIRST ORDER PERTURBATION EQUATIONS

In this section we derive the dynamical equations for the first order perturbation in subsection 3.1. Substituting Eqs. (12) and (13) into Eq. (1) results in the following terms:

(a) The time derivative of $u(x,t)$ given by

$$\frac{du}{dt}(x,t) = \tau_s \frac{du_0}{dt} \exp\left[-\frac{(x-z)^2}{4a^2}\right] + u_0 \left(\frac{\tau_s}{2a}\right) \frac{dz}{dt} \left(\frac{x-z}{a}\right) \exp\left[-\frac{(x-z)^2}{4a^2}\right]. \tag{S1}$$

(b) The integral in the right hand side of Eq. (1) given by

$$\begin{aligned}
& \frac{\rho}{B} \int dx' J(x-x')\, p(x',t)\, u(x',t)^2 \\
= & \frac{\rho J_0 u_0^2}{B\sqrt{2\pi}a} \int dx' \exp\left[-\frac{(x-x')^2}{2a^2}\right] \exp\left[-\frac{(x'-z)^2}{2a^2}\right] \\
& -\frac{\rho J_0 u_0^2}{B\sqrt{2\pi}a} p_0 \int dx' \exp\left[-\frac{(x-x')^2}{2a^2}\right] \exp\left[-\frac{(x'-z)^2}{a^2}\right] \\
& +\frac{\rho J_0 u_0^2}{B\sqrt{2\pi}a} p_1 \int dx' \exp\left[-\frac{(x-x')^2}{2a^2}\right] \left(\frac{x'-z}{a}\right) \exp\left[-\frac{(x-z)^2}{a^2}\right] \\
= & \frac{\rho J_0 u_0^2}{B\sqrt{2}} \exp\left[-\frac{(x-z)^2}{4a^2}\right] - \frac{\rho J_0 u_0^2}{B\sqrt{3}} p_0 \exp\left[-\frac{(x-z)^2}{3a^2}\right] \\
& +\frac{\rho J_0 u_0^2}{B3\sqrt{3}} p_1 \left(\frac{x-z}{a}\right) \exp\left[-\frac{(x-z)^2}{3a^2}\right].
\end{aligned} \tag{S2}$$

Since the basis functions are proportional to $\exp\left[-(x-z)^2/(4a^2)\right]$ and $\left[(x-z)/a\right]\exp\left[-(x-z)^2/(4a^2)\right]$, we multiply both sides of Eq. (1) by $\exp\left[-(x-z)^2/(4a^2)\right]$ and integrate over x. This results in the equation

$$\tau_s \frac{du_0}{dt} = Ae^{-\frac{(z_0-z)^2}{8a^2}} + \frac{\rho J_0 u_0^2}{B\sqrt{2}} - \frac{\rho J_0 u_0^2}{B\sqrt{3}} p_0 \sqrt{\frac{6}{7}} - u_0. \tag{S3}$$

After rescaling the variables, we obtain Eq. (14).

Similarly, multiplying both sides of Eq. (1) by $\left[(x-z)/a\right]\exp\left[-(x-z)^2/(4a^2)\right]$ and integrating over x, we obtain Eq. (15).

Respectively, multiplying both sides of Eq. (2) by $\exp\left[-(x-z)^2/(2a^2)\right]$ and $\left[(x-z)/a\right]\exp\left[-(x-z)^2/(2a^2)\right]$ and integrating over x, we obtain Eqs. (16) and (17).

II. DERIVATION OF THE PERFECT TRACKING SOLUTION

For perfect tracking, we have $z = z_0$. At the steady state, $d\bar{u}_0/dt$, dp_0/dt and dp_1/dt vanish, and $dz/dt = v$. Eqs. (14) to (17) reduce to

$$\bar{u}_0 = \frac{\bar{u}_0^2}{\sqrt{2}B}\left(1 - p_0\sqrt{\frac{4}{7}}\right) + \bar{A}, \tag{S4}$$

$$\frac{v\tau_s}{a} = \frac{2\bar{u}_0}{B}\left(\frac{2}{7}\right)^{3/2} p_1, \tag{S5}$$

$$p_0 = \frac{\bar{\beta}\bar{u}_0^2}{B}\left(1 - p_0\sqrt{\frac{2}{3}}\right) - \frac{v\tau_d}{2a}p_1, \tag{S6}$$

$$\frac{v\tau_d}{a}p_0 = \left[1 + \frac{\bar{\beta}\bar{u}_0^2}{B}\left(\frac{2}{3}\right)^{3/2}\right] p_1. \tag{S7}$$

From Eq. (S4), we have

$$\frac{\bar{u}_0}{B} = \frac{\sqrt{2}\left(1 - \frac{\bar{A}}{\bar{u}_0}\right)}{1 - \sqrt{\frac{4}{7}}p_0}. \tag{S8}$$

Substituting this expression into Eq. (S5) yields

$$\frac{v\tau_s}{a} = \left(\frac{4}{7}\right)^{3/2} p_1 \frac{1 - \frac{\bar{A}}{\bar{u}_0}}{1 - \sqrt{\frac{4}{7}}p_0}. \tag{S9}$$

Using Eq. (S7) to eliminate p_1, we obtain an equation for p_0,

$$\frac{v\tau_s}{a} = \left(\frac{4}{7}\right)^{3/2} \frac{1 - \frac{\bar{A}}{\bar{u}_0}}{1 - \sqrt{\frac{4}{7}}p_0} \frac{v\tau_d}{a} \frac{p_0}{1 + \frac{\bar{\beta}\bar{u}_0^2}{B}\left(\frac{2}{3}\right)^{3/2}}. \tag{S10}$$

The solution for p_0 is

$$p_0 = \frac{\frac{\tau_s}{\tau_d}\left[1 + \frac{\bar{\beta}\bar{u}_0^2}{B}\left(\frac{2}{3}\right)^{3/2}\right]}{\left(\frac{4}{7}\right)^{3/2}\left(1 - \frac{\bar{A}}{\bar{u}_0}\right) + \sqrt{\frac{4}{7}}\frac{\tau_s}{\tau_d}\left[1 + \frac{\bar{\beta}\bar{u}_0^2}{B}\left(\frac{2}{3}\right)^{3/2}\right]}. \tag{S11}$$

Substituting Eq. (S11) into Eq. (S8), we obtain the expression for $\bar{u}_0/B$, given by

$$\frac{\bar{u}_0}{B} = \sqrt{2}\left(1 - \frac{\bar{A}}{\bar{u}_0}\right) + \frac{7}{\sqrt{8}}\frac{\tau_s}{\tau_d}\left[1 + \frac{\bar{\beta}\bar{u}_0^2}{B}\left(\frac{2}{3}\right)^{3/2}\right]. \tag{S12}$$

Substituting Eq. (S12) into Eq. (S5), we obtain an expression for $v\tau_s/(ap_1)$. Substituting Eq. (S11) into Eq. (S6), we obtain an expression for $v\tau_s p_1/a$. The quotient and product of these two expressions yield, respectively,

$$p_1 = \sqrt{2\frac{\tau_s}{\tau_d}}\frac{\sqrt{\frac{\bar{\beta}\bar{u}_0^2}{B}\left(\frac{4}{7}\right)^{3/2}\left(1 - \frac{\bar{A}}{\bar{u}_0}\right) - \frac{\tau_s}{\tau_d}\left[1 + \frac{\bar{\beta}\bar{u}_0^2}{B}\left(\frac{2}{3}\right)^{3/2}\right]\left[1 + \frac{\bar{\beta}\bar{u}_0^2}{B}\left(\sqrt{\frac{2}{3}} - \sqrt{\frac{4}{7}}\right)\right]}}{\left(\frac{4}{7}\right)^{3/2}\left(1 - \frac{\bar{A}}{\bar{u}_0}\right) + \sqrt{\frac{4}{7}}\frac{\tau_s}{\tau_d}\left[1 + \frac{\bar{\beta}\bar{u}_0^2}{B}\left(\frac{2}{3}\right)^{3/2}\right]}, \tag{S13}$$

$$\frac{v\tau_s}{a} = \sqrt{2\frac{\tau_s}{\tau_d}\left\{\frac{\bar{\beta}\bar{u}_0^2}{B}\left(\frac{4}{7}\right)^{3/2}\left(1 - \frac{\bar{A}}{\bar{u}_0}\right) - \frac{\tau_s}{\tau_d}\left[1 + \frac{\bar{\beta}\bar{u}_0^2}{B}\left(\frac{2}{3}\right)^{3/2}\right]\left[1 + \frac{\bar{\beta}\bar{u}_0^2}{B}\left(\sqrt{\frac{2}{3}} - \sqrt{\frac{4}{7}}\right)\right]\right\}}. \tag{S14}$$

Equations (S11) to (S14) enable us to express the variables p_0, $\bar{u}_0/B$, p_1 and $v\tau_s/a$ in terms of the parameters $\bar{\beta}\bar{u}_0^2/B$, τ_s/τ_d and $\bar{A}/\bar{u}_0$. Since $\bar{u}_0^2/B$ is the rescaled firing rate at the peak of the bump, the parameter $\bar{\beta}\bar{u}_0^2/B$ is the rate of neurotransmitter consumption at the bump peak after rescaling.

Real solutions exist only if the expression on the right hand side of Eq. (S14) is real. This implies that

$$\frac{\bar{\beta}\bar{u}_0^2}{B} \geq 2\left\{\sqrt{\frac{64}{343}\frac{\tau_d}{\tau_s}}\left(1-\frac{\bar{A}}{\bar{u}_0}\right)-\sqrt{\frac{8}{27}}-\sqrt{\frac{2}{3}}+\sqrt{\frac{4}{7}} + \sqrt{\left[\sqrt{\frac{64}{343}\frac{\tau_d}{\tau_s}}\left(1-\frac{\bar{A}}{\bar{u}_0}\right)-\sqrt{\frac{8}{27}}-\sqrt{\frac{2}{3}}+\sqrt{\frac{4}{7}}\right]^2-4\sqrt{\frac{8}{27}}\left(\sqrt{\frac{2}{3}}-\sqrt{\frac{4}{7}}\right)}\right\}^{-1}. \quad \text{(S15)}$$

Perfect tracking in the limit of vanishing stimulus speed is obtained by equating $v\tau_s/a$ to zero. Hence the expression for $\bar{\beta}_{\text{perfect}}\bar{u}_0^2/B$ is given by the right hand side of Eq. (S15).

III. DEVIATIONS FROM PERFECT TRACKING

For the case that the displacement s/a is non-vanishing but small, Eqs. (14) to (17) become

$$\bar{u}_0 = \frac{\bar{u}_0^2}{\sqrt{2}B}\left(1-p_0\sqrt{\frac{4}{7}}\right)+\bar{A}, \quad \text{(S16)}$$

$$\frac{v\tau_s}{a} = \frac{2\bar{u}_0}{B}\left(\frac{2}{7}\right)^{3/2}p_1-\frac{\bar{A}}{\bar{u}_0}\left(\frac{s}{a}\right), \quad \text{(S17)}$$

$$p_0 = \frac{\bar{\beta}\bar{u}_0^2}{B}\left(1-p_0\sqrt{\frac{2}{3}}\right)-\frac{v\tau_d}{2a}p_1, \quad \text{(S18)}$$

$$\frac{v\tau_d}{a}p_0 = \left[1+\frac{\bar{\beta}\bar{u}_0^2}{B}\left(\frac{2}{3}\right)^{3/2}\right]p_1. \quad \text{(S19)}$$

Since τ_s only appears in Eq. (S17), we can recover the equations (S4) to (S7) if we substitute τ_s by $\tau_s\left[1+\left(\bar{A}/\bar{u}_0\right)\left(s/(v\tau_s)\right)\right]$. Hence the solution is given by

$$\frac{\bar{u}_0}{B}=\sqrt{2}\left(1-\frac{\bar{A}}{\bar{u}_0}\right)+\frac{7}{\sqrt{8}}\frac{\tau_s}{\tau_d}\left[1+\frac{\bar{\beta}\bar{u}_0^2}{B}\left(\frac{2}{3}\right)^{3/2}\right]\left(1+\frac{\bar{A}}{\bar{u}_0}\frac{s}{v\tau_s}\right), \quad \text{(S20)}$$

$$p_0=\frac{\frac{\tau_s}{\tau_d}\left[1+\frac{\bar{\beta}\bar{u}_0^2}{B}\left(\frac{2}{3}\right)^{3/2}\right]}{\left(\frac{4}{7}\right)^{3/2}\left(\frac{1-\bar{A}/\bar{u}_0}{1+\bar{A}s/(\bar{u}_0 v\tau_s)}\right)+\sqrt{\frac{4}{7}}\frac{\tau_s}{\tau_d}\left[1+\frac{\bar{\beta}\bar{u}_0^2}{B}\left(\frac{2}{3}\right)^{3/2}\right]}, \quad \text{(S21)}$$

$$p_1=\sqrt{2\frac{\tau_s}{\tau_d}}\frac{\sqrt{\frac{\bar{\beta}\bar{u}_0^2}{B}\left(\frac{4}{7}\right)^{3/2}\left(\frac{1-\bar{A}/\bar{u}_0}{1+\bar{A}s/(\bar{u}_0 v\tau_s)}\right)-\frac{\tau_s}{\tau_d}\left[1+\frac{\bar{\beta}\bar{u}_0^2}{B}\left(\frac{2}{3}\right)^{3/2}\right]\left[1+\frac{\bar{\beta}\bar{u}_0^2}{B}\left(\sqrt{\frac{2}{3}}-\sqrt{\frac{4}{7}}\right)\right]}}{\left(\frac{4}{7}\right)^{3/2}\left(\frac{1-\bar{A}/\bar{u}_0}{1+\bar{A}s/(\bar{u}_0 v\tau_s)}\right)+\sqrt{\frac{4}{7}}\frac{\tau_s}{\tau_d}\left[1+\frac{\bar{\beta}\bar{u}_0^2}{B}\left(\frac{2}{3}\right)^{3/2}\right]}, \quad \text{(S22)}$$

$$\frac{v\tau_s}{a} = \left\{2\frac{\tau_s}{\tau_d}\left\{\frac{\bar{\beta}\bar{u}_0^2}{B}\left(\frac{4}{7}\right)^{3/2}\left(\frac{1-\bar{A}/\bar{u}_0}{1+\bar{A}s/(\bar{u}_0 v\tau_s)}\right) -\frac{\tau_s}{\tau_d}\left[1+\frac{\bar{\beta}\bar{u}_0^2}{B}\left(\frac{2}{3}\right)^{3/2}\right]\left[1+\frac{\bar{\beta}\bar{u}_0^2}{B}\left(\sqrt{\frac{2}{3}}-\sqrt{\frac{4}{7}}\right)\right]\right\}^{1/2}\right. . \quad \text{(S23)}$$

We can invert Eq. (S23) to yield an expression for s/a, which reads

$$\frac{s}{a} = -\frac{\bar{u}_0}{\bar{A}}\frac{v\tau_s}{a}\left\{1 - \frac{2\frac{\tau_s}{\tau_d}\frac{\bar{\beta}\bar{u}_0^2}{B}\left(\frac{4}{7}\right)^{3/2}\left(1-\frac{\bar{A}}{\bar{u}_0}\right)}{\left(\frac{v\tau_s}{a}\right)^2 + 2\left(\frac{\tau_s}{\tau_d}\right)^2\left[1+\frac{\bar{\beta}\bar{u}_0^2}{B}\left(\frac{2}{3}\right)^{3/2}\right]\left[1+\frac{\bar{\beta}\bar{u}_0^2}{B}\left(\sqrt{\frac{2}{3}}-\sqrt{\frac{4}{7}}\right)\right]}\right\}. \tag{S24}$$

When $v\tau_s/a \ll 1$, we can approximate Eq. (S24) by the series expansion

$$\begin{aligned}\frac{s}{a} \approx& -\frac{\bar{u}_0}{\bar{A}}\frac{v\tau_s}{a}\left\{1 - \frac{2\frac{\tau_s}{\tau_d}\frac{\bar{\beta}\bar{u}_0^2}{B}\left(\frac{4}{7}\right)^{3/2}\left(1-\frac{\bar{A}}{\bar{u}_0}\right)}{2\left(\frac{\tau_s}{\tau_d}\right)^2\left[1+\frac{\bar{\beta}\bar{u}_0^2}{B}\left(\frac{2}{3}\right)^{3/2}\right]\left[1+\frac{\bar{\beta}\bar{u}_0^2}{B}\left(\sqrt{\frac{2}{3}}-\sqrt{\frac{4}{7}}\right)\right]}\right.\\ &\left.+\frac{2\frac{\tau_s}{\tau_d}\frac{\bar{\beta}\bar{u}_0^2}{B}\left(\frac{4}{7}\right)^{3/2}\left(1-\frac{\bar{A}}{\bar{u}_0}\right)}{\left\{2\left(\frac{\tau_s}{\tau_d}\right)^2\left[1+\frac{\bar{\beta}\bar{u}_0^2}{B}\left(\frac{2}{3}\right)^{3/2}\right]\left[1+\frac{\bar{\beta}\bar{u}_0^2}{B}\left(\sqrt{\frac{2}{3}}-\sqrt{\frac{4}{7}}\right)\right]\right\}^2}\left(\frac{v\tau_s}{a}\right)^2\right\}.\end{aligned} \tag{S25}$$

For the case of perfect tracking in the limit of vanishing stimulus speed, the lowest order term in Eq. (S25) vanishes, and we have

$$\frac{s}{a} \approx -\frac{\bar{u}_0}{2\bar{A}}\frac{\tau_s}{\tau_d}\left[1+\frac{\bar{\beta}\bar{u}_0^2}{B}\left(\frac{2}{3}\right)^{3/2}\right]^{-1}\left[1+\frac{\bar{\beta}\bar{u}_0^2}{B}\left(\sqrt{\frac{2}{3}}-\sqrt{\frac{4}{7}}\right)\right]^{-1}\left(\frac{v\tau_s}{a}\right)^3. \tag{S26}$$

Hence we arrive at Eq. (18), with

$$C = \left[1+\frac{\bar{\beta}\bar{u}_0^2}{B}\left(\frac{2}{3}\right)^{3/2}\right]^{-1}\left[1+\frac{\bar{\beta}\bar{u}_0^2}{B}\left(\sqrt{\frac{2}{3}}-\sqrt{\frac{4}{7}}\right)\right]^{-1}, \tag{S27}$$

which is less than 1.