Approximating Equilibria in Sequential Auctions with Incomplete Information and Multi-Unit Demand: Supplemental Material

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1 Three economics models

This section briefly summarizes the three economic models we implemented in the Experiments section in the paper. All those models involve drawing valuations from a fixed distribution, which we took to be $Unif[0,1]$ in experiments. Please refer to the original papers for more information.

- **Weber** [3]: $n \geq 3$ bidders, each demanding one unit of the good and valuing it at θ_i, an independent draw from a bounded distribution.

  Take the perspective of a bidder with valuation x and rank the valuation of her opponents as $Y^1 \geq ... \geq Y^{n-1}$. Then bidding $b(x) = E[Y^2|Y^2 < x]$ in the first round (and bidding truthfully in the second round) constitute a symmetric equilibrium. We denote this equilibrium strategy σ^W.

- **Katzman**[1]: $n \geq 2$ bidders. Valuation of each is determined by two independent draws from a twice differentiable, atomless, bounded distribution. Each bidder ranks her two draws into the higher one H_i and the lower one L_i, each representing her valuation for the first and the second good obtained.

  Take the perspective of bidder 1 who has valuation (H_1, L_1), and define Y^2 as the second highest among all opponent valuations $\{H_2, L_2, ..., H_n, L_n\}$. Then bidding $b(H_1) = E[Y^2|Y^2 < H_1]$[1] (and bidding truthfully in the second round) constitute a symmetric equilibrium. We denote this equilibrium strategy σ^K.

- **Menezes** [2]: $n \geq 2$ bidders, each drawing her type x from a bounded distribution. The bidder then values one good at x and two goods at $\delta(x)$, where $\delta()$ is a publicly known, strictly increasing function. Note that $\delta(x)-x$ is the bidders' marginal value for the second good, and the two goods have positive synergy (complements) if $\delta(x) > 2x$ and negative synergy (substitutes) if $\delta(x) < 2x^2$.

[1] We express the same function differently from the authors' initial form to illustrate the similarity, in our mind, to Weber's result.

[2] In simulations, we take $\delta(x) - x = x^2 \leq x$ for negative synergy cases and $\delta(x) - x = \sqrt{x} \geq x$ for positive synergy cases.

Take the perspective of a bidder with type x and rank opponent types into $Y^1 \geq ... \geq Y^{N-1}$. Then bidding as follows in the first round (and truthfully in the second round) characterize a symmetric equilibrium:

– $b(x) = \delta(x) - x$ if $n = 2$
– $b(x) = E[\max(\delta(x) - x, Y^2)|Y^2 < x]$[3] if $n > 2$

We call this strategy profile σ^M.

We also extended the Menezes model to three rounds in the Experiment section simply by adding adding an extra function $\delta_2(x)$ for the valuation of 3 goods. Thus the marginal value of the third good will be $\delta_2(x) - \delta(x)$.

2 Derivation: Menezes equilibrium σ^M unstable

In the Menezes model, we derive the set of best responses for the case of $n > 3$ and decreasing marginal values ($\delta(x) < 2x$); the case for $n = 2$ and increasing marginal values can be derived analogously. We show that a bidder with valuation x maximize her utility when pretending to be any type within $[\delta(x) - x, x]$.

Take the perspective of a bidder with valuation x and rank opponent valuations into order statistics $Y_1 \geq ... \geq Y_{n-1}$. We will use succinct notation for their cdfs and pdfs:

- cdfs $F_1(y_1) \doteq P(Y_1 < y_1)$, $F_2(y_2) \doteq P(Y_2 < y_2)$, $F_{1,2}(y_1, y_2) = P(Y_1 < y_1, Y_2 < y_2)$
- pdfs $f_1(y_1) \doteq dF_1(y_1)/dy_1$, $f_2(y_2) \doteq dF_2(y_2)/dy_2$
- joint pdf $\bar{f}_1(y_1)\bar{f}_2(y_2) = dF_{1,2}(y_1, y_2)/dy_1 dy_2$. (Note that $\bar{f}_1(y_1) \neq f_1(y_1)$)

Valuation cdf F is bounded, so we assume $F(0) = 0$ and $F(1) = 1$ without loss of generality. Let's denote $a \vee b \doteq \max(a, b)$ and $a^+ \doteq \max(a, 0)$ for all $a, b \in \mathbb{R}$. Finally we write $\delta'(x) \doteq \delta(x) - x$. Suppose the bidder pretends to be type w, then the corresponding profit is:

$$\begin{aligned}
h(w) =& E[(x - \delta'(Y_1)) \vee Y_2)^+ | Y_1 > w] + E[x - b(Y_1) | Y_1 > w] + E[(\delta'(x) - Y_1)^+ | Y_1 < w] && (1)\\
=& E[(x - \delta'(Y_1) \vee Y_2)^+ | Y_1 > w] + E[x - E[\delta'(Y_1) \vee Y_2^* | Y_2^* < Y_1] | Y_1 > w] + E[(\delta'(x) - Y_1)^+ | Y_1 < w] && (2)\\
=& E[(x - (\delta'(Y_1) - Y_1) \vee Y_2)^+ | Y_1 > w] + E[x - \delta'(Y_1) \vee Y_2^* | Y_1 > w] + E[(\delta'(x) - Y_1)^+ | Y_1 < w] && (3)\\
=& E[(x - \delta'(Y_1) \vee Y_2)^+ | Y_1 > w] + E[x - \delta'(Y_1) \vee Y_2 | Y_1 > w] + E[(\delta'(x) - Y_1)^+ | Y_1 < w] && (4)\\
=& \int_w^1 \int_0^{y_1} [x - \delta'(y_1) \vee y_2]^+ \bar{f}_1(y_1)\bar{f}_2(y_2) dy_2 dy_1 && (5)\\
&+ \int_0^w \int_0^{y_1} [x - \delta'(y_1) \vee y_2] \bar{f}_1(y_1)\bar{f}_2(y_2) dy_2 dy_1 + \int_0^w [\delta'(x) - y_1]^+ f_1(y_1) dy_1 && (6)
\end{aligned}$$

Differentiating with respect to w:

$$\begin{aligned}
h'(w) =& -\int_0^w [x - \delta'(w) \vee y_2]^+ \bar{f}_1(w)\bar{f}_2(y_2) dy_2 + \int_0^w [x - \delta'(w) \vee y_2] \bar{f}_1(w)\bar{f}_2(y_2) dy_2\\
&+ [\delta'(x) - w]^+ f_1(w) && (7)\\
=& \underbrace{\int_0^w \left\{-[x - \delta'(w) \vee y_2]^+ + [x - \delta'(w) \vee y_2]\right\} \bar{f}_1(w)\bar{f}_2(y_2) dy_2}_{A} + \underbrace{[\delta'(x) - w]^+ f_1(w)}_{B}\\
& && (8)
\end{aligned}$$

All densities are positive by assumption, so it suffices to look at other parts. Two observations about (9):

[3] We changed the notation a bit to show relationship with Weber's result.

- Term $A \leq 0$, and $A = 0$ when $w \in [0, x]$. That can be seen in the integrand. Since $w < x \Rightarrow \delta'(w) < x$ and $y_2 < w < x$, so $-[x - \delta'(w) \vee y_2] + [x - \delta'(w) \vee y_2] = -[x - \delta'(w) \vee y_2] + [x - \delta'(w) \vee y_2] = 0$.
- Term $B \geq 0$ and $= 0$ when $w \in [\delta'(x), 1]$.

- If $w < \delta'(x)$, $h'(w) > 0$, so the bidder will want to increase w.
- if $w \in [\delta'(x), x] = [\delta(x) - x, x]$, $h'(w) = 0$, which means that the agent is indifferent.
- If $w > x$, $h'(w) < 0$, so the bidder will want to decrease w.

References

[1] B. Katzman. A Two Stage Sequential Auction with Multi-Unit Demands,. *Journal of Economic Theory*, 86(1):77–99, May 1999.

[2] F. M. Menezes and P. K. Monteiro. Synergies and Price Trends in Sequential Auctions. *Review of Economic Design*, 8:85–98, 2003.

[3] R. J. Weber. Multiple-Object Auctions. In R. Engelbrecht-Wiggans, R. M. Stark, and M. Shubik, editors, *Competitive Bidding, Auctions, and Procurement*, pages 165–191. New York University Press, 1983.