Appendix

Proof of Lemma 3: Since, $\mathrm{poi}(\lambda, \mu) = e^{-\lambda}\lambda^{\mu}/\mu!$,

$$\begin{aligned}
\mathbb{E}_{\mu\sim\mathrm{poi}(\lambda_1)}\left[\frac{\mathrm{poi}(\lambda_2,\mu)}{\mathrm{poi}(\lambda_0,\mu)}\right] &= \sum_{\mu=0}^{\infty}\frac{\mathrm{poi}(\lambda_1,\mu)\mathrm{poi}(\lambda_2,\mu)}{\mathrm{poi}(\lambda_0,\mu)}\\
&= \exp\left(\lambda_0-\lambda_2-\lambda_1\right)\sum_{\mu=0}^{\infty}\left(\frac{\lambda_1\lambda_2}{\lambda_0}\right)^{\mu}\frac{1}{\mu!}\\
&= \exp\left(\lambda_0-\lambda_1-\lambda_2\right)\exp\left(\frac{\lambda_1\lambda_2}{\lambda_0}\right)\\
&= \exp\left(\frac{(\lambda_1-\lambda_0)(\lambda_2-\lambda_0)}{\lambda_0}\right). \qquad \blacksquare
\end{aligned}$$

Proof of Lemma 4: Let $Q^* = \mathrm{argmin}_Q \max_P D(P||Q)$, then $Q^{*\prime}$ satisfies $D(P'||Q^{*\prime}) \le \overline{R}(\mathcal{P})$ for all $P' \in \mathcal{P}$. $\blacksquare$

Proof of Lemma 5: Let P_1 and Q_1 be distributions achieving redundancy $\overline{R}(\mathcal{P}_{\mathcal{A}})$ and $\overline{R}(\mathcal{P}_{\mathcal{B}})$. Then,

$$\begin{aligned}
\overline{R}(\mathcal{P}_{\mathcal{A}})+\overline{R}(\mathcal{P}_{\mathcal{B}}) &= \sup_P\left(\sum_{a\in\mathcal{A}}P(a)\log\frac{P(a)}{P_1(a)}\right)+\sup_Q\left(\sum_{b\in\mathcal{B}}Q(b)\log\frac{Q(b)}{Q_1(b)}\right)\\
&\ge \sup_{P\cdot Q\in\mathcal{P}}\left(\sum_{a\in\mathcal{A}}P(a)\log\frac{P(a)}{P_1(a)}+\sum_{b\in\mathcal{B}}Q(b)\log\frac{Q(b)}{Q_1(b)}\right)\\
&= \sup_{P\cdot Q\in\mathcal{P}}\left(\sum_{(a,b)\in(\mathcal{A},\mathcal{B})}P(a)Q(b)\log\frac{P(a)Q(b)}{P_1(a)Q_1(b)}\right)\\
&= \sup_{P\cdot Q\in\mathcal{P}} D(P\cdot Q||P_1\cdot Q_1)\\
&\ge \overline{R}(\mathcal{P}). \qquad \blacksquare
\end{aligned}$$

Proof of Lemma 7: For $1 \le i \le T$, let the distribution $Q_i = \mathrm{argmin}_Q \max_{P\in\mathcal{P}_i} D(P||Q)$ achieve the redundancy bound $\overline{R}(\mathcal{P}_i)$, and $Q = \frac{1}{T}\sum_{i=1}^{T} Q_i$Then for all i and all $a \in \mathcal{A}$, $Q(a) \ge Q_i(a)/T$, and hence for any $P \in P_1 \cup P_2 \ldots \cup P_T$,

$$D(P||Q) = \mathrm{E}_{X\sim P}\Big[\log\frac{P(X)}{Q(X)}\Big] \le \log T + \mathrm{E}_P\Big[\log\frac{P(X)}{Q_i(X)}\Big] \le \log T + \overline{R}(\mathcal{P}_i).$$

Considering maximum over all distributions in $\mathcal{P}$ yields the desired result. $\blacksquare$

Proof of Lemma 9: As mentioned in the proof sketch, we construct the distribution Λ^* as follows. Pick any distribution $\Lambda' \in \mathcal{I}_\tau$ and let $\Lambda^* = \bigcup_{j=1}^{B}\Lambda_j^*$, where $\Lambda_j^* = \{\lambda_{j,1}^*, \ldots, \lambda_{j,m_j}^*\}$ is such that

$$\lambda_{j,1}^* \stackrel{\mathrm{def}}{=} \lambda_{j,2}^* \stackrel{\mathrm{def}}{=} \cdots \stackrel{\mathrm{def}}{=} \lambda_{j,m_j}^* \stackrel{\mathrm{def}}{=} \lambda_j^* \stackrel{\mathrm{def}}{=} \frac{1}{m_j}\sum_{l=1}^{m_j}\lambda'_{j,l},$$

for $j = 1, 2, \ldots, B$. In other words, $\Lambda^* \in \mathcal{I}_\tau$ is obtained from an arbitrary $\Lambda' \in \mathcal{I}_\tau$ by making all the m_j elements of Λ_j^* equal to $\sum_{l=1}^{m_j}\lambda'_{l,j}/m_j$, the average of the elements in Λ'_j, for $j = 1, 2, \ldots, B$.

For any $\Lambda \in \mathcal{I}_\tau$, we analyze $D(\Lambda||\Lambda^*)$. Let $\overline{\varphi}_j$ be the profile generated by $\Lambda_j = \{\lambda_{j,1}, \ldots, \lambda_{j,m_j}\}$. Since $\overline{\varphi}_{\mathrm{med}} = \overline{\varphi}_1 \cup \ldots \overline{\varphi}_B = f((\overline{\varphi}_1, \overline{\varphi}_2, \ldots, \overline{\varphi}_B))$, a function of the $B-$tuple $(\overline{\varphi}_1, \overline{\varphi}_2, \ldots, \overline{\varphi}_B)$. By independence of sampling, $(\overline{\varphi}_1, \overline{\varphi}_2, \ldots, \overline{\varphi}_B)$ is distributed as $\Lambda_1 \times \ldots \Lambda_B$. U

$$D(\Lambda||\Lambda^*) \le D\big(\prod_{j=1}^{B}\Lambda_j \;||\; \prod_{j=1}^{B}\Lambda_j^*\big) \;=\; \sum_{j=1}^{B}D(\Lambda_j||\Lambda_j^*) \;=\; \sum_{j=1}^{B}\mathrm{E}_{\Lambda_j}\Big[\log\frac{\Lambda_j(\overline{\varphi}_j)}{\Lambda_j^*(\overline{\varphi}_j)}\Big] \;\le\; \sum_{j=1}^{B}\log\Big(\mathrm{E}_{\Lambda_j}\Big[\frac{\Lambda_j(\overline{\varphi}_j)}{\Lambda_j^*(\overline{\varphi}_j)}\Big]\Big),$$

where the first inequality follows from Lemma (4), and the last from the concavity of the logarithms.

We bound each of the summands as follows. Let $\overline{\varphi}_j = \{\mu_1, \mu_2, \ldots, \mu_{m_j}\}$, where μ_j is generated by λ_j. Using Equation (5), if $\overline{\varphi}_j = \{\mu_1, \mu_2, \ldots, \mu_{m_j}\}$, then

$$\Lambda_j(\overline{\varphi}_j) = F(\overline{\varphi}_j) \sum_{\sigma \in S_{m_j}} \prod_{l=1}^{m_j} \mathrm{poi}(\lambda_{j,\sigma(l)}, \mu_l).$$

For Λ_j^*, all the summands are identical, so $\Lambda_j^*(\overline{\varphi}_j) = F(\overline{\varphi}_j) \prod_{l=1}^{m_j} \mathrm{poi}(\lambda_j^*, \mu_l)$. Hence,

$$\frac{\Lambda_j(\overline{\varphi}_j)}{\Lambda_j^*(\overline{\varphi}_j)} = \frac{1}{m!} \sum_{\sigma \in S_{m_j}} \prod_{l=1}^{m_j} \frac{\mathrm{poi}(\lambda_{j,\sigma(l)}, \mu_l)}{\mathrm{poi}(\lambda_j^*, \mu_l)}.$$

By the linearity of expectations and the independence of μ's,

$$\mathrm{E}_{\Lambda_j}\Big[\frac{\Lambda_j(\overline{\varphi}_j)}{\Lambda_j^*(\overline{\varphi}_j)}\Big] \;=\; \frac{1}{m!} \sum_{\sigma \in S_{m_j}} \prod_{l=1}^{m_j} \mathrm{E}_{\Lambda_j}\Big[\frac{\mathrm{poi}(\lambda_{j,\sigma(l)}, \mu_l)}{\mathrm{poi}(\lambda_j^*, \mu_l)}\Big],$$

and since that μ_j is distributed $\lambda_{j,l}$, invoking Lemma 3,

$$\begin{aligned}
\mathrm{E}_{\Lambda_j}\Big[\frac{\Lambda_j(\overline{\varphi}_j)}{\Lambda_j^*(\overline{\varphi}_j)}\Big] \;&=\; \frac{1}{m!} \sum_{\sigma \in S_{m_j}} \prod_{l=1}^{m_j} \mathrm{E}_{\mu_j \sim \mathrm{poi}(\lambda_{j,l})}\Big[\frac{\mathrm{poi}(\lambda_{j,\sigma(l)}, \mu_l)}{\mathrm{poi}(\lambda_j^*, \mu_l)}\Big] \\
&=\; \frac{1}{m!} \sum_{\sigma \in S_{m_j}} \exp\Big(\sum_{l=1}^{m_j} \frac{(\lambda_{j,l} - \lambda_j^*)(\lambda_{j,\sigma(l)} - \lambda_j^*)}{\lambda_j^*}\Big) \\
&\le\; \exp\Big(m_j \frac{\Delta_j^2}{\lambda_j^-}\Big),
\end{aligned}$$

where in the last inequality, we use $|\lambda_j^* - \lambda_{j,l}^*| \le \Delta_j$ and $\lambda_j^* \ge \lambda_j^-$. Combining the above inequalities, for all $\Lambda \in \mathcal{I}_\tau$,

$$D(\Lambda||\Lambda^*) \;\le\; \sum_{j=1}^{B} \log\Big(\mathrm{E}_{\Lambda_j}\Big[\frac{\Lambda_j(\overline{\varphi}_j)}{\Lambda_j^*(\overline{\varphi}_j)}\Big]\Big) \;\le\; \sum_{j=1}^{B} m_j \frac{\Delta_j^2}{\lambda_j^-}.$$

So,

$$\overline{R}(\mathcal{I}_\tau) = \min_Q \max_{\Lambda \in \mathcal{I}_\tau} D(\Lambda||Q) \le \max_{\Lambda \in \mathcal{I}_\tau} D(\Lambda||\Lambda^*) \;\le\; \sum_{j=1}^{B} m_j \frac{\Delta_j^2}{\lambda_j^-}. \qquad \blacksquare$$

Proof of Lemma 14: We construct a map f from the set of profiles to $\mathcal{L}$ and then show that for any $\Lambda \in \mathcal{L}$, if $\overline{\varphi} \sim \Lambda$

$$P(f(\overline{\varphi}) \neq \Lambda) < \epsilon. \tag{6}$$

Let $\overline{\varphi} = \{\mu_1, \mu_2, \ldots\}$ be a profile. For each $j = 1, 2, \ldots, K$, let

$$x_i = \begin{cases} 1 & \text{if } \exists j \text{ such that } i = \operatorname*{argmin}_r |\mu_j - \lambda_r^*| \\ 0 & \text{otherwise.} \end{cases}$$

In other words, for each multiplicity μ_j we set the coordinate x_i to 1 if μ_j is closest to λ_i^*. Let $\overline{x} = x_1 \ldots x_K$. Let $\hat{c} \in \mathcal{C}$ be the code with minimum Hamming distance from $\overline{x}$. Then,

$$f(\overline{\varphi}) = \Lambda_{\hat{c}}.$$

Let $\overline{c} \in \mathcal{C}$. We now analyze Equation 6 for $\Lambda_{\overline{c}}$. Two adjacent λ^*'s are separated by

$$\Delta_i \triangleq \lambda_{i+1}^* - \lambda_i^* = (2i+1)C > 2\sqrt{C\lambda_i^*}. \tag{7}$$

Let $\lambda_i^* \in \Lambda_{\overline{c}}$ be any element. Let Y_i be a random variable that is 1 if the multiplicity μ_i generated by λ_i^* is closest to a λ_j^*, $j \neq i$ and 0 otherwise. Using the fact that the minimum distance of the code is αK, the probability of error is at most the probability that $\sum Y_i \geq \frac{\alpha K}{2}$. So, for $\overline{\varphi} \sim \Lambda_{\overline{c}}$

$$P(f(\overline{\varphi}) \neq \Lambda_{\overline{c}}) \leq P\left(\sum Y_i \geq \frac{\alpha K}{2}\right).$$

Using Equation (7), an application of Chernoff bound,

$$P(Y_i = 1) \leq P\left(|\mu_i - \lambda_i^*| \geq \frac{\Delta_i}{2}\right) \leq e^{-C/4}.$$

So,

$$\mathbb{E}\left[\sum Y_i\right] \leq e^{-C/4} K.$$

By Markov's Inequality,

$$P(f(\overline{\varphi}) \neq \Lambda_{\overline{c}}) \leq \frac{2e^{-C/4}}{\alpha},$$

thus proving the result. ■