Supplementary material for "Coding efficiency and detectability of rate fluctuations with non-Poisson neuronal firing"

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In this supplementary material, we derive the KL divergence of the IG distribution (14) and lognormal distribution (15), and the formula for the lower bound (21) in the main manuscript.

S-1 Derivation of the KL divergence

S-1.1 Inverse Gaussian distribution

Inserting Eq. (7) into equation Eq. (5) and assuming that the time scale of the rate fluctuation $\lambda(t)$ is longer than the mean ISI, so that the firing rate in each ISI can be approximated to be constant, we obtain

$$\begin{aligned} D_\kappa(\lambda(t)||\mu) &= \frac{1}{2}\lim_{n\to\infty}\frac{1}{t_n - t_0}\sum_{i=1}^{n}\left\{-\log\lambda(t_i) - \frac{\kappa}{\Lambda(t_i)-\Lambda(t_{i-1})} + \frac{\kappa}{\mu(t_i - t_{i-1})}\right\} \\ &\quad + \frac{\mu}{2}\log\mu. \end{aligned} \tag{S-1.1}$$

Each terms in Eq. (S-1.1) is evaluated as follows:

i) By assuming a long time scale in which a serial correlation of spikes is negligible, the first term in rhs of Eq. (S-1.1) can be evaluated in the same way as (12):

$$\lim_{n\to\infty}\frac{1}{t_n - t_0}\sum_{i=1}^{n}\log\lambda(t_i) = \langle\lambda\log\lambda\rangle_\lambda. \tag{S-1.2}$$

ii) Under the assumption that the rate fluctuation has a long time scale, the second term in rhs of (S-1.1) becomes

$$\frac{1}{t_n - t_0}\sum_{i=1}^{n}\frac{1}{\Lambda(t_i)-\Lambda(t_{i-1})} = \frac{n}{t_n - t_0}\frac{1}{n}\sum_{i=1}^{n}\frac{1}{\lambda(t_i)(t_i - t_{i-1})}. \tag{S-1.3}$$

In order to evaluate the rhs of the above equation, we use the fact that the expectation of $1/(\lambda x)$ with respect to the inverse Gaussian distribution $\lambda f_\kappa(\lambda x)$ is given by

$$\int_0^\infty \frac{1}{\lambda x}\lambda f_\kappa(\lambda x)dx = \frac{1}{\kappa} + 1. \tag{S-1.4}$$

Note that this does not depend on λ. Thus, we obtain

$$\lim_{n\to\infty} \frac{n}{t_n - t_0} \frac{1}{n} \sum_{i=1}^{n} \frac{1}{\lambda(t_i)(t_i - t_{i-1})} = \mu\left(\frac{1}{\kappa} + 1\right). \tag{S-1.5}$$

iii) To evaluate the third term in rhs of Eq. (S-1.1), let $\{t_i^{(\lambda)} - t_{i-1}^{(\lambda)}\}$ be a set of ISIs that are generated with the rate λ, and n_λ be the number of the ISIs included in this set. Using Eq. (S-1.4), we obtain

$$\frac{1}{n_\lambda} \sum_{i=1}^{n_\lambda} \frac{1}{t_i^{(\lambda)} - t_{i-1}^{(\lambda)}} \to \lambda\left(\frac{1}{\kappa} + 1\right), \quad \text{as } n_\lambda \to \infty. \tag{S-1.6}$$

On the other hand, the ratio of n_λ to n converges to

$$\frac{n_\lambda}{n} \to \frac{\lambda p(\lambda) d\lambda}{\mu}, \quad \text{as } n \to \infty, \tag{S-1.7}$$

from the law of large number. Using Eqs. (S-1.6) and (S-1.7), we obtain

$$\frac{1}{n} \sum_{i=1}^{n} \frac{1}{t_i - t_{i-1}} \to \frac{1}{\mu}\left(\frac{1}{\kappa} + 1\right) \int_0^\infty \lambda^2 p(\lambda) d\lambda = \left(\frac{1}{\kappa} + 1\right)\left(\mu + \frac{\langle(\lambda - \mu)^2\rangle_\lambda}{\mu}\right), \quad \text{as } n \to \infty. \tag{S-1.8}$$

Substituting Eqs. (S-1.2), (S-1.5) and (S-1.8) into Eq. (S-1.1), the KL divergence for IG distribution is obtained as

$$D_\kappa(\lambda(t)||\mu) \quad = \quad \frac{\mu}{2} \log \mu - \frac{1}{2}\langle \lambda \log \lambda \rangle_\lambda + \frac{\kappa + 1}{2\mu}\langle(\lambda - \mu)^2\rangle_\lambda. \tag{S-1.9}$$

S-1.2 Lognormal distribution

Substituting Eq. (8) into Eq. (5), and using the same approximation leads to

$$\begin{aligned} & D_\kappa(\lambda(t)||\mu) \\ = \; & \frac{\mu}{2} \log \mu + \frac{\mu}{2\kappa}(\log \mu)^2 + \frac{\log \mu}{\kappa} \lim_{n\to\infty} \frac{1}{t_n - t_0} \sum_{i=1}^{n} \log(t_i - t_{i-1}) \\ & - \lim_{n\to\infty} \frac{1}{t_n - t_0} \sum_{i=1}^{n} \left[\frac{1}{2} \log \lambda(t_i) + \frac{1}{2\kappa}\{\log \lambda(t_i)\}^2 + \frac{1}{\kappa} \log \lambda(t_i) \cdot \log(t_i - t_{i-1})\right] \end{aligned} \tag{S-1.10}$$

Each term in Eq. (S-1.10) is evaluated as follows:

i) In the same way as Eq. (12), we obtain

$$\lim_{n\to\infty} \frac{1}{t_n - t_0} \sum_{i=1}^{n} \log \lambda(t_i) = \langle \lambda \log \lambda \rangle_\lambda, \tag{S-1.11}$$

and

$$\lim_{n\to\infty} \frac{1}{t_n - t_0} \sum_{i=1}^{n} \{\log \lambda(t_i)\}^2 = \langle \lambda (\log \lambda)^2 \rangle_\lambda. \tag{S-1.12}$$

ii) Using the fact that the expectation of $\log x$ with respect to the lognormal distribution $\lambda f_\kappa(\lambda x)$ is given by $-\log \lambda - \frac{\kappa}{2}$, and the same argument as the inverse IG distribution (i.e., using Eq. (S-1.7)), we obtain

$$\begin{aligned}
&\lim_{n\to\infty}\frac{1}{t_n-t_0}\sum_{i=1}^{n}\log(t_i-t_{i-1}) \\
&\quad= \lim_{n\to\infty}\frac{n}{t_n-t_0}\frac{1}{n}\sum_{i=1}^{n}\log(t_i-t_{i-1}) \\
&\quad= \int_0^\infty\Big(-\log\lambda-\frac{\kappa}{2}\Big)\lambda p(\lambda)d\lambda, \qquad \text{(S-1.13)}
\end{aligned}$$

and

$$\begin{aligned}
&\lim_{n\to\infty}\frac{1}{t_n-t_0}\sum_{i=1}^{n}\log\lambda(t_i)\cdot\log(t_i-t_{i-1}) \\
&\quad= \lim_{n\to\infty}\frac{n}{t_n-t_0}\frac{1}{n}\sum_{i=1}^{n}\log\lambda(t_i)\cdot\log(t_i-t_{i-1}) \\
&\quad= \int_0^\infty\log\lambda\Big(-\log\lambda-\frac{\kappa}{2}\Big)\lambda p(\lambda)d\lambda. \qquad \text{(S-1.14)}
\end{aligned}$$

Substituting Eqs. (S-1.11)-(S-1.14) into Eq. (S-1.10), the KL divergence for the lognormal distribution is obtained as

$$D_\kappa(\lambda(t)||\mu) = \frac{\mu}{2\kappa}(\log\mu)^2 - \frac{\log\mu}{\kappa}\langle\lambda\log\lambda\rangle_\lambda + \frac{1}{2\kappa}\langle\lambda(\log\lambda)^2\rangle_\lambda. \qquad \text{(S-1.15)}$$

S-2 Derivation of the formula for the lower bound

Here, we derive the formula (21) in the main manuscript. In this derivation, we assume the following conditions for asymptotic analysis:

(A) The time scale of the rate fluctuation is longer than the mean ISI, so that the serial correlation of spikes are negligible.

(B) The amplitude of rate fluctuation relative to the mean rate is small.

(C) A large observation interval $T \gg 1$.

S-2.1 Formulation of the empirical Bayes method

Here, we briefly summarize the empirical Bayes method. We suppose that a spike train $\{t_i\} := \{t_1, t_2, \ldots, t_n\}$ in the interval $[0, T]$ is derived from the time-rescaled renewal process with the gamma ISI distribution, whose firing rate and the shape parameter are given by $\lambda(t)$ and κ, respectively. We also suppose that the firing rate is given by the form:

$$\lambda(t) = e^{x(t)} = \mu + \sigma f(t),$$

where μ is the mean firing rate, and $f(t)$ represent the rate fluctuation such that $\langle f(t)\rangle = 0$ and $\langle f(t)f(t')\rangle = \phi(t-t')$. $x(t) \in \mathbb{R}$ denotes the latent process.

From the given spike train $\{t_i\}$, we wish to evaluate the marginal likelihood function:

$$p_{\nu,\gamma}(\{t_i\}) = \int p_\nu(\{t_i\}|\{x(t)\})p_\gamma(\{x(t)\})\mathcal{D}\{x(t)\}, \qquad \text{(S-2.1)}$$

where

$$p_\nu(\{t_i\}|\{x(t)\}) = \prod_{i=1}^{n}\lambda(t_i)f_\nu(\Lambda(t_i)-\Lambda(t_{i-1})), \quad f_\nu(y) = \nu^\nu y^{\nu-1}e^{-\nu y}/\Gamma(\nu), \qquad \text{(S-2.2)}$$

is the likelihood function of the time-rescaled renewal process with the gamma ISI distribution, and

$$p_\gamma(\{x(t)\}) = \frac{1}{Z(\gamma)} \exp\left[-\frac{1}{2\gamma^2}\int_0^T \left(\frac{dx(t)}{dt}\right)^2 dt \right]$$

is the prior distribution of $x(t)$. $Z(\gamma)$ is the normalization constant given by

$$Z(\gamma) = \frac{1}{\sqrt{2\pi\gamma^2 T}} \exp\left[-\frac{\{x(T)-x(0)\}^2}{2\gamma^2 T} \right].$$

Then, the estimators for κ and γ are obtained by maximizing the marginal likelihood:

$$(\hat{\kappa}, \hat{\gamma}) = \arg\max_{\nu,\gamma} p_{\nu,\gamma}(\{t_i\}).$$

S-2.2 Evaluation of the marginal likelihood

The log of likelihood function (S-2.2) is explicitly given by

$$\begin{aligned}
\log p_\nu(\{t_i\}|\{x(t)\}) &= \sum_i \big[x(t_i) + \nu\log\nu + (\nu-1)\log(\Lambda(t_i)-\Lambda(t_{i-1})) \\
&\quad - \nu(\Lambda(t_i)-\Lambda(t_{i-1})) - \log\Gamma(\nu)\big]. \\
&= \sum_i \big[x(t_i) + (\nu-1)\log(\Lambda(t_i)-\Lambda(t_{i-1}))\big] \\
&\quad + n\nu\log\nu - \nu\int_0^T e^{x(t)}dt - n\log\Gamma(\nu).
\end{aligned}$$

From the condition (A), the firing rate in each ISI can be approximated to be constant, and thus we obtain

$$\log(\Lambda(t_i)-\Lambda(t_{i-1})) \approx x(t_i) + \log(t_i - t_{i-1}).$$

Under this approximation, the log likelihood is rewritten as

$$\begin{aligned}
\log p_\nu(\{t_i\}|\{x(t)\}) &= \sum_i \big[\nu x(t_i) + (\nu-1)\log(t_i-t_{i-1})\big] \\
&\quad + n\nu\log\nu - \nu\int_0^T e^{x(t)}dt - n\log\Gamma(\nu).
\end{aligned}$$

We decompose the state $x(t)$ into the mean $\log\mu$ and fluctuation $y(t)$, as

$$x(t) = \log\mu + y(t).$$

Accordingly, the log likelihood function is decomposed into two parts, as

$$\log p_\nu(\{t_i\}|\{x(t)\}) = \mathcal{H} + \mathcal{I}, \tag{S-2.3}$$

where $\mathcal{H}$ represents the log likelihood function of the gamma distribution,

$$\mathcal{H} = -T\nu\mu + n\nu\log\mu + n\nu\log\nu - n\log\Gamma(\nu) + (\nu-1)\sum_i \log(t_i - t_{i-1}), \tag{S-2.4}$$

whereas $\mathcal{I}$ represents the contribution of rate fluctuation,

$$\mathcal{I} = \nu\int_0^T \left[\sum_i \delta(t-t_i)y(t) - \mu(e^{y(t)}-1)\right]dt.$$

Substituting Eq. (S-2.3) into Eq. (S-2.1), the marginal likelihood function is obtained as

$$p_{\nu,\gamma}(\{t_i\}) = e^{\mathcal{H}}\mathcal{F}.$$

The contribution of the rate fluctuation can be represented in the form of a path integral [2],

$$\mathcal{F} = \frac{1}{Z(\gamma)}\int \exp\left[-\int_0^T L(y,\dot{y})dt\right]\mathcal{D}\{y(t)\}, \tag{S-2.5}$$

where $L(\dot{x},x)$ is a "Lagrangian" of the form,

$$L(y,\dot{y}) = \frac{1}{2\gamma^2}\dot{y}^2 + \nu\mu(e^y - 1) - \nu\sum_i \delta(t-t_i)y. \tag{S-2.6}$$

Using Eq. (12) the Lagrangian (S-2.6) is represented as

$$L(y,\dot{y}) = \frac{1}{2\gamma^2}\dot{y}^2 + \nu\mu(e^y - 1) - \nu\left[\lambda(t) + \sqrt{\lambda(t)/\kappa}\xi(t)\right]y.$$

Under the condition (B), we can take the terms up to second-order with respect to y, and the Lagrangian is approximated to

$$L(y,\dot{y}) = \frac{1}{2\gamma^2}\dot{y}^2 + \frac{\nu\mu}{2}y^2 - \nu\left[\sigma f(t) + \sqrt{\lambda(t)/\kappa}\xi(t)\right]y. \tag{S-2.7}$$

Note that this approximation is valid in $O((\sigma/\mu)^{3/2})$.

The MAP estimate $\hat{y}$ is obtained by solving the Euler-Lagrange equation,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\hat{y}}}\right) - \frac{\partial L}{\partial \hat{y}} = 0.$$

The Euler-Lagrange equation associated with Eq. (S-2.7) is obtained as

$$\frac{1}{\gamma^2}\ddot{\hat{y}} - \nu\mu\hat{y} + \nu\left[\lambda(t) - \mu + \sqrt{\lambda(t)/\kappa}\xi(t)\right] = 0. \tag{S-2.8}$$

By decomposing $y(t) = \hat{y}(t) + \phi(t)$ and approximate the path integral (S-2.5) to the range quadratic with respect to $\phi(t)$, the marginal likelihood function can be evaluated as

$$p_{\nu,\gamma}(\{t_i\}) = \frac{e^{\mathcal{H}}}{Z(\gamma)} R \exp\left[-\int_0^T L(\hat{y},\dot{\hat{y}})dt\right], \tag{S-2.9}$$

where R is the fluctuation factor:

$$R \;=\; \int \exp\left[-\int_0^T \left(\frac{1}{2\gamma^2}\dot{\phi}^2 + \frac{\nu\mu}{2}\phi^2\right)dt\right]\mathcal{D}\{\phi(t)\}. \tag{S-2.10}$$

In the following, we evaluate the three factors in Eq. (S-2.9).

Contribution of the MAP path

First, we evaluate the factor $\exp[-\int_0^T L(\hat{y},\dot{\hat{y}})dt]$ in Eq. (S-2.9). The MAP path is obtained by solving Eq. (S-2.8) as

$$\hat{y}(t) \;=\; \frac{\gamma}{2}\sqrt{\frac{\nu}{\mu}}\int_0^T e^{-\gamma\sqrt{\nu\mu}|t-s|}\left[\sigma f(s) + \sqrt{\lambda(s)/\kappa}\xi(s)\right]ds. \tag{S-2.11}$$

By using Eqs. (S-2.7) and (S-2.8), we obtain

$$\frac{1}{T}\int_0^T L(\hat{y},\dot{\hat{y}})dt \;=\; \int_0^T \left[\frac{1}{2\gamma^2}\frac{d}{dt}(\hat{y}\dot{\hat{y}}) - \frac{\nu}{2}\{\sigma f(t) + \sqrt{\lambda(t)/\kappa}\xi(t)\}\hat{y}\right]dt.$$

For $T \gg 1$, the boundary effect is negligible so that the first-term in the rhs of the above equation vanishes. Substituting the MAP path (S-2.11) into the above equation leads to

$$\frac{1}{T}\int_0^T L(\hat{y},\dot{\hat{y}})dt = -\frac{\gamma\sqrt{\nu\mu}}{4}\left\{\frac{\nu}{\kappa} + \frac{2\nu\sigma^2}{\mu}\int_0^\infty \phi(u)e^{-\gamma\sqrt{\nu\mu}u}du\right\}, \tag{S-2.12}$$

for $T \to \infty$.

Fluctuation factor

Next, we evaluate the fluctuation factor (S-2.10). This factor can be represented by the ratio of determinants [1]:

$$R = \frac{1}{\sqrt{2\pi\gamma^2 T}}\left[\frac{\det(-\partial_t^2 + \nu\mu\gamma^2)}{\det(-\partial_t^2)}\right]^{-\frac{1}{2}} = \frac{1}{\sqrt{2\pi\gamma^2 T}}\left[\frac{\varphi_1(T)}{\varphi_2(T)}\right]^{-\frac{1}{2}},$$

which can be computed by solving the associated differential equations:

$$\begin{aligned}(-\partial_t^2 + \nu\mu\gamma^2)\varphi_1(t) = 0, &\quad \varphi_1(0) = 0,\ \dot{\varphi}_1(0) = 1,\\ -\partial_t^2\varphi_2(t) = 0, &\quad \varphi_2(0) = 0,\ \dot{\varphi}_2(0) = 1.\end{aligned}$$

The above differential equations are solved as

$$\varphi_1(t) = \frac{1}{2\gamma\sqrt{\nu\mu}}\left\{e^{\gamma\sqrt{\nu\mu}t} - e^{-\gamma\sqrt{\nu\mu}t}\right\}, \quad \varphi_2(t) = t,$$

from which the fluctuation factor is obtained as

$$\frac{1}{T}\log R = -\frac{\gamma\sqrt{\nu\mu}}{2}, \tag{S-2.13}$$

for $T \to \infty$.

Log likelihood of gamma distribution

In order to evaluate the log likelihood function of the gamma distribution (S-2.4), we need to evaluate $\frac{1}{n}\sum_i \log(t_i - t_{i-1})$. Let $\{t_i^{(\lambda)} - t_{i-1}^{(\lambda)}\}$ be a set of ISIs in the whole ISIs $\{t_i - t_{i-1}\}$ derived from the gamma distribution with the rate λ, and n_λ be the number of the ISIs in this set. Then, we obtain

$$\frac{1}{n_\lambda}\sum_{i=1}^{n_\lambda}\log(t_i^{(\lambda)} - t_{i-1}^{(\lambda)}) \to \psi(\kappa) - \log\kappa - \log\lambda, \quad \text{as } n_\lambda \to \infty,$$

where $\psi(\kappa) = \frac{d}{d\kappa}\log\Gamma(\kappa)$ is the digamma function. On the other hand, n_λ/n converges to

$$\frac{n_\lambda}{n} \to \frac{\lambda p(\lambda)d\lambda}{\mu}, \quad \text{as } n \to \infty,$$

from the law of large number. Using these, we obtain

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\log(t_i - t_{i-1}) = \int_0^\infty [\psi(\kappa) - \log\kappa - \log\lambda]\frac{\lambda p(\lambda)}{\mu}d\lambda.$$

By using the condition (B) and expanding up to the second-order with respect to σ/μ, the above equation can be evaluated as

$$\psi(\kappa) - \log\kappa - \log\mu - \frac{\sigma^2}{2\mu^2}\int_{-\infty}^\infty f^2 p(f)df = \psi(\kappa) - \log\kappa - \log\mu - \frac{\sigma^2\phi(0)}{2\mu^2}. \tag{S-2.14}$$

Substituting (S-2.14) into Eq.(S-2.4), the log likelihood of the gamma distribution is obtained as

$$\frac{1}{T}\mathcal{H} = \mu\left\{\log\mu - \nu + \nu\log\nu - \log\Gamma(\nu) + (\nu-1)\left[\psi(\kappa) - \log\kappa - \frac{\sigma^2\phi(0)}{2\mu^2}\right]\right\}. \tag{S-2.15}$$

Log marginal likelihood function

Summing the contribution of the MAP path (S-2.12), fluctuation factor (S-2.13) and the likelihood function (S-2.15), the log marginal likelihood function is written as

$$\begin{aligned}
\mathcal{L}(\gamma,\nu) &:= \frac{1}{T}\log p_{\nu,\gamma}(\{t_i\}) = \frac{1}{T}\left(\log R - \log Z(\gamma) - \int_0^T L(\dot{\hat{y}},\hat{y})dt + \mathcal{H}\right) \\
&= -\frac{\gamma\sqrt{\nu\mu}}{4}\left\{2 - \frac{\nu}{\kappa}\left(1 + \frac{2\kappa\sigma^2}{\mu}\int_0^\infty \phi(u)e^{-\gamma\sqrt{\nu\mu}u}du\right)\right\} \\
&\quad + \sqrt{\mu\sigma}\left\{\left(\log\mu - \nu + \nu\log\nu - \log\Gamma(\nu) + (\nu-1)[\psi(\kappa)-\log\kappa]\right)\left(\frac{\sigma}{\mu}\right)^{\frac{1}{2}}\right. \\
&\quad \left. - \frac{(\nu-1)\phi(0)}{2}\left(\frac{\sigma}{\mu}\right)^{\frac{3}{2}}\right\}, \qquad \text{(S-2.16)}
\end{aligned}$$

in the limit of $T \to \infty$. Note that Eq. (S-2.16) is valid in $O((\sigma/\mu)^{3/2})$.

S-2.3 Lower bound

In the range of parameter space (γ,ν) that is valid for the asymptotic analysis (in which $o((\sigma/\mu)^{3/2})$ is negligible), the log marginal likelihood function can have a maximum at $(\gamma,\nu) = (0,\hat{\kappa}_0)$ or at $(\hat{\gamma},\hat{\kappa})$, $\hat{\gamma} > 0$, which correspond to constant and fluctuating rate estimations, respectively.

For the case of $\gamma = 0$, the fluctuation in the rate estimation (S-2.11) vanishes, and thus the rate estimation becomes constant $\hat{\lambda}(t) = \mu$. Taking $\partial\mathcal{L}/\partial\nu = 0$ leads to

$$\psi(\hat{\kappa}_0) - \log\hat{\kappa}_0 - \left[\psi(\kappa) - \log\kappa - \frac{\sigma^2\phi(0)}{2\mu^2}\right] = 0.$$

The solution of the above equation is obtained as

$$\hat{\kappa}_0 = \kappa - \frac{\sigma^2\phi(0)}{2\mu^2 I(\kappa)} + o((\sigma/\mu)^2), \qquad \text{(S-2.17)}$$

where $I(\kappa) = \dot{\psi}(\kappa) - 1/\kappa$ is the Fisher information of the gamma distribution.

We next evaluate the fluctuating rate estimation if it exists. From (S-2.17), $\hat{\kappa}$ must be $\kappa + O((\sigma/\mu)^2)$, and the log marginal likelihood function becomes

$$\begin{aligned}
\mathcal{L}(\gamma,\hat{\kappa}) &= \mathcal{L}(\gamma,\kappa) \\
&= -\frac{\eta}{4}\left[1 - 2\frac{\kappa\sigma^2}{\mu}\int_0^\infty \phi(u)e^{-\eta u}du\right] + \mathcal{L}(0,\hat{\kappa}_0), \qquad \text{(S-2.18)}
\end{aligned}$$

in $O((\sigma/\mu)^{3/2})$, where $\eta := \gamma\sqrt{\kappa\mu}$. $\mathcal{L}(\gamma,\hat{\kappa})$ satisfies $\mathcal{L}(0,\hat{\kappa}) = 0$ and $\mathcal{L}(\infty,\hat{\kappa}) = -\infty$, and has the global maximum either at $\gamma = \hat{\gamma} > 0$ or $\gamma = 0$, depending on the value of $\kappa\sigma^2/\mu$. $\mathcal{L}(\gamma,\hat{\kappa})$ has the global maximum at $\gamma = \hat{\gamma} > 0$ if $\kappa\sigma^2/\mu$ exceeds the critical value:

$$\frac{1}{2\max_\eta \int_0^\infty \phi(u)e^{-\eta u}du}. \qquad \text{(S-2.19)}$$

On the other hand, from Eq. (14), the KL divergence of the time-rescaled gamma renewal process with the rate $\lambda(t) = \mu + \sigma f(t)$ is obtained as

$$D_\kappa(\lambda(t)||\mu) = \frac{\kappa\sigma^2}{2\mu}\phi(0) + o((\sigma/\mu)^{3/2}). \qquad \text{(S-2.20)}$$

From eqn. (S-2.19) and (S-2.20), we obtain the formula (22) that the critical point satisfies as

$$D_\kappa(\lambda(t)||\mu) = \frac{\phi(0)}{4\max_\eta \int_0^\infty \phi(u)e^{-\eta u}du}. \qquad \text{(S-2.21)}$$

References

[1] S. Coleman. *Aspects of Symmetry.* Cambridge University Press, 1988.

[2] R. P. Feynman and A. R. Hibbs. *Quantum Mechanics and Path Integrals.* McGraw-Hill, 1965.