Proof of convergence

Under the assumption that the drift $a(\cdot)$ is Lipschitz, we will show that the approximation

$$\mathbf{X}_t = \mathbf{X}_t^{\text{NL}} + \mathbf{X}_t^{L} \tag{1}$$

converges to the solution of the SDE

$$d\mathbf{X}_t = a(\mathbf{X}_t)dt + Bd\mathbf{W}_t \tag{2}$$

in $L^2([0,T]\times\Omega)$. We can apply Parseval's theorem to equation (14) in our paper to see that $\mathbf{X}^L$ tends to 0 in L^2. It remains to show that $\mathbf{X}^{\text{NL}}$ converges to $\mathbf{X}$ in L^2. Our strategy will be to exploit the Lipschitz assumption so that we can bound the error using Gronwall's lemma.

Suppose $\mathbf{X}_0 = \mathbf{X}^{\text{NL}} = x_0$. For each t in $[0,T]$,

$$\begin{aligned}\mathbf{X}_t - \mathbf{X}_t^{\text{NL}} &= \int_0^t \left(a(\mathbf{X}_u) - a(\mathbf{X}_u^{\text{NL}})\right)du + B\left(\mathbf{W}_t - \sum_{i=1}^N \mathbf{Z}_i \int_0^t \phi_i(u)du\right) \\ &= \int_0^t \left(a(\mathbf{X}_u) - a(\mathbf{X}_u^{\text{NL}})\right)du + B\mathbf{R}_t^{(N)}.\end{aligned} \tag{3}$$

Since $\|a+b\|^2 \le 2\|a\|^2 + 2\|b\|^2$, we have

$$\mathbb{E}\left[\|\mathbf{X}_t - \mathbf{X}_t^{\text{NL}}\|^2\right] \le 2\mathbb{E}\left[\left\|\int_0^t \left(a(\mathbf{X}_u) - a(\mathbf{X}_u^{\text{NL}})\right)du\right\|^2\right] + 2\mathbb{E}\left[\left\|\mathbf{R}_t^{(N)}\right\|^2\right]. \tag{4}$$

The Cauchy-Schwarz inequality tells us

$$\mathbb{E}\left[\left\|\int_0^t \left(a(\mathbf{X}_u) - a(\mathbf{X}_u^{\text{NL}})\right)du\right\|^2\right] \le t\int_0^t \mathbb{E}\left[\left\|a(\mathbf{X}_u) - a(\mathbf{X}_u^{\text{NL}})\right\|^2\right]du. \tag{5}$$

Thus, by the Lipschitz assumption, there exists $L > 0$ such that

$$\mathbb{E}\left[\left\|\int_0^t \left(a(\mathbf{X}_u) - a(\mathbf{X}_u^{\text{NL}})\right)du\right\|^2\right] \le tL^2\int_0^t \mathbb{E}\left[\left\|\mathbf{X}_u - \mathbf{X}_u^{\text{NL}}\right\|^2\right]du. \tag{6}$$

Set

$$C^N(t) = 2E[\|\mathbf{R}_t^N\|^2] \tag{7}$$

$$\psi(t) = E[\|\mathbf{X}_t - \mathbf{X}_t^{\text{NL}}\|^2] \tag{8}$$

Combining (6) with (4), we see that

$$\psi(t) \le 2TL^2\int_0^t \psi(u)du + C^N(t) \tag{9}$$

By Gronwall's inequality,

$$\psi(t) \le C^N(t) + 2TL^2\exp(2T^2L^2)\int_0^t C^N(u)du \tag{10}$$

Since C^N is non-negative,

$$\psi(t) \le C^N(t) + k\int_0^T C^N(u)du \tag{11}$$

for an appropriate constant k. We integrate both sides over $[0,T]$:

$$\int_0^T \psi(u)du \le \int_0^T C^N(u)du + kT\int_0^T C^N(u)du. \tag{12}$$

We can apply the dominated convergence theorem to show that

$$\int_0^T C^N(u)du = 2\int_0^T E[\|R_u^N\|^2]du \to 0. \tag{13}$$

We conclude that

$$\int_0^T \psi(u)du = \int_0^T E[\|\mathbf{X}_u - \mathbf{X}_u^{\text{NL}}\|^2]du \to 0. \tag{14}$$

That is, $\hat{\mathbf{X}} \to \mathbf{X}$ in L^2.