Meta-Gaussian Information Bottleneck

Supplementary Material

Multi-information for meta-Gaussian random vectors. In the following we compute the multi-information for a meta-Gaussian random vector $Z = (Z_1, \dots, Z_d)$.

$$I(Z) = D_{kl}(f(z) \parallel f_0(z)) = \int_{[0,1]^d} c_Z(u) \log c_Z(u) \mathrm{d}u, \;\; u = (u_1, \dots, u_d),$$
$$= \int |P|^{-\frac{1}{2}} \exp\left\{-\tfrac{1}{2}(\Phi^{-1}(u))^T(P^{-1} - I)\Phi^{-1}(u)\right\} \log\left[|P|^{-\frac{1}{2}} \exp\left\{-\tfrac{1}{2}(\Phi^{-1}(u))^T(P^{-1} - I)\Phi^{-1}(u))\right\}\right] \mathrm{d}u,$$

where $\Phi^{-1}(u) = (\Phi^{-1}(u_1), \dots, \Phi^{-1}(u_d))$. We now use a change of variable $g(\tilde{z}) = g(\tilde{z}_1, \dots, \tilde{z}_d) := (\Phi(\tilde{z}_1), \dots, \Phi(\tilde{z}_d)) = (u_1, \dots, u_d)$. The Jacobian matrix of the transformation is diagonal with elements $\mathrm{D}g(\tilde{z})_{jj} = \Phi'(\tilde{z}_j)$ and its determinant is $\det(\mathrm{D}g) = 2\pi^{-d/2} \exp\{-\frac{1}{2}\tilde{z}^T I \tilde{z}\}$.

$$I(Z) = \int_{-\infty}^{+\infty} |P|^{-\frac{1}{2}} \exp\left\{-\tfrac{1}{2}\tilde{z}^T(P^{-1} - I)\tilde{z}\right\} \log\left[|P|^{-\frac{1}{2}} \exp\left\{-\tfrac{1}{2}\tilde{z}^T(P^{-1} - I)\tilde{z}\right\}\right] \det(\mathrm{D}g)\mathrm{d}\tilde{z}, \tag{1}$$
$$= \int |P|^{-\frac{1}{2}} \exp\left\{-\tfrac{1}{2}\tilde{z}^T P^{-1}\tilde{z}\right\} \exp\left\{\tfrac{1}{2}\tilde{z}^T I\tilde{z}\right\} \left[\log(|P|^{-\frac{1}{2}}) - \tfrac{1}{2}\tilde{z}^T(P^{-1} - I)\tilde{z}\right] 2\pi^{d/2} \exp\{-\tfrac{1}{2}\tilde{z}^T I \tilde{z}\}\mathrm{d}\tilde{z}, \tag{2}$$
$$= \int 2\pi^{d/2}|P|^{-\frac{1}{2}} \exp\left\{-\tfrac{1}{2}\tilde{z}^T P^{-1}\tilde{z}\right\} \left[\log(|P|^{-\frac{1}{2}}) - \tfrac{1}{2}\tilde{z}^T(P^{-1} - I)\tilde{z}\right] \mathrm{d}\tilde{z}, \tag{3}$$
$$= \mathrm{E}_{\mathcal{N}(0,P)}\left[\log(|P|^{-\frac{1}{2}}) - \tfrac{1}{2}\tilde{z}^T(P^{-1} - I)\tilde{z}\right], \tag{4}$$
$$= \log(|P|^{-\frac{1}{2}}) - \tfrac{1}{2}\mathrm{E}_{\mathcal{N}(0,P)}[\tilde{z}^T P^{-1}\tilde{z}] + \tfrac{1}{2}\mathrm{E}_{\mathcal{N}(0,P)}[\tilde{z}^T I \tilde{z}], \tag{5}$$
$$= \log(|P|^{-\frac{1}{2}}) - \tfrac{1}{2}d + \tfrac{1}{2}d, \tag{6}$$
$$= \log(|P|^{-\frac{1}{2}}), \tag{7}$$
$$= -\tfrac{1}{2}\log|P|. \tag{8}$$