Supplementary Material

Proposition 1. The data function sum $E_\sigma\Big(\prod_{k=1}^r h(x_{\sigma\cdot i_1^k},\cdots,x_{\sigma\cdot i_d^k})\Big)$ is invariant within each index orbit of group action $G:=S_n\times S_r\times S_d^r$ acting on the index set U_d^r as defined in definition 1, and $E_\sigma\Big(\prod_{k=1}^r h(x_{\sigma\cdot i_1^k},\cdots,x_{\sigma\cdot i_d^k})\Big)=$

$$\frac{\sum_{\{(j_1^1,\cdots,j_d^1),\cdots,(j_1^r,\cdots,j_d^r)\}\in[\{(i_1^1,\cdots,i_d^1),\cdots,(i_1^r,\cdots,i_d^r)\}]}\prod_{k=1}^r h(x_{j_1^k},\cdots,x_{j_d^k})}{\operatorname{card}\Big([\{(i_1^1,\cdots,i_d^1),\cdots,(i_1^r,\cdots,i_d^r)\}]\Big)} \tag{1}$$

where $\operatorname{card}\Big([\{(i_1^1,\cdots,i_d^1),\cdots,(i_1^r,\cdots,i_d^r)\}]\Big)$ is the cardinality of the index orbit, i.e., the number of indices within the index orbit $[\{(i_1^1,\cdots,i_d^1),\cdots,(i_1^r,\cdots,i_d^r)\}]$.

Proof: first, it is obvious that $G:=S_n\times S_r\times S_d^r$ is a group with identity $e_G=(e_n,e_r,e_d,\cdots,e_d)$, where e_n, e_r, and e_d are identity mappings of S_n, S_r, and S_d respectively. Then it is easy to verify that $e_G\cdot(\{(i_1^1,\cdots,i_d^1),\cdots,(i_1^r,\cdots,i_d^r)\})\mapsto\{(i_1^1,\cdots,i_d^1),\cdots,(i_1^r,\cdots,i_d^r)\}$.

In additon, let $g1=(\sigma1,\tau1,\pi1_1,\cdots,\pi1_r)$ and $g2=(\sigma2,\tau2,\pi2_1,\cdots,\pi2_r)\in G$, then
$g1\cdot(g2\cdot(\{(i_1^1,\cdots,i_d^1),\cdots,(i_1^r,\cdots,i_d^r)\}))$
$=\{(\sigma1\sigma2(i^{\tau2^{-1}\tau1^{-1}(1)}_{\pi2_1^{-1}\pi1_1^{-1}(1)})\cdots\sigma1\sigma2(i^{\tau2^{-1}\tau1^{-1}(1)}_{\pi2_1^{-1}\pi1_1^{-1}(d)})),\cdots,(\sigma1\sigma2(i^{\tau2^{-1}\tau1^{-1}(r)}_{\pi2_r^{-1}\pi1_r^{-1}(1)})\cdots\sigma1\sigma2(i^{\tau2^{-1}\tau1^{-1}(r)}_{\pi2_r^{-1}\pi1_r^{-1}(d)}))\}$
$=\{(\sigma1\sigma2(i^{(\tau1\tau2)^{-1}(1)}_{(\pi1_1\pi2_1)^{-1}(1)})\cdots\sigma1\sigma2(i^{(\tau1\tau2)^{-1}(1)}_{(\pi1_1\pi2_1)^{-1}(d)})),\cdots,(\sigma1\sigma2(i^{(\tau1\tau2)^{-1}(r)}_{(\pi1_r\pi2_r)^{-1}(1)})\cdots\sigma1\sigma2(i^{(\tau1\tau2)^{-1}(r)}_{(\pi1_r\pi2_r)^{-1}(d)}))\}$
$=(g1g2)\cdot(\{(i_1^1,\cdots,i_d^1),\cdots,(i_1^r,\cdots,i_d^r)\}))$.

Therefore, the action defined in definition 1 is a group action acting on the set of U_d^r. It is well known that a group action partitions the set into disjoint orbits. By the definition of orbit, for any index paragraph

$$\{(j_1^1,\cdots,j_d^1),\cdots,(j_1^r,\cdots,j_d^r)\}\in[\{(i_1^1,\cdots,i_d^1),\cdots,(i_1^r,\cdots,i_d^r)\}],$$

there exists a $(\tilde{\sigma},\tau,\pi_1,\cdots,\pi_r)\in G$ such that

$$\{(j_1^1,\cdots,j_d^1),\cdots,(j_1^r,\cdots,j_d^r)\}=(\tilde{\sigma},\tau,\pi_1,\cdots,\pi_r)\cdot\{(i_1^1,\cdots,i_d^1),\cdots,(i_1^r,\cdots,i_d^r)\}.$$

Since the data function h is symmetric and multiplication is commutative,

$$E_\sigma(\prod_{k=1}^r h(x_{\sigma(j_1^k)},\cdots,x_{\sigma(j_d^k)}))=\frac{1}{|S_n|}\sum_{\sigma\in S_n}(\prod_{k=1}^r h(x_{\sigma\tilde{\sigma}(i_1^k)},\cdots,x_{\sigma\tilde{\sigma}(i_d^k)}))=$$
$$=E_{\sigma\tilde{\sigma}}(\prod_{k=1}^r h(x_{\sigma\tilde{\sigma}(i_1^k)},\cdots,x_{\sigma\tilde{\sigma}(i_d^k)}))\}=E_\sigma(\prod_{k=1}^r h(x_{\sigma\cdot i_1^k},\cdots,x_{\sigma\cdot i_d^k})).$$

So $E_\sigma(\prod_{k=1}^r h(x_{\sigma\cdot i_1^k},\cdots,x_{\sigma\cdot i_d^k}))$ is invariant within each index orbit. In addition, since a permutation moves any obit to itself, we get

$$\sum_{\{(j_1^1,\cdots,j_d^1),\cdots,(j_1^r,\cdots,j_d^r)\}\in[\{(i_1^1,\cdots,i_d^1),\cdots,(i_1^r,\cdots,i_d^r)\}]}(\prod_{k=1}^r h(x_{\sigma(j_1^k)},\cdots,x_{\sigma(j_d^k)}))$$
$$=\sum_{\{(j_1^1,\cdots,j_d^1),\cdots,(j_1^r,\cdots,j_d^r)\}\in[\{(i_1^1,\cdots,i_d^1),\cdots,(i_1^r,\cdots,i_d^r)\}]}(\prod_{k=1}^r h(x_{j_1^k},\cdots,x_{j_d^k})),thus$$

$$card([\{(i_1^1,\cdots,i_d^1),\cdots,(i_1^r,\cdots,i_d^r)\}])E_\sigma(\prod_{k=1}^{r} h(x_{\sigma\cdot i_1^k},\cdots,x_{\sigma\cdot i_d^k}))$$

$$= \sum_{\{(j_1^1,\cdots,j_d^1),\cdots,(j_1^r,\cdots,j_d^r)\}\in[\{(i_1^1,\cdots,i_d^1),\cdots,(i_1^r,\cdots,i_d^r)\}]} E_\sigma(\prod_{k=1}^{r} h(x_{\sigma(j_1^k)},\cdots,x_{\sigma(j_d^k)}))$$

$$= E_\sigma(\sum_{\{(j_1^1,\cdots,j_d^1),\cdots,(j_1^r,\cdots,j_d^r)\}\in[\{(i_1^1,\cdots,i_d^1),\cdots,(i_1^r,\cdots,i_d^r)\}]} (\prod_{k=1}^{r} h(x_{\sigma(j_1^k)},\cdots,x_{\sigma(j_d^k)})))$$

$$= E_\sigma(\sum_{\{(j_1^1,\cdots,j_d^1),\cdots,(j_1^r,\cdots,j_d^r)\}\in[\{(i_1^1,\cdots,i_d^1),\cdots,(i_1^r,\cdots,i_d^r)\}]} (\prod_{k=1}^{r} h(x_{j_1^k},\cdots,x_{j_d^k})))$$

$$= \sum_{\{(j_1^1,\cdots,j_d^1),\cdots,(j_1^r,\cdots,j_d^r)\}\in[\{(i_1^1,\cdots,i_d^1),\cdots,(i_1^r,\cdots,i_d^r)\}]} (\prod_{k=1}^{r} h(x_{j_1^k},\cdots,x_{j_d^k}))$$

Finally, we get $E_\sigma(\prod_{k=1}^{r} h(x_{\sigma\cdot i_1^k},\cdots,x_{\sigma\cdot i_d^k})) =$

$$\frac{\sum_{\{(j_1^1,\cdots,j_d^1),\cdots,(j_1^r,\cdots,j_d^r)\}\in[\{(i_1^1,\cdots,i_d^1),\cdots,(i_1^r,\cdots,i_d^r)\}]} \prod_{k=1}^{r} h(x_{j_1^k},\cdots,x_{j_d^k})}{card([\{(i_1^1,\cdots,i_d^1),\cdots,(i_1^r,\cdots,i_d^r)\}])} \tag{2}$$

.

Proposition 2. The r-th moment of permutation statistics can be obtained by summing up the product of the data function orbit sum h_λ and the index function orbit sum w_λ over all index orbits, i.e.,

$$E_\sigma\Big(T^r(x)\Big) = \sum_{\lambda\in L} \frac{w_\lambda h_\lambda}{\mathrm{card}([\lambda])}, \tag{3}$$

where $\lambda = \{(i_1^1,\cdots,i_d^1),\cdots,(i_1^r,\cdots,i_d^r)\}$ is a representative index paragraph, $[\lambda]$ is the index orbit including index paragraph λ, and L is a transversal of all index orbits . The data function orbit sum is

$$h_\lambda = \sum_{\{(j_1^1,\cdots,j_d^1),\cdots,(j_1^r,\cdots,j_d^r)\}\in[\lambda]} \prod_{k=1}^{r} h(x_{j_1^k},\cdots,x_{j_d^k}), \tag{4}$$

and the index function orbit sum is

$$w_\lambda = \sum_{\{(j_1^1,\cdots,j_d^1),\cdots,(j_1^r,\cdots,j_d^r)\}\in[\lambda]} \prod_{k=1}^{r} w(j_1^k,\cdots,j_d^k). \tag{5}$$

Proof: With Proposition 1, $E_\sigma(\prod_{k=1}^{r} h(x_{\pi(i_1^k)},\cdots,x_{\pi(i_d^k)}))$ is invariant within each equivalent index subset, therefore,

$$E_\sigma(T^r(x)) = \sum_{i_1^1,\cdots,i_d^1,\cdots,i_1^r,\cdots,i_d^r} \{(\prod_{k=1}^{r} w(i_1^k,\cdots,i_d^k))E_\sigma(\prod_{k=1}^{r} h(x_{\sigma\cdot i_1^k},\cdots,x_{\sigma\cdot i_d^k}))\}$$

$$= \sum_{\lambda\in L} \sum_{\{(j_1^1,\cdots,j_d^1),\cdots,(j_1^r,\cdots,j_d^r)\}\in[\lambda]} \{(\prod_{k=1}^{r} w(j_1^k,\cdots,j_d^k))E_\sigma(\prod_{k=1}^{r} h(x_{\sigma\cdot j_1^k},\cdots,x_{\sigma\cdot j_d^k}))\}$$

$$= \sum_{\lambda\in L} \sum_{\{(j_1^1,\cdots,j_d^1),\cdots,(j_1^r,\cdots,j_d^r)\}\in[\lambda]} \{\prod_{k=1}^{r} w(j_1^k,\cdots,j_d^k)\frac{h_\lambda}{card([\lambda])}\} = \sum_{\lambda\in L} \frac{w_\lambda h_\lambda}{card([\lambda])}. \tag{6}$$

Proposition 3. The transversal of $G^* \setminus\!\!\setminus U_d^{r*}$ is also a transversal of $G \setminus\!\!\setminus U_d^r$.

Proof: We define a mapping $\theta : G \setminus\!\!\setminus U_d^r \mapsto G^* \setminus\!\!\setminus U_d^{r*}$ as follow:
$\forall\{(i_1^1, \cdots, i_d^1), \cdots, (i_1^r, \cdots, i_d^r)\} \in U_d^r$
$\theta[\{(i_1^1, \cdots, i_d^1), \cdots, (i_1^r, \cdots, i_d^r)\}] := [(\sigma_\theta, e_r, e_d, \cdots, e_d)\{(i_1^1, \cdots, i_d^1), \cdots, (i_1^r, \cdots, i_d^r)\}]$
$= [\{(\sigma_\theta(i_1^1), \cdots, \sigma_\theta(i_d^1)), \cdots, (\sigma_\theta(i_1^r), \cdots, \sigma_\theta(i_d^r))\}]$
$= [\{(i^{*1}_1, \cdots, i^{*1}_d), \cdots, (i^{*r}_1, \cdots, i^{*r}_d)\}] \in G^* \setminus\!\!\setminus U_d^{r*}$,
where $\sigma_\theta \in S_n : U_d^r \mapsto U_d^{r*}$ permutes the smallest index value of $\{(i_1^1, \cdots, i_d^1), \cdots, (i_1^r, \cdots, i_d^r)\}$ to 1, permutes the second smallest index value to 2, and so on.

First, we prove the mapping θ is well-defined. We choose any two index paragraphs belonging to the same orbit,$[\{(i_1^1, \cdots, i_d^1), \cdots, (i_1^r, \cdots, i_d^r)\}] = [\{(j_1^1, \cdots, j_d^1), \cdots, (j_1^r, \cdots, j_d^r)\}]$. So there exists a $g = (\sigma, \tau, \pi_1, \cdots, \pi_r) \in G$ such that,
$\{(i_1^1, \cdots, i_d^1), \cdots, (i_1^r, \cdots, i_d^r)\} = (\sigma, \tau, \pi_1, \cdots, \pi_r)\{(j_1^1, \cdots, j_d^1), \cdots, (j_1^r, \cdots, j_d^r)\}$.
By applying the mapping θ with $g1_\theta = (\sigma 1_\theta, e_r, e_d, \cdots, e_d)$ and $g2_\theta = (\sigma 2_\theta, e_r, e_d, \cdots, e_d)$, we have
$\{(i^{*1}_1, \cdots, i^{*1}_d), \cdots, (i^{*r}_1, \cdots, i^{*r}_d)\} = g1_\theta\{(i_1^1, \cdots, i_d^1), \cdots, (i_1^r, \cdots, i_d^r)\}$,
$\{(j^{*1}_1, \cdots, j^{*1}_d), \cdots, (j^{*r}_1, \cdots, j^{*r}_d)\} = g2_\theta\{(j_1^1, \cdots, j_d^1), \cdots, (j_1^r, \cdots, j_d^r)\}$, and
$\{(i^{*1}_1, \cdots, i^{*1}_d), \cdots, (i^{*r}_1, \cdots, i^{*r}_d)\} = g1_\theta g g2_\theta^{-1}\{(j^{*1}_1, \cdots, j^{*1}_d), \cdots, (j^{*r}_1, \cdots, j^{*r}_d)\}$,
where $g1_\theta g g2_\theta^{-1} = (\sigma 1_\theta \sigma \sigma 2_\theta^{-1}, \tau, \pi_1, \cdots, \pi_r)$.
Since $\sigma 1_\theta \sigma \sigma 2_\theta^{-1} \in S_{dr}$, $g1_\theta g g2_\theta^{-1} = (\sigma 1_\theta \sigma \sigma 2_\theta^{-1}, \tau, \pi_1, \cdots, \pi_r) \in G^*$, we get
$[\{(i^{*1}_1, \cdots, i^{*1}_d), \cdots, (i^{*r}_1, \cdots, i^{*r}_d)\}] = [\{(j^{*1}_1, \cdots, j^{*1}_d), \cdots, (j^{*r}_1, \cdots, j^{*r}_d)\}]$.

Secondly, we prove the mapping θ is one-to-one. For the sake of contradiction,
$\forall[\{(i_1^1, \cdots, i_d^1), \cdots, (i_1^r, \cdots, i_d^r)\}] \neq [\{(j_1^1, \cdots, j_d^1), \cdots, (j_1^r, \cdots, j_d^r)\}]$, we suppose
$[\{(i^{*1}_1, \cdots, i^{*1}_d), \cdots, (i^{*r}_1, \cdots, i^{*r}_d)\}] = [\{(j^{*1}_1, \cdots, j^{*1}_d), \cdots, (j^{*r}_1, \cdots, j^{*r}_d)\}]$,
then there exists a $g^* = (\sigma^*, \tau, \pi_1, \cdots, \pi_r) \in G^*$, such that
$\{(i^{*1}_1, \cdots, i^{*1}_d), \cdots, (i^{*r}_1, \cdots, i^{*r}_d)\} = g^*\{(j^{*1}_1, \cdots, j^{*1}_d), \cdots, (j^{*r}_1, \cdots, j^{*r}_d)\}$.
Therefore, $\{(i_1^1, \cdots, i_d^1), \cdots, (i_1^r, \cdots, i_d^r)\} = g1_\theta^{-1} g^* g2_\theta\{(j_1^1, \cdots, j_d^1), \cdots, (j_1^r, \cdots, j_d^r)\}$
Similarly, it is strightforward to verify that $g1_\theta^{-1} g^* g2_\theta \in G$, we have
$[\{(i_1^1, \cdots, i_d^1), \cdots, (i_1^r, \cdots, i_d^r)\}] = [\{(j_1^1, \cdots, j_d^1), \cdots, (j_1^r, \cdots, j_d^r)\}]$, which causes a contradiction.

Then, we prove the mapping θ is onto. This is obvious because $U_d^{r*} \subset U_d^r$. A representative of any orbit in $G^* \setminus\!\!\setminus U_d^{r*}$ itself is also a representative of the corresponding orbit in $G \setminus\!\!\setminus U_d^r$.

Proposition 4. A transversal of $S_{dr} \setminus\!\!\setminus U_d^{r*}$ can be generated by all possible mergings of $[\{(1, \cdots, d), \cdots, (d(r-1)+1, \cdots, dr)\}]^s$.

Proof: It is easy to verify that $[\{(1, \cdots, d), \cdots, (d(r-1)+1, \cdots, dr)\}]^s$ is an orbit of $S_{dr} \setminus\!\!\setminus U_d^{r*}$. Here, $[\{(1, \cdots, d), \cdots, (d(r-1)+1, \cdots, dr)\}]^s$ denotes the orbit that all dr index words have distinct index values. Since the group action $S_{dr} \times U_d^{r*} \to U_d^{r*}$ prohibits any shuffling of index words and allows permutation of index values, each orbit of $S_{dr} \setminus\!\!\setminus U_d^{r*}$ is determined by which index words have the same index values. This is equivalent to merging the corresponding index words. For example, starting from $[\{(1,2)(3,4)\}]^s$, we can get the orbit of $[\{(1,2)(1,2)\}]^s$ by merging 1 and 3, and 2 and 4.

Proposition 5. Enumerating a transversal of $S_{dr} \times S_{dr} \setminus\!\!\setminus U_d^{r*}$ is equivalent to the integer partition of dr.

Proof: Since the group action $S_{dr} \times S_{dr} \times U_d^{r*} \to U_d^{r*}$ allows free shuffling of index words and permutation of index values, the order of index words does not matter now. Each orbit of $S_{dr} \times S_{dr} \setminus\!\!\setminus U_d^{r*}$ is determined by how many (not which) index words have the same index values. For example, $[\{(1,2)(1,2)\}]^l = [\{(1,2)(2,1)\}]^l = [\{(1,1)(2,2)\}]^l$. This is equivalent to the integer partition of dr. In the previous example, $[\{(1,2)(1,2)\}]^l$ corresponds to the integer partition of $4 = 2 + 2$.

Proposition 6. The index function orbit sum w_λ can be calculated by subtracting all lower order orbit sums from the corresponding relaxed index function orbit sum w^*_λ, i.e., $w_\lambda = w^*_\lambda - \sum_{\nu \prec \lambda} w_\nu \frac{\#(\lambda)}{\#(\nu)} \#(\lambda \to \nu)$. The cardinality of $[\lambda]$ is $\#(\lambda) n(n-1) \cdots (n-q+1)$, where q is the number of distinct values in λ. The calculation of the data index function orbit sum h_λ is similar.

Proof: In order to calculate w_λ, we can calculate the relaxed orbit sum w^*_λ first, then exclude all terms violating the inequality constraint. Note that each relaxed orbit sum includes all orbit sums that have lower symmetric orders. Since each orbit $[(\lambda)]$ includes $\#(\lambda)$ different $[(\lambda)]^s$ and violating the inequality constraint in a relaxed orbit sum is equivalent to all sorts of merging operations, $w_\lambda = w^*_\lambda - \sum_{\nu \prec \lambda} w_\nu \frac{\#(\lambda)}{\#(\nu)} \#(\lambda \to \nu)$. It is obvious that there are $n(n-1) \cdots (n-q+1)$ index words within each $[(\lambda)]^s$. Therefore, the cardinality of $[\lambda]$ is $\#(\lambda) n(n-1) \cdots (n-q+1)$.

Proposition 7. For $d \geq 2$, let $m(m-1)/2 \leq rd(d-1)/2 < (m+1)m/2$, where r is the order of moment and m is an integer. For a d-th order weighted v-statistic, the computational cost of the orbit sum for the r-th moment is bounded by $O(n^m)$. When $d = 1$, the computational complexity of the orbit sum is $O(n)$.

Proof: As we know, the main computational cost comes from computing relaxed data function orbit sums. When $d = 1$, each relaxed sum can be represented by a polynomial of power sums, thus the computational complexity is $O(n)$. When $d \geq 2$, the computational complexity depends on the largest symmetric subgraph among all relaxed index orbit graph representations needed for the r-th moment. Since there are at most $rd(d-1)/2$ edges connecting distinct index words, the computational cost of the orbit sum for the r-th moment is bounded by $O(n^m)$, where m is a integer that satisfies $m(m-1)/2 \leq rd(d-1)/2 < (m+1)m/2$.

Proposition 8. We can obtain the r-th moment of bootstrapping weighted v-statistics by summing up the product of the index function orbit sum w_λ and the relaxed data function orbit sum h^*_λ over all index orbits, i.e.,

$$E_\sigma(T^r(x)) = \sum_{\lambda \in L} \frac{w_\lambda h^*_\lambda}{card([\lambda^*])}, \tag{7}$$

where $\sigma \in End_n$, $card([\lambda^*]) = \#(\lambda) n^q$, and q is the number of distinct values in λ.

Proof: Similar to proposition 1, we divide the index set U^r_d into index orbits $G \setminus\!\setminus U^r_d$ by $G := S_n \times S_r \times S^r_d$ acting on U^r_d, not monoid action $H \times U^r_d \to U^r_d$. Therefore,

$$E_\sigma(T^r(x)) = \sum_{i^1_1, \cdots, i^1_d, \cdots, i^r_1, \cdots, i^r_d} \{ (\prod_{k=1}^r w(i^k_1, \cdots, i^k_d)) E_\sigma (\prod_{k=1}^r h(x_{\sigma \cdot i^k_1}, \cdots, x_{\sigma \cdot i^k_d})) \}$$

$$= \sum_{\lambda \in L} \sum_{\{(j^1_1, \cdots, j^1_d), \cdots, (j^r_1, \cdots, j^r_d)\} \in [\lambda]} \{ (\prod_{k=1}^r w(j^k_1, \cdots, j^k_d)) E_\sigma (\prod_{k=1}^r h(x_{\sigma \cdot j^k_1}, \cdots, x_{\sigma \cdot j^k_d})) \}$$

It is obvious that each index paragraph is mapped to a same index subset (we call it bootstrap index subset) by the monoid action $H \times U^r_d \to U^r_d$ since $\sigma \in End_n$ is uniformly distributed. For resampling with replacement, there are $card([\lambda^*]) = \#(\lambda) n^q$ index paragraphs within the bootstrap index subset. $E_\sigma(\prod_{k=1}^r h(x_{\sigma \cdot j^k_1}, \cdots, x_{\sigma \cdot j^k_d}))\}$ is equivalent to the corresponding relaxed data function sum h^*_λ divided by $card([\lambda^*])$, i.e., $E_\sigma(\prod_{k=1}^r h(x_{\sigma \cdot j^k_1}, \cdots, x_{\sigma \cdot j^k_d}))\} = \frac{h^*_\lambda}{card([\lambda^*])}$. Therefore, $E_\sigma(T^r(x)) = \sum_{\lambda \in L} \frac{w_\lambda h^*_\lambda}{card([\lambda^*])}$.

First order test statistics

The first moment

$$[U_1^1] = [\{1\}] \tag{8}$$

$$w_{[\{1\}]} = \sum_{i \in [\{1\}]} w(i) = \sum_i w(i) \tag{9}$$

$$E(T) = \frac{h_{[\{1\}]}}{\#([\{1\}])} w_{[\{1\}]} = \frac{h_{[\{1\}]}}{n} w_{[\{1\}]} \tag{10}$$

The second moment

$$[U_1^2] = [\{1,1\}] \bigcup [\{1,2\}] \tag{11}$$

$$w_{[\{1,1\}]} = \sum_{i_1, i_2 \in [\{1,1\}]} w(i_1)w(i_2) = \sum_i w(i)^2 \tag{12}$$

$$w^*_{[\{1,2\}]} = \sum_{i_1, i_2} w(i_1)w(i_2) = (\sum_i w(i))^2 \tag{13}$$

$$w_{[\{1,2\}]} = \sum_{i_1, i_2 \in [\{1,2\}]} w(i_1)w(i_2) = w^*_{[\{1,2\}]} - \frac{1}{1} \times 1 \times w_{[\{1,1\}]} \tag{14}$$

$$\begin{aligned} E(T^2) &= \frac{h_{[\{1,1\}]}}{\#([\{1,1\}])} w_{[\{1,1\}]} + \frac{h_{[\{1,2\}]}}{\#([\{1,2\}])} w_{[\{1,2\}]} \\ &= \frac{h_{[\{1,1\}]}}{n} w_{[\{1,1\}]} + \frac{h_{[\{1,2\}]}}{n(n-1)} w_{[\{1,2\}]} \end{aligned} \tag{15}$$

The third moment

$$[U_1^3] = [\{1,1,1\}] \bigcup [\{1,1,2\}] \bigcup [\{1,2,3\}] \tag{16}$$

$$w_{[\{1,1,1\}]} = \sum_{i_1, i_2, i_3 \in [\{1,1,1\}]} w(i_1)w(i_2)w(i_3) = \sum_i w(i)^3 \tag{17}$$

$$w^*_{[\{1,1,2\}]} = 3 \times \sum_{i,j} w(i)^2 w(j) = 3 \times (\sum_i w(i)^2)(\sum_j w(j)) \tag{18}$$

$$w_{[\{1,1,2\}]} = \sum_{i_1, i_2, i_3 \in [\{(1,1,2)\}]} w(i_1)w(i_2)w(i_3) = w^*_{[\{1,1,2\}]} - \frac{3}{1} \times 1 \times w_{[\{(1,1,1)\}]} \tag{19}$$

$$w^*_{[\{1,2,3\}]} = \sum_{i,j,k} w(i)w(j)w(k) = (\sum_i w(i))^3 \tag{20}$$

$$w_{[\{1,2,3\}]} = w^*_{[\{1,2,3\}]} - w_{[\{1,1,2\}]} - w_{[\{1,1,1\}]} \tag{21}$$

$$\begin{aligned} E(T^3) &= \frac{h_{[\{1,1,1\}]}}{\#([\{1,1,1\}])} w_{[\{1,1,1\}]} + \frac{h_{[\{1,1,2\}]}}{\#([\{1,1,2\}])} w_{[\{1,1,2\}]} + \frac{h_{[\{1,2,3\}]}}{\#([\{1,2,3\}])} w_{[\{1,2,3\}]} \\ &= \frac{h_{[\{1,1,1\}]}}{n} w_{[\{1,1,1\}]} + \frac{h_{[\{1,1,2\}]}}{3n(n-1)} w_{[\{1,1,2\}]} + \frac{h_{[\{1,2,3\}]}}{n(n-1)(n-2)} w_{[\{1,2,3\}]} \end{aligned} \tag{22}$$

The fourth moment

$$[U_1^4] = [\{1,1,1,1\}] \bigcup [\{1,1,1,2\}] \bigcup [\{1,1,2,2\}] \bigcup [\{1,1,2,3\}] \bigcup [\{1,2,3,4\}] \tag{23}$$

$$w_{[\{1,1,1,1\}]} = \sum_{i_1, i_2, i_3, i_4 \in [\{1,1,1,1\}]} w(i_1)w(i_2)w(i_3)w(i_4) = \sum_i w(i)^4 \tag{24}$$

$$w^*_{[\{1,1,1,2\}]} = 4 \times \sum_{i,j} w(i)^3 w(j) = 4 \times (\sum_i w(i)^3)(\sum_j w(j)) \tag{25}$$

$$
\begin{aligned}
w_{[\{1,1,1,2\}]} &= \sum_{i_1,i_2,i_3,i_4 \in [\{(1,1,1,2)\}]} w(i_1)w(i_2)w(i_3)w(i_4) \\
&= w^*_{[\{1,1,1,2\}]} - \frac{4}{1} \times 1 \times w_{[\{(1,1,1,1)\}]} && (26) \\
w^*_{[\{1,1,2,2\}]} &= 3 \times \sum_{i,j} w(i)^2 w(j)^2 = 3 \times (\sum_i w(i)^2)^2 && (27)
\end{aligned}
$$

$$
\begin{aligned}
w_{[\{1,1,2,2\}]} &= \sum_{i_1,i_2,i_3,i_4 \in [\{(1,1,2,2)\}]} w(i_1)w(i_2)w(i_3)w(i_4) \\
&= w^*_{[\{1,1,2,2\}]} - \frac{3}{1} \times 1 \times w_{[\{(1,1,1)\}]} && (28) \\
w^*_{[\{1,1,2,3\}]} &= 6 \times \sum_{i,j,k} w(i)^2 w(j) w(k) = 6 \times (\sum_i w(i)^2)(\sum_j w(j))^2 && (29) \\
w_{[\{1,1,2,3\}]} &= \sum_{i_1,i_2,i_3,i_4 \in [\{(1,1,2,3)\}]} w(i_1)w(i_2)w(i_3)w(i_4)
\end{aligned}
$$

$$
= w^*_{[\{1,1,2,3\}]} - \frac{6}{3} \times 1 \times w_{[\{(1,1,2,2)\}]} - \frac{6}{4} \times 2 \times w_{[\{(1,1,1,2)\}]} - \frac{6}{1} \times 1 \times w_{[\{(1,1,1,1)\}]} \quad (30)
$$

$$
w^*_{[\{1,2,3,4\}]} = \sum_{i,j,k,l} w(i)w(j)w(k)w(l) = (\sum_i w(i))^4 \quad (31)
$$

$$
w_{[\{1,2,3,4\}]} = w^*_{[\{1,2,3,4\}]} - w_{[\{1,1,2,3\}]} - w_{[\{1,1,2,2\}]} - w_{[\{1,1,1,2\}]} - w_{[\{1,1,1,1\}]} \quad (32)
$$

second order test statistics

The first moment

$$
\begin{aligned}
[U_2^1] &= [\{(1,1)\}] \bigcup [\{(1,2)\}] && (33) \\
w_{[\{(1,1)\}]} &= \sum_{i_1,i_2 \in [\{(1,1)\}]} w(i_1,i_2) = \sum_i w(i,i) && (34) \\
w^*_{[\{(1,2)\}]} &= \sum_{i_1,i_2} w(i_1,i_2) && (35) \\
w_{[\{(1,2)\}]} &= \sum_{i_1,i_2 \in [\{(1,2)\}]} w(i_1,i_2) = w^*_{[\{(1,2)\}]} - \frac{1}{1} \times 1 \times w_{[\{(1,1)\}]} && (36)
\end{aligned}
$$

$$
\begin{aligned}
E(T) &= \frac{h_{[\{(1,1)\}]}}{\#([\{(1,1)\}])} w_{[\{(1,1)\}]} + \frac{h_{[\{(1,2)\}]}}{\#([\{(1,2)\}])} w_{[\{(1,2)\}]} \\
&= \frac{h_{[\{(1,1)\}]}}{n} w_{[\{(1,1)\}]} + \frac{h_{[\{(1,2)\}]}}{n(n-1)} w_{[\{(1,2)\}]} && (37)
\end{aligned}
$$

The second moment

$$
\begin{aligned}
[U_2^2] &= [\{(1,1)(1,1)\}] \bigcup [\{(1,1)(1,2)\}] \bigcup [\{(1,1)(2,2)\}] \bigcup [\{(1,2)(1,2)\}] \\
&\quad \bigcup [\{(1,1)(2,3)\}] \bigcup [\{(1,2)(1,3)\}] \bigcup [\{(1,2)(3,4)\}] && (38) \\
w_{[\{(1,1)(1,1)\}]} &= \sum_{i_1,i_2,i_3,i_4 \in [\{(1,1)(1,1)\}]} w(i_1,i_2)w(i_3,i_4) = \sum_i w(i,i)^2 && (39) \\
w^*_{[\{(1,1)(1,2)\}]} &= 4 \times \sum_{i,j} w(i,i) w(i,j) && (40)
\end{aligned}
$$

$$w_{[\{(1,1)(1,2)\}]} = w^*_{[\{(1,1)(1,2)\}]} - \frac{4}{1} \times 1 \times w_{[\{(1,1)(1,1)\}]} \tag{41}$$

$$w^*_{[\{(1,1)(2,2)\}]} = \sum_{i,j} w(i,i)w(j,j) = (\sum_i w(i,i))^2 \tag{42}$$

$$w_{[\{(1,1)(2,2)\}]} = w^*_{[\{(1,1)(2,2)\}]} - \frac{1}{1} \times 1 \times w_{[\{(1,1)(1,1)\}]} \tag{43}$$

$$w^*_{[\{(1,2)(1,2)\}]} = 2 \times \sum_{i,j} w(i,j)^2 \tag{44}$$

$$w_{[\{(1,2)(1,2)\}]} = w^*_{[\{(1,2)(1,2)\}]} - \frac{2}{1} \times 1 \times w_{[\{(1,1)(1,1)\}]} \tag{45}$$

$$w^*_{[\{(1,1)(2,3)\}]} = 2 \times \sum_{i,j,k} w(i,i)w(j,k) \tag{46}$$

$$\begin{aligned} w_{[\{(1,1)(2,3)\}]} = w^*_{[\{(1,1)(2,3)\}]} - \frac{2}{1} \times 1 \times w_{[\{(1,1)(2,2)\}]} - \frac{2}{4} \times 2 \times w_{[\{(1,1)(1,2)\}]} \\ - \frac{2}{1} \times 1 \times w_{[\{(1,1)(1,1)\}]} \end{aligned} \tag{47}$$

$$w^*_{[\{(1,2)(1,3)\}]} = 4 \times \sum_{i,j,k} w(i,j)w(i,k) \tag{48}$$

$$\begin{aligned} w_{[\{(1,2)(1,3))\}]} = w^*_{[\{(1,2)(1,3)\}]} - \frac{4}{2} \times 1 \times w_{[\{(1,2)(1,2)\}]} - \frac{4}{4} \times 2 \times w_{[\{(1,1)(1,2)\}]} \\ - \frac{4}{1} \times 1 \times w_{[\{(1,1)(1,1)\}]} \end{aligned} \tag{49}$$

$$w^*_{[\{(1,2)(3,4)\}]} = \sum_{i,j,k,l} w(i,j)w(k,l) = (\sum_{i,j} w(i,j))^2 \tag{50}$$

$$\begin{aligned} w_{[\{(1,2)(3,4)\}]} = w^*_{[\{(1,2)(3,4)\}]} - w_{[\{(1,2)(1,3))\}]} - w_{[\{(1,1)(2,3)\}]} - w_{[\{(1,2)(1,2)\}]} \\ - w_{[\{(1,1)(2,2)\}]} - w_{[\{(1,1)(1,2)\}]} - w_{[\{(1,1)(1,1)\}]} \end{aligned} \tag{51}$$

$$\begin{aligned} E(T^2) = & \frac{h_{[\{(1,1)(1,1)\}]}}{\#([\{(1,1)(1,1)\}])} w_{[\{(1,1)(1,1)\}]} + \frac{h_{[\{(1,1)(1,2)\}]}}{\#([\{(1,1)(1,2)\}])} w_{[\{(1,1)(1,2)\}]} \\ & + \frac{h_{[\{(1,1)(2,2)\}]}}{\#([\{(1,1)(2,2)\}])} w_{[\{(1,1)(2,2)\}]} + \frac{h_{[\{(1,2)(1,2)\}]}}{\#([\{(1,2)(1,2)\}])} w_{[\{(1,2)(1,2)\}]} \\ & + \frac{h_{[\{(1,1)(2,3)\}]}}{\#([\{(1,1)(2,3)\}])} w_{[\{(1,1)(2,3)\}]} + \frac{h_{[\{(1,2)(1,3)\}]}}{\#([\{(1,2)(1,3)\}])} w_{[\{(1,2)(1,3)\}]} \\ & + \frac{h_{[\{(1,2)(3,4)\}]}}{\#([\{(1,2)(3,4)\}])} w_{[\{(1,2)(3,4)\}]} \\ = & \frac{h_{[\{(1,1)(1,1)\}]}}{n} w_{[\{(1,1)(1,1)\}]} + \frac{h_{[\{(1,1)(1,2)\}]}}{4n(n-1)} w_{[\{(1,1)(1,2)\}]} \\ & + \frac{h_{[\{(1,1)(2,2)\}]}}{n(n-1)} w_{[\{(1,1)(2,2)\}]} + \frac{h_{[\{(1,2)(1,2)\}]}}{2n(n-1)} w_{[\{(1,2)(1,2)\}]} \\ & + \frac{h_{[\{(1,1)(2,3)\}]}}{2n(n-1)(n-2)} w_{[\{(1,1)(2,3)\}]} + \frac{h_{[\{(1,2)(1,3)\}]}}{4n(n-1)(n-2)} w_{[\{(1,2)(1,3)\}]} \\ & + \frac{h_{[\{(1,2)(3,4)\}]}}{n(n-1)(n-2)(n-3)} w_{[\{(1,2)(3,4)\}]} \end{aligned} \tag{52}$$

The third moment

$$\begin{aligned} [U_2^3] = & [\{(1,1)(1,1)(1,1)\}] \bigcup [\{(1,1)(1,1)(1,2)\}] \bigcup [\{(1,1)(1,1)(2,2)\}] \\ & \bigcup [\{(1,1)(1,2)(1,2)\}] \bigcup [\{(1,1)(1,2)(2,2)\}] \bigcup [\{(1,2)(1,2)(1,2)\}] \\ & \bigcup [\{(1,1)(1,1)(2,3)\}] \bigcup [\{(1,1)(1,2)(1,3)\}] \bigcup [\{(1,1)(1,2)(3,3)\}] \end{aligned}$$

$$\bigcup[\{(1,2)(1,2)(1,3)\}]\bigcup[\{(1,1)(1,2)(2,3)\}]\bigcup[\{(1,1)(2,2)(3,3)\}]$$
$$\bigcup[\{(1,1)(2,3)(2,3)\}]\bigcup[\{(1,2)(1,3)(2,3)\}]\bigcup[\{(1,2)(1,3)(1,4)\}]$$
$$\bigcup[\{(1,1)(1,2)(3,4)\}]\bigcup[\{(1,1)(2,2)(3,4)\}]\bigcup[\{(1,1)(2,3)(2,4)\}]$$

$$\bigcup[\{(1,2)(1,2)(3,4)\}]\bigcup[\{(1,2)(1,3)(2,4)\}]\bigcup[\{(1,1)(2,3)(4,5)\}]$$
$$\bigcup[\{(1,2)(1,3)(4,5)\}]\bigcup[\{(1,2)(3,4)(5,6)\}] \tag{53}$$

$$w_{[\{(1,1)(1,1)(1,1)\}]} = \sum_{i_1,i_2,i_3,i_4,i_5,i_6 \in [\{(1,1)(1,1)(1,1)\}]} w(i_1,i_2)w(i_3,i_4)w(i_5,i_6) = \sum_i w(i,i)^3 \tag{54}$$

$$w^*_{[\{(1,1)(1,1)(1,2)\}]} = 6 \times \sum_{i,j} w(i,i)^2 w(i,j) = 6 \times \sum_i (w(i,i)^2 \sum_j w(i,j)) \tag{55}$$

$$w_{[\{(1,1)(1,1)(1,2)\}]} = w^*_{[\{(1,1)(1,1)(1,2)\}]} - \frac{6}{1} \times 1 \times w_{[\{(1,1)(1,1)(1,1)\}]} \tag{56}$$

$$w^*_{[\{(1,1)(1,1)(2,2)\}]} = 3 \times \sum_{i,j} w(i,i)^2 w(j,j) = 3 \times (\sum_i w(i,i)^2)(\sum_i w(i,i)) \tag{57}$$

$$w_{[\{(1,1)(1,1)(2,2)\}]} = w^*_{[\{(1,1)(1,1)(2,2)\}]} - \frac{3}{1} \times 1 \times w_{[\{(1,1)(1,1)(1,1)\}]} \tag{58}$$

$$w^*_{[\{(1,1)(1,2)(1,2)\}]} = 12 \times \sum_{i,j} w(i,i) w(i,j)^2 = 12 \times \sum_i (w(i,i) \sum_j w(i,j)^2) \tag{59}$$

$$w_{[\{(1,1)(1,2)(1,2)\}]} = w^*_{[\{(1,1)(1,2)(1,2)\}]} - \frac{12}{1} \times 1 \times w_{[\{(1,1)(1,1)(1,1)\}]} \tag{60}$$

$$\begin{aligned} w^*_{[\{(1,1)(1,2)(2,2)\}]} &= 6 \times \sum_{i,j} w(i,i)w(i,j)w(j,j) \\ &= 6 \times \sum_i (w(i,i) \sum_j w(i,j)w(j,j)) \end{aligned} \tag{61}$$

$$w_{[\{(1,1)(1,2)(2,2)\}]} = w^*_{[\{(1,1)(1,2)(2,2)\}]} - \frac{6}{1} \times 1 \times w_{[\{(1,1)(1,1)(1,1)\}]} \tag{62}$$

$$w^*_{[\{(1,2)(1,2)(1,2)\}]} = 4 \times \sum_{i,j} w(i,j)^3 \tag{63}$$

$$w_{[\{(1,2)(1,2)(1,2)\}]} = w^*_{[\{(1,2)(1,2)(1,2)\}]} - \frac{4}{1} \times 1 \times w_{[\{(1,1)(1,1)(1,1)\}]} \tag{64}$$

$$w^*_{[\{(1,1)(1,1)(2,3)\}]} = 3 \times \sum_{i,j,k} w(i,i)^2 w(j,k) = 3 \times (\sum_i w(i,i)^2)(\sum_{j,k} w(j,k)) \tag{65}$$

$$\begin{aligned} w_{[\{(1,1)(1,1)(2,3)\}]} &= w^*_{[\{(1,1)(1,1)(2,3)\}]} - \frac{3}{3} \times 1 \times w_{[\{(1,1)(1,1)(2,2)\}]} \\ &- \frac{3}{6} \times 2 \times w_{[\{(1,1)(1,1)(1,2)\}]} - \frac{3}{1} \times 1 \times w_{[\{(1,1)(1,1)(1,1)\}]} \end{aligned} \tag{66}$$

$$w^*_{[\{(1,1)(1,2)(1,3)\}]} = 12 \times \sum_{i,j,k} w(i,i)w(i,j)w(i,k) = 12 \times \sum_i (w(i,i))(\sum_j w(i,j)^2) \tag{67}$$

$$\begin{aligned} w_{[\{(1,1)(1,2)(1,3)\}]} &= w^*_{[\{(1,1)(1,2)(1,3)\}]} - \frac{12}{12} \times 1 \times w_{[\{(1,1)(1,2)(1,2)\}]} \\ &- \frac{12}{6} \times 2 \times w_{[\{(1,1)(1,1)(1,2)\}]} - \frac{12}{1} \times 1 \times w_{[\{(1,1)(1,1)(1,1)\}]} \end{aligned} \tag{68}$$

$$
\begin{aligned}
w^*_{[\{(1,1)(1,2)(3,3)\}]} &= 12 \times \sum_{i,j,k} w(i,i)w(i,j)w(k,k) \\
&= 12 \times (\sum_i (w(i,i) \sum_j w(i,j)))(\sum_k w(k,k))
\end{aligned} \tag{69}
$$

$$
\begin{aligned}
w_{[\{(1,1)(1,2)(3,3)\}]} &= w^*_{[\{(1,1)(1,2)(3,3)\}]} - \frac{12}{6} \times 1 \times w_{[\{(1,1)(1,2)(2,2)\}]} \\
-\frac{12}{3} \times 1 \times w_{[\{(1,1)(1,1)(2,2)\}]} &- \frac{12}{6} \times 1 \times w_{[\{(1,1)(1,1)(1,2)\}]} \\
&- \frac{12}{1} \times 1 \times w_{[\{(1,1)(1,1)(1,1)\}]}
\end{aligned} \tag{70}
$$

$$
w^*_{[\{(1,2)(1,2)(1,3)\}]} = 24 \times \sum_{i,j,k} w(i,j)^2 w(i,k) = 24 \times \sum_i ((\sum_j w(i,j)^2)(\sum_k w(i,k))) \tag{71}
$$

$$
\begin{aligned}
w_{[\{(1,2)(1,2)(1,3)\}]} &= w^*_{[\{(1,2)(1,2)(1,3)\}]} - \frac{24}{4} \times 1 \times w_{[\{(1,2)(1,2)(1,2)\}]} \\
-\frac{24}{12} \times 1 \times w_{[\{(1,1)(1,2)(1,2)\}]} &- \frac{24}{6} \times 1 \times w_{[\{(1,1)(1,1)(1,2)\}]} \\
&- \frac{24}{1} \times 1 \times w_{[\{(1,1)(1,1)(1,1)\}]}
\end{aligned} \tag{72}
$$

$$
\begin{aligned}
w^*_{[\{(1,1)(1,2)(2,3)\}]} &= 24 \times \sum_{i,j,k} w(i,i)w(i,j)w(j,k) \\
&= 24 \times \sum_j ((\sum_i w(i,i)w(i,j))(\sum_k w(j,k)))
\end{aligned} \tag{73}
$$

$$
\begin{aligned}
w_{[\{(1,1)(1,2)(2,3)\}]} &= w^*_{[\{(1,1)(1,2)(2,3)\}]} - \frac{24}{6} \times 1 \times w_{[\{(1,1)(1,2)(2,2)\}]} \\
-\frac{24}{12} \times 1 \times w_{[\{(1,1)(1,2)(1,2)\}]} &- \frac{24}{6} \times 1 \times w_{[\{(1,1)(1,1)(1,2)\}]} \\
&- \frac{24}{1} \times 1 \times w_{[\{(1,1)(1,1)(1,1)\}]}
\end{aligned} \tag{74}
$$

$$
w^*_{[\{(1,1)(2,2)(3,3)\}]} = 1 \times \sum_{i,j,k} w(i,i)w(j,j)w(k,k) = (\sum_i w(i,i))^3 \tag{75}
$$

$$
\begin{aligned}
w_{[\{(1,1)(2,2)(3,3)\}]} &= w^*_{[\{(1,1)(2,2)(3,3)\}]} - \frac{1}{3} \times 3 \times w_{[\{(1,1)(1,1)(2,2)\}]} \\
&- \frac{1}{1} \times 1 \times w_{[\{(1,1)(1,1)(1,1)\}]}
\end{aligned} \tag{76}
$$

$$
w^*_{[\{(1,1)(2,3)(2,3)\}]} = 6 \times \sum_{i,j,k} w(i,i)w(j,k)w(j,k) = 6 \times (\sum_i w(i,i))(\sum_{j,k} w(j,k)^2) \tag{77}
$$

$$
\begin{aligned}
w_{[\{(1,1)(2,3)(2,3)\}]} &= w^*_{[\{(1,1)(2,3)(2,3))\}]} - \frac{6}{3} \times 1 \times w_{[\{(1,1)(1,1)(2,2)\}]} \\
-\frac{6}{12} \times 2 \times w_{[\{(1,1)(1,2)(1,2)\}]} &- \frac{6}{1} \times 1 \times w_{[\{(1,1)(1,1)(1,1)\}]}
\end{aligned} \tag{78}
$$

$$
\begin{aligned}
w^*_{[\{(1,2)(1,3)(2,3)\}]} &= 8 \times \sum_{i,j,k} w(i,j)w(j,k)w(i,k) \\
&= 8 \times \sum_{i,j} w(i,j)(\sum_k w(j,k)w(i,k))
\end{aligned} \tag{79}
$$

$$
\begin{aligned}
w_{[\{(1,2)(1,3)(2,3)\}]} &= w^*_{[\{(1,2)(1,3)(2,3)\}]} - \frac{8}{12} \times 3 \times w_{[\{(1,1)(1,2)(1,2)\}]} \\
&- \frac{8}{1} \times 1 \times w_{[\{(1,1)(1,1)(1,1)\}]}
\end{aligned} \tag{80}
$$

$$w^*_{[\{(1,2)(1,3)(1,4)\}]} = 8 \times \sum_{i,j,k,l} w(i,j)w(i,k)w(i,l) = 8 \times \sum_i (\sum_j w(i,j))^3 \tag{81}$$

$$\begin{aligned} w_{[\{(1,2)(1,3)(1,4)\}]} = w^*_{[\{(1,2)(1,3)(1,4)\}]} - \frac{8}{24} \times 3 \times w_{[\{(1,2)(1,2)(1,3)\}]} \\ -\frac{8}{12} \times 3 \times w_{[\{(1,1)(1,2)(1,3)\}]} - \frac{8}{4} \times 1 \times w_{[\{(1,2)(1,2)(1,2)\}]} \\ -\frac{8}{12} \times 3 \times w_{[\{(1,1)(1,2)(1,2)\}]} - \frac{8}{6} \times 3 \times w_{[\{(1,1)(1,1)(1,2)\}]} \\ -\frac{8}{1} \times 1 \times w_{[\{(1,1)(1,1)(1,1)\}]} \end{aligned} \tag{82}$$

$$\begin{aligned} w^*_{[\{(1,1)(1,2)(3,4)\}]} &= 12 \times \sum_{i,j,k,l} w(i,i)w(i,j)w(k,l) \\ &= 12 \times (\sum_{i,j} w(i,i)w(i,j))(\sum_{k,l} w(k,l)) \end{aligned} \tag{83}$$

$$\begin{aligned} w_{[\{(1,1)(1,2)(3,4)\}]} = w^*_{[\{(1,1)(1,2)(3,4)\}]} - \frac{12}{24} \times 2 \times w_{[\{(1,1)(1,2)(2,3)\}]} \\ -\frac{12}{12} \times 1 \times w_{[\{(1,1)(1,2)(3,3)\}]} - \frac{12}{3} \times 1 \times w_{[\{(1,1)(1,1)(2,3)\}]} \\ -\frac{12}{12} \times 2 \times w_{[\{(1,1)(1,2)(1,3)\}]} - \frac{12}{6} \times 1 \times w_{[\{(1,1)(1,2)(2,2)\}]} \\ -\frac{12}{3} \times 1 \times w_{[\{(1,1)(1,1)(2,2)\}]} - \frac{12}{12} \times 2 \times w_{[\{(1,1)(1,2)(1,2)\}]} \\ -\frac{12}{6} \times 3 \times w_{[\{(1,1)(1,1)(1,2)\}]} - \frac{12}{1} \times 1 \times w_{[\{(1,1)(1,1)(1,1)\}]} \end{aligned} \tag{84}$$

$$\begin{aligned} w^*_{[\{(1,1)(2,2)(3,4)\}]} &= 3 \times \sum_{i,j,k,l} w(i,i)w(j,j)w(k,l) \\ &= 3 \times (\sum_i w(i,i))^2(\sum_{k,l} w(k,l)) \end{aligned} \tag{85}$$

$$\begin{aligned} w_{[\{(1,1)(2,2)(3,4)\}]} = w^*_{[\{(1,1)(2,2)(3,4)\}]} - \frac{3}{1} \times 1 \times w_{[\{(1,1)(2,2)(3,3)\}]} \\ -\frac{3}{12} \times 4 \times w_{[\{(1,1)(1,2)(3,3)\}]} - \frac{3}{3} \times 1 \times w_{[\{(1,1)(1,1)(2,3)\}]} \\ -\frac{3}{6} \times 2 \times w_{[\{(1,1)(1,2)(2,2)\}]} - \frac{3}{3} \times 3 \times w_{[\{(1,1)(1,1)(2,2)\}]} \\ -\frac{3}{6} \times 2 \times w_{[\{(1,1)(1,1)(1,2)\}]} - \frac{3}{1} \times 1 \times w_{[\{(1,1)(1,1)(1,1)\}]} \end{aligned} \tag{86}$$

$$\begin{aligned} w^*_{[\{(1,1)(2,3)(2,4)\}]} &= 12 \times \sum_{i,j,k,l} w(i,i)w(j,k)w(j,l) \\ &= 12 \times (\sum_i w(i,i))(\sum_j(\sum_k w(j,k))^2) \end{aligned} \tag{87}$$

$$\begin{aligned} w_{[\{(1,1)(2,3)(2,4)\}]} = w^*_{[\{(1,1)(2,3)(2,4)\}]} - \frac{12}{6} \times 1 \times w_{[\{(1,1)(2,3)(2,3)\}]} \\ -\frac{12}{24} \times 2 \times w_{[\{(1,1)(1,2)(2,3)\}]} - \frac{12}{12} \times 2 \times w_{[\{(1,1)(1,2)(3,3)\}]} \\ -\frac{12}{12} \times 1 \times w_{[\{(1,1)(1,2)(1,3)\}]} - \frac{12}{6} \times 2 \times w_{[\{(1,1)(1,2)(2,2)\}]} \end{aligned}$$

$$-\frac{12}{3}\times 1\times w_{[\{(1,1)(1,1)(2,2)\}]}-\frac{12}{6}\times 2\times w_{[\{(1,1)(1,1)(1,2)\}]}$$
$$-\frac{12}{1}\times 1\times w_{[\{(1,1)(1,1)(1,1)\}]} \tag{88}$$
$$w^*_{[\{(1,2)(1,2)(3,4)\}]}=6\times\sum_{i,j,k,l}w(i,j)w(i,j)w(k,l)$$
$$=6\times(\sum_{i,j}w(i,j)^2)(\sum_{k,l}w(k,l)) \tag{89}$$
$$w_{[\{(1,2)(1,2)(3,4)\}]}=w^*_{[\{(1,2)(1,2)(3,4)\}]}-\frac{6}{6}\times 1\times w_{[\{(1,1)(2,3)(2,3)\}]}$$
$$-\frac{6}{24}\times 4\times w_{[\{(1,2)(1,2)(1,3)\}]}-\frac{6}{3}\times 1\times w_{[\{(1,1)(1,1)(2,3)\}]}$$
$$-\frac{6}{4}\times 2\times w_{[\{(1,2)(1,2)(1,2)\}]}-\frac{6}{12}\times 2\times w_{[\{(1,1)(1,2)(1,2)\}]}$$
$$-\frac{6}{3}\times 1\times w_{[\{(1,1)(1,1)(2,2)\}]}-\frac{6}{6}\times 2\times w_{[\{(1,1)(1,1)(1,2)\}]}$$
$$-\frac{6}{1}\times 1\times w_{[\{(1,1)(1,1)(1,1)\}]} \tag{90}$$
$$w^*_{[\{(1,2)(1,3)(2,4)\}]}=24\times\sum_{i,j,k,l}w(i,j)w(i,k)w(j,l)$$
$$=24\times\sum_{i,j}(w(i,j)(\sum_{k}w(i,k))(\sum_{l}w(j,l))) \tag{91}$$
$$w_{[\{(1,2)(1,3)(2,4)\}]}=w^*_{[\{(1,2)(1,3)(2,4)\}]}-\frac{24}{8}\times 1\times w_{[\{(1,2)(1,3)(2,3)\}]}$$
$$-\frac{24}{24}\times 2\times w_{[\{(1,1)(1,2)(2,3)\}]}-\frac{24}{24}\times 2\times w_{[\{(1,2)(1,2)(1,3)\}]}$$
$$-\frac{24}{12}\times 1\times w_{[\{(1,1)(1,2)(1,3)\}]}-\frac{24}{6}\times 1\times w_{[\{(1,1)(1,2)(2,2)\}]}$$
$$-\frac{24}{4}\times 1\times w_{[\{(1,2)(1,2)(1,2)\}]}-\frac{24}{12}\times 3\times w_{[\{(1,1)(1,2)(1,2)\}]}$$
$$-\frac{24}{6}\times 2\times w_{[\{(1,1)(1,1)(1,2)\}]}-\frac{24}{1}\times 1\times w_{[\{(1,1)(1,1)(1,1)\}]} \tag{92}$$
$$w^*_{[\{(1,1)(2,3)(4,5)\}]}=3\times\sum_{i,j,k,l,m}w(i,i)w(j,k)w(l,m)$$
$$=3\times(\sum_{i}(w(i,i))(\sum_{j,k}w(j,k))^2 \tag{93}$$
$$w_{[\{(1,1)(2,3)(4,5)\}]}=w^*_{[\{(1,1)(2,3)(4,5)\}]}-\frac{3}{3}\times 2\times w_{[\{(1,1)(2,2)(3,4)\}]}$$
$$-\frac{3}{12}\times 4\times w_{[\{(1,1)(2,3)(2,4)\}]}-\frac{3}{12}\times 4\times w_{[\{(1,1)(1,2)(3,4)\}]}$$
$$-\frac{3}{1}\times 1\times w_{[\{(1,1)(2,2)(3,3)\}]}-\frac{3}{6}\times 2\times w_{[\{(1,1)(2,3)(2,3)\}]}$$
$$-\frac{3}{12}\times 4\times w_{[\{(1,1)(1,2)(3,3)\}]}-\frac{3}{3}\times 2\times w_{[\{(1,1)(1,1)(2,3)\}]}$$
$$-\frac{3}{12}\times 4\times w_{[\{(1,1)(1,2)(1,3)\}]}-\frac{3}{6}\times 4\times w_{[\{(1,1)(1,2)(2,2)\}]}$$

$$
\begin{aligned}
&-\frac{3}{3}\times 3\times w_{[\{(1,1)(1,1)(2,2)\}]}-\frac{3}{12}\times 4\times w_{[\{(1,1)(1,2)(1,2)\}]}\\
&-\frac{3}{6}\times 4\times w_{[\{(1,1)(1,1)(1,2)\}]}-\frac{3}{1}\times 1\times w_{[\{(1,1)(1,1)(1,1)\}]} \qquad (94)\\
w^*_{[\{(1,2)(1,3)(4,5)\}]} &= 12\times\sum_{i,j,k,l,m} w(i,j)w(i,k)w(l,m)\\
&= 12\times(\sum_i(\sum_j w(i,j))^2)(\sum_{l,m} w(l,m)) \qquad (95)
\end{aligned}
$$

$$
\begin{aligned}
w_{[\{((1,2)(1,3)(4,5)\}]} &= w^*_{[\{(1,2)(1,3)(4,5)\}]}-\frac{12}{12}\times 1\times w_{[\{(1,1)(2,3)(2,4)\}]}\\
&-\frac{12}{6}\times 1\times w_{[\{(1,2)(1,2)(3,4)\}]}-\frac{12}{24}\times 4\times w_{[\{(1,2)(1,3)(2,4)\}]}\\
&-\frac{12}{8}\times 2\times w_{[\{(1,2)(1,3)(1,4)\}]}-\frac{12}{12}\times 2\times w_{[\{(1,1)(1,2)(3,4)\}]}\\
&-\frac{12}{6}\times 1\times w_{[\{(1,1)(2,3)(2,3)\}]}-\frac{12}{8}\times 2\times w_{[\{(1,2)(1,3)(2,3)\}]}\\
&-\frac{12}{12}\times 2\times w_{[\{(1,1)(1,2)(3,3)\}]}-\frac{12}{24}\times 4\times w_{[\{(1,2)(1,2)(1,3)\}]}\\
&-\frac{12}{24}\times 6\times w_{[\{(1,1)(1,2)(2,3)\}]}-\frac{12}{3}\times 1\times w_{[\{(1,1)(1,1)(2,3)\}]}\\
&-\frac{12}{12}\times 5\times w_{[\{(1,1)(1,2)(1,3)\}]}-\frac{12}{6}\times 2\times w_{[\{(1,1)(1,2)(2,2)\}]}\\
&-\frac{12}{4}\times 2\times w_{[\{(1,2)(1,2)(1,2)\}]}-\frac{12}{3}\times 1\times w_{[\{(1,1)(1,1)(2,2)\}]}\\
&-\frac{12}{12}\times 5\times w_{[\{(1,1)(1,2)(1,2)\}]}-\frac{12}{6}\times 4\times w_{[\{(1,1)(1,1)(1,2)\}]}\\
&-\frac{12}{1}\times 1\times w_{[\{(1,1)(1,1)(1,1)\}]} \qquad (96)
\end{aligned}
$$

$$
\begin{aligned}
w_{[\{(1,2)(3,4)(5,6)\}]} &= w^*_{[\{(1,2)(3,4)(5,6)\}]}-w_{[\{(1,1)(1,1)(1,1)\}]}-w_{[\{(1,1)(1,1)(1,2)\}]}\\
&-w_{[\{(1,1)(1,1)(2,2)\}]}-w_{[\{(1,1)(1,2)(1,2)\}]}-w_{[\{(1,1)(1,2)(2,2)\}]}\\
&-w_{[\{(1,2)(1,2)(1,2)\}]}-w_{[\{(1,1)(1,1)(2,3)\}]}-w_{[\{(1,1)(1,2)(1,3)\}]}\\
&-w_{[\{(1,1)(1,2)(3,3)\}]}-w_{[\{(1,2)(1,2)(1,3)\}]}-w_{[\{(1,1)(1,2)(2,3)\}]}\\
&-w_{[\{(1,1)(2,2)(3,3)\}]}-w_{[\{(1,1)(2,3)(2,3)\}]}-w_{[\{(1,2)(1,3)(2,3)\}]}\\
&-w_{[\{(1,2)(1,3)(1,4)\}]}-w_{[\{(1,1)(1,2)(3,4)\}]}-w_{[\{(1,1)(2,2)(3,4)\}]}\\
&-w_{[\{(1,1)(2,3)(2,4)\}]}-w_{[\{(1,2)(1,2)(3,4)\}]}-w_{[\{(1,2)(1,3)(2,4)\}]}\\
&-w_{[\{(1,1)(2,3)(4,5)\}]}-w_{[\{(1,2)(1,3)(4,5)\}]} \qquad (97)\\
& \qquad (98)
\end{aligned}
$$

$$
\begin{aligned}
E(T^3) &= \frac{h_{[\{(1,1)(1,1)(1,1)\}]}}{\#([\{(1,1)(1,1)(1,1)\}])}w_{[\{(1,1)(1,1)(1,1)\}]}\\
&+\frac{h_{[\{(1,1)(1,1)(1,2)\}]}}{\#([\{(1,1)(1,1)(1,2)\}])}w_{[\{(1,1)(1,1)(1,2)\}]}+\frac{h_{[\{(1,1)(1,1)(2,2)\}]}}{\#([\{(1,1)(1,1)(2,2)\}])}w_{[\{(1,1)(1,1)(2,2)\}]}\\
&+\frac{h_{[\{(1,1)(1,2)(1,2)\}]}}{\#([\{(1,1)(1,2)(1,2)\}])}w_{[\{(1,1)(1,2)(1,2)\}]}+\frac{h_{[\{(1,1)(1,2)(2,2)\}]}}{\#([\{(1,1)(1,2)(2,2)\}])}w_{[\{(1,1)(1,2)(2,2)\}]}\\
&+\frac{h_{[\{(1,2)(1,2)(1,2)\}]}}{\#([\{(1,2)(1,2)(1,2)\}])}w_{[\{(1,2)(1,2)(1,2)\}]}+\frac{h_{[\{(1,1)(1,1)(2,3)\}]}}{\#([\{(1,1)(1,1)(2,3)\}])}w_{[\{(1,1)(1,1)(2,3)\}]}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{h_{[\{(1,1)(1,2)(1,3)\}]}}{\#([\{(1,1)(1,2)(1,3)\}])}w_{[\{(1,1)(1,2)(1,3)\}]}+\frac{h_{[\{(1,1)(1,2)(3,3)\}]}}{\#([\{(1,1)(1,2)(3,3)\}])}w_{[\{(1,1)(1,2)(3,3)\}]}\\
&+\frac{h_{[\{(1,2)(1,2)(1,3)\}]}}{\#([\{(1,2)(1,2)(1,3)\}])}w_{[\{(1,2)(1,2)(1,3)\}]}+\frac{h_{[\{(1,1)(1,2)(2,3)\}]}}{\#([\{(1,1)(1,2)(2,3)\}])}w_{[\{(1,1)(1,2)(2,3)\}]}\\
&+\frac{h_{[\{(1,1)(2,2)(3,3)\}]}}{\#([\{(1,1)(2,2)(3,3)\}])}w_{[\{(1,1)(2,2)(3,3)\}]}+\frac{h_{[\{(1,1)(2,3)(2,3)\}]}}{\#([\{(1,1)(2,3)(2,3)\}])}w_{[\{(1,1)(2,3)(2,3)\}]}\\
&+\frac{h_{[\{(1,2)(1,3)(2,3)\}]}}{\#([\{(1,2)(1,3)(2,3)\}])}w_{[\{(1,2)(1,3)(2,3)\}]}+\frac{h_{[\{(1,2)(1,3)(1,4)\}]}}{\#([\{(1,2)(1,3)(1,4)\}])}w_{[\{(1,2)(1,3)(1,4)\}]}\\
&+\frac{h_{[\{(1,1)(1,2)(3,4)\}]}}{\#([\{(1,1)(1,2)(3,4)\}])}w_{[\{(1,1)(1,2)(3,4)\}]}+\frac{h_{[\{(1,1)(2,2)(3,4)\}]}}{\#([\{(1,1)(2,2)(3,4)\}])}w_{[\{(1,1)(2,2)(3,4)\}]}\\
&+\frac{h_{[\{(1,1)(2,3)(2,4)\}]}}{\#([\{(1,1)(2,3)(2,4)\}])}w_{[\{(1,1)(2,3)(2,4)\}]}+\frac{h_{[\{(1,2)(1,2)(3,4)\}]}}{\#([\{(1,2)(1,2)(3,4)\}])}w_{[\{(1,2)(1,2)(3,4)\}]}\\
&+\frac{h_{[\{(1,2)(1,3)(2,4)\}]}}{\#([\{(1,2)(1,3)(2,4)\}])}w_{[\{(1,2)(1,3)(2,4)\}]}+\frac{h_{[\{(1,1)(2,3)(4,5)\}]}}{\#([\{(1,1)(2,3)(4,5)\}])}w_{[\{(1,1)(2,3)(4,5)\}]}\\
&+\frac{h_{[\{(1,2)(1,3)(4,5)\}]}}{\#([\{(1,2)(1,3)(4,5)\}])}w_{[\{(1,2)(1,3)(4,5)\}]}+\frac{h_{[\{(1,2)(3,4)(5,6)\}]}}{\#([\{(1,2)(3,4)(5,6)\}])}w_{[\{(1,2)(3,4)(5,6)\}]}\\
&=\frac{h_{[\{(1,1)(1,1)(1,1)\}]}}{n}w_{[\{(1,1)(1,1)(1,1)\}]}+\frac{h_{[\{(1,1)(1,1)(1,2)\}]}}{6n(n_1)}w_{[\{(1,1)(1,1)(1,2)\}]}\\
&+\frac{h_{[\{(1,1)(1,1)(2,2)\}]}}{3n(n_1)}w_{[\{(1,1)(1,1)(2,2)\}]}+\frac{h_{[\{(1,1)(1,2)(1,2)\}]}}{12n(n_1)}w_{[\{(1,1)(1,2)(1,2)\}]}\\
&+\frac{h_{[\{(1,1)(1,2)(2,2)\}]}}{6n(n_1)}w_{[\{(1,1)(1,2)(2,2)\}]}+\frac{h_{[\{(1,2)(1,2)(1,2)\}]}}{4n(n_1)}w_{[\{(1,2)(1,2)(1,2)\}]}\\
&+\frac{h_{[\{(1,1)(1,1)(2,3)\}]}}{3n(n_1)(n-2)}w_{[\{(1,1)(1,1)(2,3)\}]}+\frac{h_{[\{(1,1)(1,2)(1,3)\}]}}{12n(n_1)(n-2)}w_{[\{(1,1)(1,2)(1,3)\}]}\\
&+\frac{h_{[\{(1,1)(1,2)(3,3)\}]}}{12n(n_1)(n-2)}w_{[\{(1,1)(1,2)(3,3)\}]}+\frac{h_{[\{(1,2)(1,2)(1,3)\}]}}{24n(n_1)(n-2)}w_{[\{(1,2)(1,2)(1,3)\}]}\\
&+\frac{h_{[\{(1,1)(1,2)(2,3)\}]}}{24n(n_1)(n-2)}w_{[\{(1,1)(1,2)(2,3)\}]}+\frac{h_{[\{(1,1)(2,2)(3,3)\}]}}{n(n_1)(n-2)}w_{[\{(1,1)(2,2)(3,3)\}]}\\
&+\frac{h_{[\{(1,1)(2,3)(2,3)\}]}}{6n(n_1)(n-2)}w_{[\{(1,1)(2,3)(2,3)\}]}+\frac{h_{[\{(1,2)(1,3)(2,3)\}]}}{8n(n_1)(n-2)}w_{[\{(1,2)(1,3)(2,3)\}]}\\
&+\frac{h_{[\{(1,2)(1,3)(1,4)\}]}}{8n(n_1)(n-2)(n-3)}w_{[\{(1,2)(1,3)(1,4)\}]}+\frac{h_{[\{(1,1)(1,2)(3,4)\}]}}{12n(n_1)(n-2)(n-3)}w_{[\{(1,1)(1,2)(3,4)\}]}\\
&+\frac{h_{[\{(1,1)(2,2)(3,4)\}]}}{3n(n_1)(n-2)(n-3)}w_{[\{(1,1)(2,2)(3,4)\}]}+\frac{h_{[\{(1,1)(2,3)(2,4)\}]}}{12n(n_1)(n-2)(n-3)}w_{[\{(1,1)(2,3)(2,4)\}]}\\
&+\frac{h_{[\{(1,2)(1,2)(3,4)\}]}}{6n(n_1)(n-2)(n-3)}w_{[\{(1,2)(1,2)(3,4)\}]}+\frac{h_{[\{(1,2)(1,3)(2,4)\}]}}{24n(n_1)(n-2)(n-3)}w_{[\{(1,2)(1,3)(2,4)\}]}\\
&+\frac{h_{[\{(1,1)(2,3)(4,5)\}]}}{3n(n_1)(n-2)(n-3)(n-4)}w_{[\{(1,1)(2,3)(4,5)\}]}+\frac{h_{[\{(1,2)(1,3)(4,5)\}]}}{12n(n_1)(n-2)(n-3)(n-4)}w_{[\{(1,2)(1,3)(4,5)\}]}\\
&+\frac{h_{[\{(1,2)(3,4)(5,6)\}]}}{n(n_1)(n-2)(n-3)(n-4)(n-5)}w_{[\{(1,2)(3,4)(5,6)\}]}
\end{aligned}
\tag{99}
$$