A Appendix

The problem in Equation (2) is a convex optimization problem. It can be rewritten as

$$\underset{\alpha}{\text{minimize}} \quad \max\left\{\frac{a_1}{b_1+\alpha}, \cdots, \frac{a_n}{b_n+\alpha}\right\} + \lambda\alpha, \qquad \text{subject to} \quad \alpha > 0. \tag{A.1}$$

Equation (A.1) is a convex optimization problem because the first term of the objective function is the sum of a pointwise maximum of convex functions, $a_i/(b_i+\alpha)$, which is convex in α.

Theorem 1. *The sequence $\{f(\mathbf{x}^j)\}_{j=1,2,\cdots}$ provided by Equation (7) converges to the local maximum of the density field.*

Proof. $f(\mathbf{x})$ shown in Equation (3) is a bounded function because it is the sum of finite bounded kernel density functions. To prove the theorem, it is sufficient to show that the sequence $\{f(\mathbf{x}^j)\}_{j=1,2,\cdots}$ is strictly monotonically increasing, i.e., $f(\mathbf{x}^j) < f(\mathbf{x}^{j+1})$, if $\mathbf{x}^j \neq \mathbf{x}^{j+1}$.

From Equation (3),

$$f(\mathbf{x}^{j+1}) - f(\mathbf{x}^j) = \frac{c}{N}\sum_{i=1}^{N}\frac{1}{h_i}\left(k\left(\frac{\|\mathbf{d}(\mathbf{l}_i,\mathbf{x}^{j+1})\|^2}{h_i^2}\right) - k\left(\frac{\|\mathbf{d}(\mathbf{l}_i,\mathbf{x}^{j})\|^2}{h_i^2}\right)\right). \tag{A.2}$$

If a function, $\phi(z)$, is convex, the following inequality holds:

$$\phi(z_2) - \phi(z_1) \geq \phi'(z_1)(z_2 - z_1), \tag{A.3}$$

where ϕ' is the derivative of ϕ.

The profile, $k(z)$, of the Gaussian kernel density function is convex and it satisfies Equation (A.3):

$$k(z_2) - k(z_1) \quad \geq \quad k'(z_1)(z_2 - z_1). \tag{A.4}$$

$\|\mathbf{z}\|^2$ is a convex function in $\mathbf{z}$ where $\mathbf{z}$ is a vector and thus,

$$\|\mathbf{z}_2\|^2 - \|\mathbf{z}_1\|^2 \geq 2\mathbf{z}^\mathsf{T}\left(\mathbf{z}_2 - \mathbf{z}_1\right). \tag{A.5}$$

The perspective distance vector function, $\mathbf{d}(\mathbf{l_i}, \mathbf{z})$, is also convex in $\mathbf{z}$ because it is a linear-fractional function $[1, 2]$ which perserves the convexity and thus,

$$\mathbf{d}(\mathbf{l}_i, \mathbf{z}_2) - \mathbf{d}(\mathbf{l}_i, \mathbf{z}_1) \geq \nabla_{\mathbf{z}}(\mathbf{d}(\mathbf{l}_i, \mathbf{z}_1))\left(\mathbf{z}_2 - \mathbf{z}_1\right). \tag{A.6}$$

From Equation (A.5) and (A.6), the following inequality holds:

$$\begin{aligned}\left\|\mathbf{d}(\mathbf{l}_i,\mathbf{x}^{j+1})\right\|^2 - \left\|\mathbf{d}(\mathbf{l}_i,\mathbf{x}^{j})\right\|^2 \quad &\geq \quad 2\mathbf{d}(\mathbf{l}_i,\mathbf{x}^j)^\mathsf{T}\left(\mathbf{d}(\mathbf{l}_i,\mathbf{x}^{j+1}) - \mathbf{d}(\mathbf{l}_i,\mathbf{x}^j)\right) \\ &\geq \quad 2\mathbf{d}(\mathbf{l}_i,\mathbf{x}^j)^\mathsf{T}\left(\left(\nabla_{\mathbf{x}}\mathbf{d}(\mathbf{l}_i,\mathbf{x}^j)\right)\left(\mathbf{x}^{j+1} - \mathbf{x}^j\right)\right).\end{aligned} \tag{A.7}$$

Equation (A.2) can be rewritten as,

$$
\begin{aligned}
f(\mathbf{x}^{j+1}) - f(\mathbf{x}^j) &\geq \frac{c}{N}\sum_{i=1}^{N}\frac{1}{h_i^3}k'\left(\left\|\frac{\mathbf{d}(\mathbf{l}_i,\mathbf{x}^j)}{h}\right\|^2\right)\left[\left\|\mathbf{d}(\mathbf{l}_i,\mathbf{x}^{j+1})\right\|^2 - \left\|\mathbf{d}(\mathbf{l}_i,\mathbf{x}^j)\right\|^2\right] \\
&\qquad\qquad\qquad\qquad\qquad \text{by Inequality (A.4)} \qquad\qquad (\text{A.8}) \\
&\geq \frac{2c}{N}\sum_{i=1}^{N}\frac{1}{h_i^3}k'\left(\left\|\frac{\mathbf{d}(\mathbf{l}_i,\mathbf{x}^j)}{h}\right\|^2\right)\mathbf{d}(\mathbf{l}_i,\mathbf{x}^j)^\mathsf{T}\left(\nabla_\mathbf{x}\mathbf{d}(\mathbf{l}_i,\mathbf{x}^j)\right)\left(\mathbf{x}^{j+1}-\mathbf{x}^j\right) \\
&\qquad\qquad\qquad\qquad\qquad \text{by Inequality (A.7)} \qquad\qquad (\text{A.9}) \\
&= \frac{2c}{N}\sum_{i=1}^{N} w_i^j\left(\widetilde{\mathbf{x}}_i^j - \mathbf{x}^j\right)^\mathsf{T}\left(\mathbf{x}^{j+1}-\mathbf{x}^j\right) \quad \text{by Equation (5)} \qquad (\text{A.10}) \\
&= \frac{2c}{N}\left[\left(\sum_{i=1}^{N} w_i^j(\widetilde{\mathbf{x}}_i^j)^\mathsf{T}\right)\mathbf{x}^{j+1} - \left(\sum_{i=1}^{N} w_i^j(\widetilde{\mathbf{x}}_i^j)^\mathsf{T}\right)\mathbf{x}^j\right. \qquad (\text{A.11}) \\
&- \left.\sum_{i=1}^{N} w_i^j(\mathbf{x}^j)^\mathsf{T}\mathbf{x}^{j+1} + \sum_{i=1}^{N} w_i^j(\mathbf{x}^j)^\mathsf{T}\mathbf{x}^j\right] \qquad \text{by expansion} \qquad (\text{A.12}) \\
&= \frac{2c}{N}\sum_{i=1}^{N} w_i^j\left(\|\mathbf{x}^{j+1}\|^2 - 2\left(\mathbf{x}^{j+1}\right)^\mathsf{T}\mathbf{x}^j + \|\mathbf{x}^j\|^2\right) \\
&\qquad \text{because } \sum_{i=1}^{N} w_i^j\mathbf{x}^{j+1} = \sum_{i=1}^{N} w_i^j\widetilde{\mathbf{x}}_i^j \text{ from Equation (6)} \qquad (\text{A.13}) \\
&= \frac{2c}{N}\|\mathbf{x}^{j+1}-\mathbf{x}^j\|^2\sum_{i=1}^{N} w_i^j. \qquad (\text{A.14})
\end{aligned}
$$

Since the profile, $k(x)$, is monotonically decreasing, $k'(x) < 0$. This leads the weight w_i^j to be strictly positive. As a result, the right hand side of Inequality (A.14) is strictly positive if $\mathbf{x}^j \neq \mathbf{x}^{j+1}$. Thus, $f(\mathbf{x}^{j+1}) - f(\mathbf{x}^j) > 0$. □

References

[1] H. Hindi. A tutorial on convex optimization. In *American Control Conference*, 2004.

[2] S. Boyd and L. Vandenberghe. *Convex Optimization*. Cambridge University Press, 2004.