Supplementary Material

Here we recall how to calculate the moment-generating function $\Lambda(n)$ via zeta-function [24] and periodic orbits [3, 1]. Let $\lambda[A]$ be the maximal eigenvalue of matrix A with non-negative elements [13]. Since AB and BA have identical eigenvalues, we get $\lambda[A^d] = (\lambda[A])^d$, $\lambda[AB] = \lambda[BA]$ (d is an integer).

Recall the content of section 4. Eqs. (15, 3, 4) lead to

$$\Lambda^m(n,m) = \sum\nolimits_{x_1,\dots,x_m} \phi[x_1,\dots,x_m], \tag{42}$$

$$\phi[x_1,\dots,x_m] \equiv \lambda\left[\prod\nolimits_{k=1}^m T_{x_k}\right]\lambda^n\left[\prod\nolimits_{k=1}^m \hat{T}_{x_k}\right] \tag{43}$$

where we have introduced a notation $T_x = T(x)$ for better readability. We obtain

$$\phi[\mathbf{x}', \mathbf{x}''] = \phi[\mathbf{x}'', \mathbf{x}'], \qquad \phi[\mathbf{x}', \mathbf{x}'] = \phi^2[\mathbf{x}'], \tag{44}$$

where $\mathbf{x}'$ and $\mathbf{x}''$ are arbitrary sequences of symbols x_i. One can prove for $\Lambda^m(n,m)$ [24]:

$$\Lambda^m(n,m) = \sum_{k|m} \sum_{(\gamma_1,\dots,\gamma_k)\in \mathrm{Per}(k)} k\,[\,\phi[\gamma_1,\dots,\gamma_k]\,]^{\frac{m}{k}}\,,$$

where $\gamma_i = 1,\dots,M$ are the indices referring to realizations of the HMM, and where $\sum_{k|m}$ means that the summation goes over all k that divide m, e.g., $k = 1,2,4$ for $m = 4$. Here $\mathrm{Per}(k)$ contains sequences $\Gamma = (\gamma_1,\dots,\gamma_k)$ selected according to the following rules: *i)* Γ turns to itself after k successive cyclic permutations, but does not turn to itself after any smaller (than k) number of successive cyclic permutations; *ii)* if Γ is in $\mathrm{Per}(k)$, then $\mathrm{Per}(k)$ contains none of those $k-1$ sequences obtained from Γ under $k-1$ successive cyclic permutations. Starting from (45) and introducing notations $p = k$, $q = \frac{m}{k}$, we transform $\xi(z,n)$ as

$$\xi(z,n) = \exp\left[-\sum_{p=1}^{\infty}\sum_{\Gamma\in\mathrm{Per}(p)}\sum_{q=1}^{\infty}\frac{z^{pq}}{q}\phi^q[\gamma_1,\dots,\gamma_p]\right].$$

The summation over q, $\sum_{q=1}^{\infty}\frac{z^{pq}}{q}\phi^q[\gamma_1,\dots,\gamma_p] = -\ln\left[1 - z^p\phi[\gamma_1,\dots,\gamma_p]\right]$, yields

$$\begin{aligned}\xi(z,n) &= \prod\nolimits_{p=1}^{\infty}\prod\nolimits_{\Gamma\in\mathrm{Per}(p)}\left[1 - z^p\phi[\gamma_1,\dots,\gamma_p]\,\right]\\ &= 1 - z\sum\nolimits_{l=1}^{M}\lambda_l\hat{\lambda}_l^n + \sum\nolimits_{k=2}^{\infty}\varphi_k z^k,\end{aligned} \tag{45}$$

where $\lambda_{\alpha\dots\beta} \equiv \lambda[T_{x_\alpha}\dots T_{x_\beta}]$, $\lambda_{\alpha+\beta} \equiv \lambda[T_{x_\alpha}]\lambda[T_{x_\beta}]$ (all the notations introduced generalize—via introducing a hat—to functions with trial values of the parameters, e.g., $\hat{T}_2$). φ_k are obtained from (45). We write them down assuming that $M = 2$ (two realizations of the observed process)

$$\varphi_2 = -\lambda_{12}\hat{\lambda}_{12}^n + \lambda_{1+2}\hat{\lambda}_{1+2}^n, \tag{46}$$

$$\varphi_3 = \lambda_{2+21}\hat{\lambda}_{2+21}^n - \lambda_{221}\hat{\lambda}_{221}^n + \lambda_{1+12}\hat{\lambda}_{1+12}^n - \lambda_{112}\hat{\lambda}_{112}^n, \tag{47}$$

$$\begin{aligned}\varphi_4 &= -\lambda_{1222}\hat{\lambda}_{1222}^n + \lambda_{2+122}\hat{\lambda}_{2+122}^n + \lambda_{1+122}\hat{\lambda}_{1+122}^n - \lambda_{1122}\hat{\lambda}_{1122}^n\\ &+ \lambda_{2+211}\hat{\lambda}_{2+211}^n - \lambda_{1+2+12}\hat{\lambda}_{1+2+12}^n + \lambda_{1+211}\hat{\lambda}_{1+211}^n - \lambda_{1112}\hat{\lambda}_{1112}^n.\end{aligned} \tag{48}$$

The algorithm for calculating $\varphi_{k\geq 5}$ is straighforward [1]. Eqs. (46–48) for $\varphi_{k\geq 4}$ suffice for approximate calculation of (45), where the infinite sum $\sum_{k=2}^{\infty}$ is approximated by its first few terms.

We now calculate $\xi(z,n)$ for the specific model considered in Section 5.1. For this model, only the first row of T_1 consists of non-zero elements, so we have

$$\lambda_{1\chi 1\sigma} = \lambda_{1\chi+1\sigma}, \qquad \hat{\lambda}_{1\chi 1\sigma} = \hat{\lambda}_{1\chi+1\sigma}, \tag{49}$$

where χ and σ are arbitrary sequences of 1's and 2's. The origin of (49) is that the transfer-matrices $T(1)T(\chi_1)T(\chi_2)\dots$ and $T(1)T(\sigma_1)T(\sigma_2)\dots$ that correspond to 1χ and 1σ, respectively, have the same structure as $T(1)$, where only the first row differs from zero. For φ_k in (45) the feature (49) implies

$$\begin{aligned} \varphi_k &= -\lambda^n[\hat{T}_1\hat{T}_2^{k-1}]\,\lambda[T_1T_2^{k-1}] \\ &+ \lambda^n[\hat{T}_1\hat{T}_2^{k-2}]\,\lambda[T_1T_2^{k-2}]\,\lambda^n[\hat{T}_2]\lambda[T_2]. \end{aligned} \tag{50}$$

To calculate $\lambda\,[T_1T_2^p]$ for an integer p one diagonalizes T_2 [13] (the eigenvalues of T_2 are generically not degenerate, hence it is diagonalizable),

$$T_2 = \sum\nolimits_{\alpha=1}^{L} \tau_\alpha |R_\alpha\rangle\langle L_\alpha|, \tag{51}$$

where τ_α are the eigenvalues of T_2, and where $|R_\alpha\rangle$ and $|L_\alpha\rangle$ are, respectively, the right and left eigenvectors:

$$T_2|R_\alpha\rangle = \tau_\alpha|R_\alpha\rangle,\ \langle L_\alpha|T_2 = \tau_\alpha\langle L_\alpha|,\ \langle L_\alpha|R_\beta\rangle = \delta_{\alpha\beta}.$$

Here $\delta_{\alpha\beta}$ is the Kronecker delta. Note that generically $\langle L_\alpha|L_\beta\rangle \neq \delta_{\alpha\beta}$ and $\langle R_\alpha|R_\beta\rangle \neq \delta_{\alpha\beta}$. Here $\langle L_\alpha|$ is the transpose of $|L_\alpha\rangle$, while $|R_\alpha\rangle\langle L_\alpha|$ is the outer product.

Now $\lambda\,[T_1T_2^p]$ reads from (22):

$$\lambda\,[T_1T_2^p] = \sum\nolimits_{\alpha=1}^{L} \tau_\alpha^p \psi_\alpha, \quad \psi_\alpha \equiv \langle 1|T_1|R_\alpha\rangle\langle L_\alpha|1\rangle, \tag{52}$$

where $\langle 1| = (1, 0, \ldots, 0)$. Combining (52, 50) and (45) we arrive at (23).