Lagrangian analysis of the "swimmer" dynamical system

Model of friction

Given a small one-dimensional stick of length dλ in a two-dimensional fluid medium, we postulate two forces that act upon it: a "viscous" friction force normal to the direction of the stick given by $d\vec{F}_n = -k_1\vec{n}(\vec{n}\cdot\vec{v})d\lambda$, and a "laminar" friction force tangent to the direction of the stick given by $d\vec{F}_t = -k_2\vec{t}\,(\vec{t}\cdot\vec{v})d\lambda$, where $\vec{t}$, $\vec{n}$ and $\vec{v}$ are the tangent, normal, and velocity vectors.

Given a stick of length *l*, we can decompose its velocity into the linear center-of-mass velocity and the angular velocity relative to the cm: $\vec{v}(\lambda) = \vec{v}_{cm} + \lambda\vec{n}\,\dot{\theta}$, where $\lambda \in [\frac{-l}{2}, \frac{l}{2}]$. The normal and tangential frictional forces resulting from the linear part will be $\vec{F}_n = -k_1\vec{n}(\vec{n}\cdot\vec{v}_{cm})l$ and $\vec{F}_t = -k_2\vec{t}\,(\vec{t}\cdot\vec{v}_{cm})l$, while the total moment arising from the angular part will be:

$$\tau = \int_{\frac{-l}{2}}^{\frac{l}{2}} \lambda\cdot df = \int_{\frac{-l}{2}}^{\frac{l}{2}} \lambda\cdot(-k_1\lambda\dot{\theta})d\lambda = -\tfrac{k_1}{12}l^3\dot{\theta}$$

Later we will derive these forces from the dissipation function, which is a measure of the rate of energy being lost to friction.

Generalized coordinates

We now connect ***k*** sticks together, and choose our generalized coordinates **q** to be the location of the center of mass and the angles of the sticks relative to the ***x*** direction.

$$\boldsymbol{\theta} = \begin{bmatrix}\theta_1 & \theta_2 & \dots & \theta_k\end{bmatrix}^T \quad \mathbf{q} = \begin{bmatrix}\vec{r}_{cmx} \\ \vec{r}_{cmy} \\ \boldsymbol{\theta}\end{bmatrix}$$

We define the unit tangent and normal vectors,

$$\vec{t}_i = \begin{bmatrix}\cos(\theta_i) \\ \sin(\theta_i)\end{bmatrix} \quad \vec{n}_i = \begin{bmatrix}-\sin(\theta_i) \\ \cos(\theta_i)\end{bmatrix}$$

Defining $\tilde{r}_i$ to be the location of the ***i***th stick relative to the center-of-mass, we can now write the equations that relate the $\tilde{r}_i$'s to the θ_i's:

$$\begin{bmatrix}\tilde{r}_{i+1} - \tilde{r}_i = \frac{l_i}{2}\vec{t}_i + \frac{l_{i+1}}{2}\vec{t}_{i+1} \\ \sum m_i\tilde{r}_i = 0\end{bmatrix}$$

The first k-1 equation can be understood from Figure 1, while the last equation stems from the $\tilde{r}_i$'s being defined in the frame of the center-of-mass.

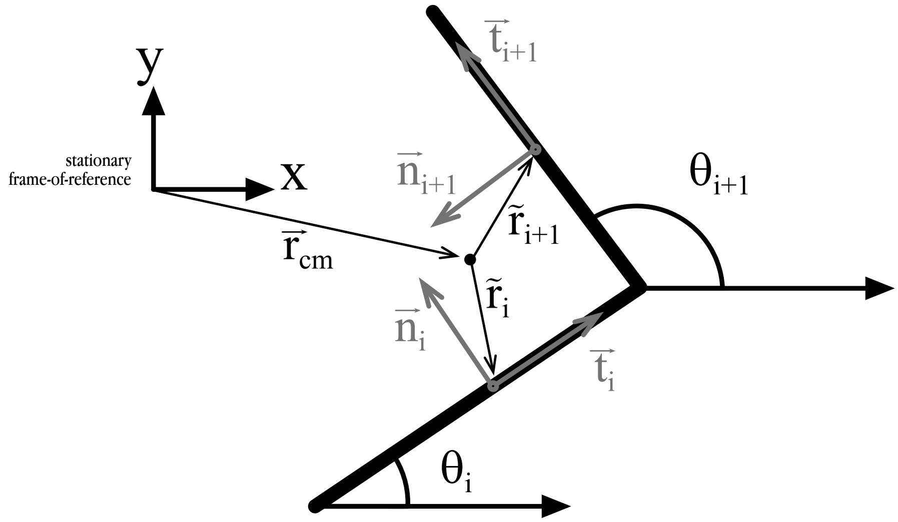

Figure 1

By taking the derivative with respect to time, we get the equations which give the velocities relative to the center-of-mass:

(1) $$\begin{bmatrix} \tilde{v}_{i+1} - \tilde{v}_i = \frac{l_i}{2}\vec{n}_i\dot{\theta}_i + \frac{l_{i+1}}{2}\vec{n}_{i+1}\dot{\theta}_{i+1} \\ \sum_i m_i \tilde{v}_i = 0 \end{bmatrix}$$

<u>Dynamics</u>

We are now in a position to write the equations of the dynamics. The Lagrangian L has only the kinetic energy component and is given by

$$L = \tfrac{1}{2}\vec{v}_{cm}^2 \sum_i m_i + \tfrac{1}{2}\sum_i m_i \tilde{v}_i^{\,2} + \tfrac{1}{2}\sum_i I_i \dot{\theta}_i^2$$

Where I_i is the ith moment-of-inertia $I_i = \frac{1}{12} m_i l_i^2$. The dissipation function is given by

$$F = \tfrac{1}{2}k_1 \sum_i [l_i(\vec{v}_i \cdot \vec{n}_i)^2 + \tfrac{l_i^3}{12}\dot{\theta}_i^2] + \tfrac{1}{2}k_2 \sum_i l_i(\vec{v}_i \cdot \vec{t}_i)^2$$

Note that the velocities in the dissipation function are relative to the fluid, i.e. $\vec{v}_i = \tilde{v}_i + \vec{v}_{cm}$. The dynamics are now given by the E-L equations

$$\tfrac{d}{dt}(\tfrac{\partial}{\partial \dot{q}_i} L) - \tfrac{\partial}{\partial q_i} L + \tfrac{\partial}{\partial \dot{q}_i} F = 0$$

The E-L equations for the center-of-mass are given simply by

$$\dot{\vec{v}}_{cm} \sum_i m_i + k_1 \sum_i (\vec{v}_i \cdot \vec{n}_i)\vec{n}_i + k_2 \sum_i (\vec{v}_i \cdot \vec{t}_i)\vec{t}_i = 0$$

However, for the angular variables θ_i, we will require some better notation.

Matrix definitions

We now define several vectors and matrices which will help us reformulate the problem in a more convenient and legible notation. First we define a set of diagonal matrices:

$$\begin{array}{ll} \Theta = \mathrm{diag}(\boldsymbol{\theta}) & \mathbf{L} = \mathrm{diag}(l_i) \\ \mathbf{M} = \mathrm{diag}(m_i) & \mathbf{I} = \tfrac{1}{12}\mathbf{ML}^2 \\ \mathbf{T}_x = \mathrm{diag}(\cos(\theta_i)) & \mathbf{T}_y = \mathrm{diag}(\sin(\theta_i)) \\ \mathbf{N}_x = \mathrm{diag}(-\sin(\theta_i)) & \mathbf{N}_y = \mathrm{diag}(\cos(\theta_i)) \end{array}$$

Note that for both the x and y dimensions $\dot{\mathbf{N}} = -\mathbf{T}\dot{\Theta}$. Now, defining the relative and absolute velocity vectors,

$$\tilde{\mathbf{v}}_x = \begin{bmatrix} \tilde{v}_{1x} \\ \tilde{v}_{2x} \\ \vdots \\ \tilde{v}_{kx} \end{bmatrix} \quad \tilde{\mathbf{v}}_y = \begin{bmatrix} \tilde{v}_{1y} \\ \tilde{v}_{2y} \\ \vdots \\ \tilde{v}_{ky} \end{bmatrix} \quad \mathbf{v}_x = \begin{bmatrix} \tilde{v}_{1x}+v_{cmx} \\ \tilde{v}_{2x}+v_{cmx} \\ \vdots \\ \tilde{v}_{kx}+v_{cmx} \end{bmatrix} \quad \mathbf{v}_y = \begin{bmatrix} \tilde{v}_{1y}+v_{cmy} \\ \tilde{v}_{2y}+v_{cmy} \\ \vdots \\ \tilde{v}_{ky}+v_{cmy} \end{bmatrix}$$

and the coefficient matrices,

$$\mathbf{Q} = \begin{bmatrix} -1 & 1 & & & \\ & -1 & 1 & & \\ & & -1 & 1 & \\ & & & \ddots & \ddots \\ m_1 & m_2 & m_3 & \cdots & m_k \end{bmatrix} \quad \mathbf{A} = \begin{bmatrix} 1 & 1 & & \\ & 1 & 1 & \\ & & \ddots & \ddots \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

we can now rewrite equations (1) as

$$\begin{array}{l} \mathbf{Q}\tilde{\mathbf{v}}_x = \tfrac{1}{2}\mathbf{ALN}_x\dot{\boldsymbol{\theta}} \\ \mathbf{Q}\tilde{\mathbf{v}}_y = \tfrac{1}{2}\mathbf{ALN}_y\dot{\boldsymbol{\theta}} \end{array} \Rightarrow \begin{array}{l} \tilde{\mathbf{v}}_x = \mathbf{PN}_x\dot{\boldsymbol{\theta}} \\ \tilde{\mathbf{v}}_y = \mathbf{PN}_y\dot{\boldsymbol{\theta}} \end{array} \quad with \quad \mathbf{P} \overset{d}{=} \tfrac{1}{2}\mathbf{Q}^{-1}\mathbf{AL}$$

So that $\frac{\partial}{\partial\dot{\boldsymbol{\theta}}}\tilde{\mathbf{v}}_n = (\mathbf{PN}_x)^T = \mathbf{N}_x\mathbf{P}^T$, and similarly for the tangential velocities.

We will additionally define the normal and tangential velocity vectors (relative to the fluid) as

$$\mathbf{v}_n = \begin{bmatrix} (\tilde{v}_1+\vec{v}_{cm})\cdot\vec{n}_1 \\ (\tilde{v}_2+\vec{v}_{cm})\cdot\vec{n}_2 \\ \vdots \\ (\tilde{v}_k+\vec{v}_{cm})\cdot\vec{n}_k \end{bmatrix} = \mathbf{N}_x\mathbf{v}_x + \mathbf{N}_y\mathbf{v}_y \qquad \mathbf{v}_t = \begin{bmatrix} (\tilde{v}_1+\vec{v}_{cm})\cdot\vec{t}_1 \\ (\tilde{v}_2+\vec{v}_{cm})\cdot\vec{t}_2 \\ \vdots \\ (\tilde{v}_k+\vec{v}_{cm})\cdot\vec{t}_k \end{bmatrix} = \mathbf{T}_x\mathbf{v}_x + \mathbf{T}_y\mathbf{v}_y$$

Also, it will be found convenient to define the constant matrix

$$\mathbf{G} = \mathbf{P}^T\mathbf{MP}$$

The θ dynamics

The Lagrangian is now given by

$$\begin{aligned} L &= \tfrac{1}{2}\vec{v}_{cm}^2\,\mathrm{tr}(\mathbf{M}) + \tfrac{1}{2}\tilde{\mathbf{v}}_x^T\mathbf{M}\tilde{\mathbf{v}}_x + \tfrac{1}{2}\tilde{\mathbf{v}}_y^T\mathbf{M}\tilde{\mathbf{v}}_y + \tfrac{1}{2}\dot{\boldsymbol{\theta}}^T\mathbf{I}\dot{\boldsymbol{\theta}} \\ &= \tfrac{1}{2}\vec{v}_{cm}^2\,\mathrm{tr}(\mathbf{M}) + \tfrac{1}{2}\dot{\boldsymbol{\theta}}^T(\mathbf{N}_x\mathbf{GN}_x + \mathbf{N}_y\mathbf{GN}_y + \mathbf{I})\dot{\boldsymbol{\theta}} \end{aligned}$$

The generalized momenta for the θ coordinates are

$$\tfrac{\partial}{\partial\dot{\boldsymbol{\theta}}}L = [\mathbf{I} + \mathbf{N}_x\mathbf{GN}_x + \mathbf{N}_y\mathbf{GN}_y]\dot{\boldsymbol{\theta}}$$

and their time derivatives

$$\tfrac{d}{dt}(\tfrac{\partial}{\partial\dot{\boldsymbol{\theta}}}L) = [\mathbf{I} + \mathbf{N}_x\mathbf{GN}_x + \mathbf{N}_x\mathbf{GN}_y]\ddot{\boldsymbol{\theta}} - 2[\mathbf{N}_x\mathbf{GT}_x + \mathbf{N}_y\mathbf{GT}_y]\dot{\boldsymbol{\theta}}^2$$

Where $\dot{\boldsymbol{\theta}}^2 = \dot{\Theta}\dot{\theta}$ is the vector of squared velocities.

The partial derivative with respect to the θs is identically zero: $\frac{\partial}{\partial\boldsymbol{\theta}}L = 0$.

The dissipation function is given by

$$F = \tfrac{1}{2}k_1\left(\mathbf{v}_n^T\mathbf{L}\mathbf{v}_n + \tfrac{1}{12}\dot{\boldsymbol{\theta}}^T\mathbf{L}^3\dot{\boldsymbol{\theta}}\right) + \tfrac{1}{2}k_2\mathbf{v}_t^T\mathbf{L}\mathbf{v}_t$$

And since

$$\tfrac{\partial}{\partial\dot{\boldsymbol{\theta}}}\mathbf{v}_n = \tfrac{\partial}{\partial\dot{\boldsymbol{\theta}}}(\mathbf{N}_x\tilde{\mathbf{v}}_x + \mathbf{N}_y\tilde{\mathbf{v}}_y) = \mathbf{N}_x\mathbf{P}^T\mathbf{N}_x + \mathbf{N}_y\mathbf{P}^T\mathbf{N}_y$$

$$\tfrac{\partial}{\partial\dot{\boldsymbol{\theta}}}\mathbf{v}_t = \tfrac{\partial}{\partial\dot{\boldsymbol{\theta}}}(\mathbf{T}_x\tilde{\mathbf{v}}_x + \mathbf{T}_y\tilde{\mathbf{v}}_y) = \mathbf{N}_x\mathbf{P}^T\mathbf{T}_x + \mathbf{N}_y\mathbf{P}^T\mathbf{T}_y$$

the generalized forces are

$$\tfrac{\partial}{\partial\dot{\boldsymbol{\theta}}}F = k_1(\mathbf{N}_x\mathbf{P}^T\mathbf{N}_x + \mathbf{N}_y\mathbf{P}^T\mathbf{N}_y)\mathbf{L}\mathbf{v}_n + \tfrac{k_1}{12}\mathbf{L}^3\dot{\boldsymbol{\theta}} + k_2(\mathbf{N}_x\mathbf{P}^T\mathbf{T}_x + \mathbf{N}_y\mathbf{P}^T\mathbf{T}_y)\mathbf{L}\mathbf{v}_t$$

So that finally the dynamics are given by solving

$$\tfrac{d}{dt}(\tfrac{\partial}{\partial\dot{\boldsymbol{\theta}}}L) - \tfrac{\partial}{\partial\boldsymbol{\theta}}L + \tfrac{\partial}{\partial\dot{\boldsymbol{\theta}}}F = \begin{bmatrix} u_1 \\ u_2 - u_1 \\ \vdots \\ u_{k-1} - u_{k-2} \\ -u_{k-1} \end{bmatrix}$$

for $\ddot{\boldsymbol{\theta}}$.